

NUMERICAL STUDY OF ONE-DIMENSIONAL STEFAN PROBLEM WITH PERIODIC BOUNDARY CONDITIONS

by

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A finite difference approach to a one-dimensional Stefan problem with periodic boundary conditions is studied. The evolution of the moving boundary and the temperature field are simulated numerically, and the effects of the Stefan number and the periodical boundary condition on the temperature distribution and the evolution of the moving boundary are analyzed.

Key words: *Stefan problem, heat transfer, finite difference method, moving interface, temperature field, numerical analysis*

Introduction

This paper considers the following 1-D melting problem in a semi-infinite plane due to the periodically oscillating boundary temperature. This problem can be formulated as [1]:

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial u}{\partial t} \quad t > 0, \quad 0 < r < R \quad (1)$$

$$\frac{dR(t)}{dt} = -\text{Ste} \frac{\partial u(R, t)}{\partial r} \quad t > 0 \quad (2)$$

subject to:

$$u(R, t) = 0 \quad t > 0 \quad (3)$$

$$u(0, t) = 1 + \varepsilon \sin(\omega t) \quad t > 0, 1 > \varepsilon > -1 \quad (4)$$

$$R(0) = 0 \quad (5)$$

where ε is the amplitude of the periodically oscillating boundary temperature, ω – the oscillation frequency, and Ste – the Stefan number given by $(C\Delta u_{\text{ref}})/l$, where C is the specific heat capacity, l – the latent heat, and Δu_{ref} – the reference temperature. In the case of $\varepsilon = 0$, this melting problem corresponds to 1-D Stefan problem with time-independent boundary condition.

Finite difference scheme

We establish a finite difference scheme to solve the above system (1)-(5) by using the invariant-space-grid method. Let Δr be the forward-moving distance of the phase change

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interface each time, namely the constant space grid size. Thus $R = N\Delta r$ is the position of the moving interface at $t = t_N$, with $t_0 = 0$ and $N = 0, 1, \dots$. Using a backward difference scheme for the time derivative and a central difference scheme for the space derivative, eq. (1) in discrete form can be expressed:

$$\frac{u_{j+1}^N - 2u_j^N + u_{j-1}^N}{(\Delta r)^2} = \frac{u_j^N - u_j^{N-1}}{\Delta t}, \quad j = 1, 2, \dots, N-1 \quad (6)$$

where $r_j = j\Delta r$ and $\Delta t = t_N - t_{N-1}$.

The Stefan condition (2) at $r_N = N\Delta r$ in discrete form is:

$$\frac{\Delta r}{\Delta t} = -\text{Ste} \frac{3u_N^N - 4u_{N-1}^N + u_{N-2}^N}{2\Delta r} \quad (7)$$

Introducing the two variables $a = 1/\Delta r$ and $b = \Delta r/\Delta t$, eqs. (6) and (7) can then be transformed into:

$$au_{j+1}^N - (2a + b)u_j^N + au_{j-1}^N = -bu_j^{N-1}, \quad j = 1, 2, \dots, N-1 \quad (8)$$

$$b = -\text{Ste} \frac{3u_N^N - 4u_{N-1}^N + u_{N-2}^N}{2\Delta r} \quad (9)$$

Boundary conditions (3)-(5) in discrete form, respectively, are:

$$u_N^N = 0 \quad (10)$$

$$u_0^N = 1 + \varepsilon \sin(\omega t_N) \quad (11)$$

$$R(t_0) = 0 \quad (12)$$

The set of eqs. (8)-(12) are the finite difference scheme of the set of partial differential eqs. (1)-(5) for the melting problem in the half plane.

Numerical experiment and sensitivity analysis

For $\varepsilon = 0$, the exact solution of the Stefan problem defined by eqs. (1)-(5) is [2-4]:

$$u(r, t) = 1 - \frac{1}{\text{erf}(\lambda)} \text{erf}(r\lambda) \quad (13)$$

$$R(t) = 2\lambda\sqrt{t} \quad (14)$$

where erf is the error function, and the value of λ is determined from the following transcendental equation:

$$\sqrt{\pi} \lambda \exp(\lambda^2) \text{erf}(\lambda) = \text{Ste} \quad (15)$$

A constant space grid size $\Delta r = 0.01$ is used for the numerical calculations. In order to initialize our numerical procedures and to circumvent the singularity at $t = 0$, *i. e.* $R(t) = 0$, the temperature distribution and the position of the moving interface obtained by eqs. (13)-(15) are used to approximate the corresponding physical quantities at t_1 and t_2 after $t = 0$ for 1-D melting problem with $u(r = 0, t) = 1 + \varepsilon \sin(\omega t)$.

