Stability and Vibrations of Geometrically Nonlinear Cylindrically Orthotropic Circular Plates

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The stability analysis of axisymmetrical equilibrium states of geometrically nonlinear, orthotropic, circular plates that are deformed by multiparameter loading, including thermal influence, is presented. The dynamic method (method of small vibrations) is used to accomplish this purpose. The behavior of the plate in different cases is revealed. In particular, it is shown that two different types of snapping processes can occur. The values of frequencies of small eigenvibrations from various cases have been calculated. These investigations are realized by numerical and qualitative methods. Here only the numerical results are presented.

Introduction

Axisymmetrical deformations of geometrically nonlinear cylindrically orthotropic circular plates under a multiparametric system of loading where thermal stresses are also taken into account are investigated. In theses cases there may be nonuniqueness of equilibrium states, i.e., for the same parameter of loading or temperature, there can exist a number of equilibrium states for the plate. This effect may lead to a loss of stability by snapping of different kinds. Therefore, there is a necessity to study the stability of all the possible equilibrium states.

The numerical method used for investigating the stability of the equilibrium states is the well-known dynamical method (method of small vibrations). To the best of our knowledge, this method was used for the first time for nonlinear shells in [1] and systematically for isotropic geometrically nonlinear plates and shells in [2–5].

Since eigenfrequencies (eigenvalues) are the basis for this method (when the eigenfrequencies are real the examined equilibrium states are stable in the corresponding sense, and when they are imaginary these states are unstable) it is necessary to find them for plates under different conditions. As opposed to the linear case the eignfrequencies in question are dependent on the values and character of the external cross forces and temperature.

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In this paper we present some of the results obtained from the numerical investgations of both the aforementioned problems in axisymmetrical state of the art. Equilibrium states, buckling modes or vibrations of nonsymmetrical types are not considered here.

Basic Equations

The constitutive Hooke-Neumann law for the material is given by (1) [