Consider the finite difference scheme for solving the Stefan problem with periodic boundary condition by using iteration method. An initial estimate $b^{(0)}$ of the unknown b is given from equation $b = -\text{Ste}(3u_{N-1}^{N-1} - 4u_{N-2}^{N-1} + u_{N-3}^{N-1})/(2\Delta r)$. The unknown u_j^N ($j=1, 2, \dots, N-1$) can be obtained by substituting $b^{(0)}$ for b in eq. (8). Then the correcting value $b^{(1)}$ is gained from eq. (9), and the correcting temperature values (u_j^N , $j=1, 2, \dots, N-1$) are obtained by substituting $b^{(1)}$ for b in eq. (8). The iterative procedure can be implemented repeatedly until the condition $|b^{(m+1)} - b^{(m)}| < 10^{-10}$ is satisfied. Thus the time t_N is given by $t_N = t_{N-1} + \Delta t$ and the temperature distribution at $t = t_N$ can be obtained.

Effect of the Stefan number

Figure 1 shows that the evolution and the velocity of the moving interface as a function of time for three different values of the Stefan number for an oscillation amplitude of 0.5 and frequency of $\pi/2$. Under the same periodic boundary conditions, the position of the moving interface is close to $R(t) = 1.982$ at $t = 10.0$, and approximately $R(t) = 2.761$ at $t = 20.0$ for $\text{Ste} = 0.2$. But in the case of $\text{Ste} = 2.0$, the position of the moving interface is approximately $R(t) = 5.124$ at $t = 10.0$, and about $R(t) = 7.209$ at $t = 20.0$. Thus one may conclude that the motion of the moving interface relies very strongly on the Stefan number. For larger Stefan number, the velocity of the moving interface is larger and the growth rate of the moving interface is also influenced more strongly. On the other hand, in the case of the same Stefan number, the growth of the moving interface is faster when the phase change domain is smaller. When the domain size is very large, the velocity of the moving interface is small and almost diminishes linearly as the domain grows. Moreover, when the phase change domain size increases, the oscillating amplitude of the velocity of the moving boundary decreases and the effect of the periodic surface temperature on the velocity of the moving boundary also lessens, as shown in fig. 1(b).

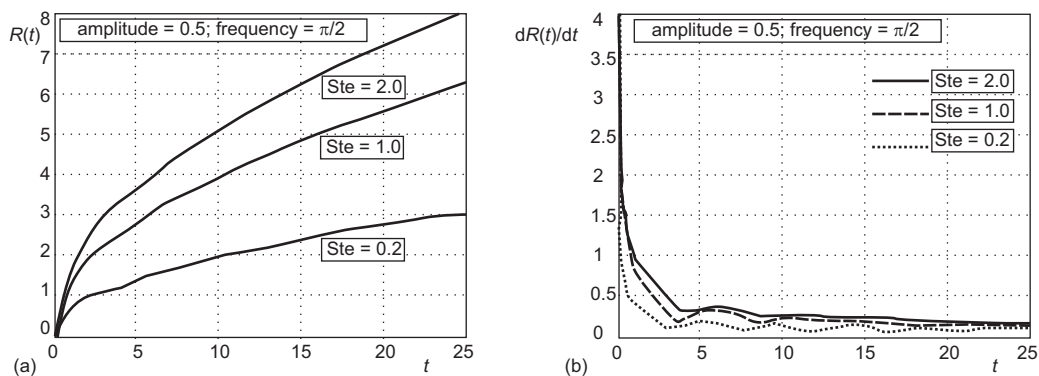


Figure 1. Evolution and velocity of the moving interface as a function of time for $\varepsilon = 0.5$ and $\omega = \pi/2$

Effect of the amplitude

Figure 2 shows that the motion and the velocity of the moving interface as a function of time for an oscillation amplitude of 0.9 and frequency of $\pi/2$. By comparing with fig. 1, in the case of the same Stefan number and the same oscillation frequency, the oscillation amplitude of the velocity of the moving boundary for $\varepsilon = 0.9$ is larger than that of the moving boundary for $\varepsilon = 0.5$. When the oscillation amplitude of the periodic surface temperature is

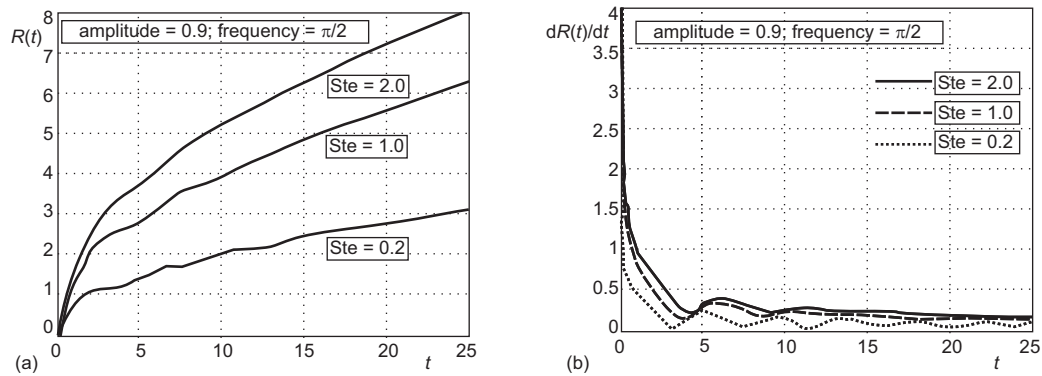


Figure 2. Evolution and velocity of the moving interface as a function of time for $\varepsilon = 0.9$ and $\omega = \pi/2$

larger, its effect on the velocity of the moving boundary can last for a longer time, namely the affected phase change domain size is also larger. Moreover, it can be seen from figs. 1(a) and 2(a) that the positions of the moving interfaces are very close at any given time point for two different amplitudes $\varepsilon = 0.5$ and $\varepsilon = 0.9$, and can overlap periodically. On the other hand, it is just in a small region near the fixed boundary that the evolution of the moving interface is influenced more strongly by the oscillation amplitude. When the phase change domain size is sufficiently large, the effects of two different amplitudes on the evolutions of the moving interfaces become almost negligible, as shown in figs. 1(a) and 2(a).

Figure 3 shows that the temperature distribution in the phase change domain for two different values of the amplitude of the periodically oscillating boundary temperature for $Ste = 1.0$ and $\omega = \pi/2$. For smaller phase change domain size, the temperature changes rapidly in the whole domain, while for larger domain size, the temperature is changing more clearly in only about the left half of the domain. The oscillating characteristic of the temperature distribution is noticeable only in a small region near the fixed boundary. Moreover, the larger oscillation amplitude $\varepsilon = 0.9$ can lead to a more pronounced change in the temperature distribution. On the other hand, for larger domain size, the temperature in the right half of the domain essentially declines linearly, *i. e.* the rate of temperature change is relatively small and almost remains constant, as can be seen from figs. 3(a) and 3(b). Consequently, for given oscillation

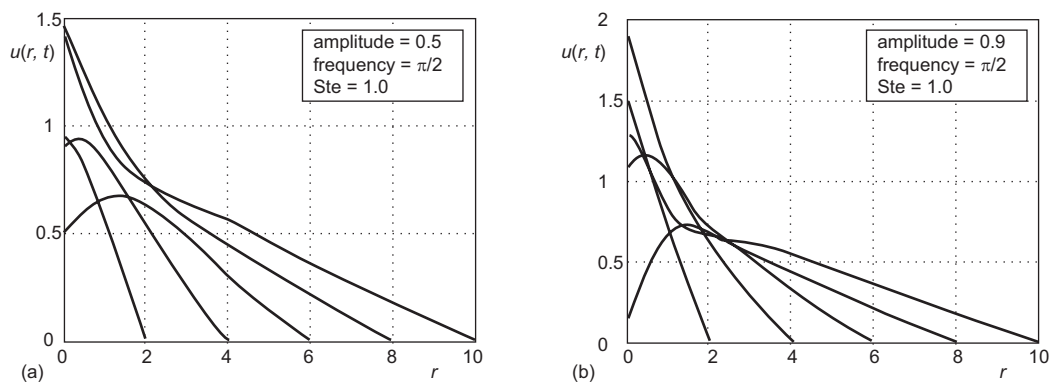


Figure 3. Temperature distributions for two different oscillation amplitudes for $Ste = 1.0$ and $\omega = \pi/2$
(for color image see journal web site)

frequency, not only the oscillation amplitude of the periodically oscillating boundary temperature but also the sizes of the phase change domain strongly influence the temperature distribution. The response of the temperature field to the periodically oscillating boundary temperature is more rapid only in a smaller region near the fixed boundary.

Effect of the frequency

Figure 4 shows that for larger oscillation frequency of the periodically oscillating surface temperature, its effect on the evolution of the moving interface is smaller and the oscillating period of the velocity of the moving interface is also smaller. In the time interval $0 \leq t \leq 20$, the oscillating characteristic of the velocity of the moving interface is very obvious and the oscillating amplitude of the velocity diminishes rapidly with time, especially for larger frequency of the periodically oscillating surface temperature. In fig. 4(a), the motion of the moving boundary for $\omega = \pi/2$ agrees well with the motion of the moving boundary for $\omega = \pi$. The motion of the moving boundary for $\omega = \pi/10$ is markedly different from the motions of the moving boundaries for two cases of $\omega = \pi/2$ and $\omega = \pi$. But it is worthwhile to note that the locations of the moving interfaces overlap periodically after a time step for the three cases of $\omega = \pi/10$, $\omega = \pi/2$ and $\omega = \pi$. Furthermore it is easily found that the time step is about 20 corresponding to the forcing period of the smaller frequency ($\omega = \pi/10$).

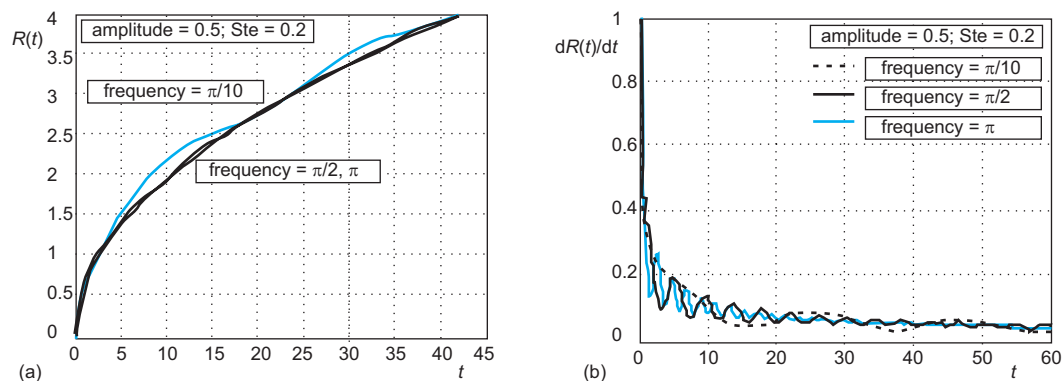


Figure 4. Evolution and velocity of the moving interface for three different frequencies for $\varepsilon = 0.5$ and $Ste = 0.2$ (for color image see journal web site)

Conclusions

A finite difference approach is established to solve 1-D phase change problem with periodic boundary condition by using an invariant-space-grid method. The position of the moving boundary can be tracked by considering the forward-moving distance of the phase change interface during a small time interval. The evolution of the moving boundary and the temperature field are simulated numerically. The effects of the Stefan number, the amplitude and frequency of the periodically oscillating boundary temperature on the evolution of the moving boundary and the temperature distribution are analyzed. Numerical experiments show that, for given amplitude and frequency, the Stefan number strongly influences the temperature distribution and the evolution of the moving boundary, especially for larger Stefan number. But the effect of the oscillating surface temperature on the evolution of the moving boundary is very pronounced when the phase change domain is small and diminishes as the do-

main grows. The response of the temperature field to the periodically oscillating surface temperature is more rapid only in a smaller region near the fixed boundary. Moreover, the oscillating characteristic of the velocity of the moving boundary strongly relies on the oscillation frequency of the periodically oscillating surface temperature.

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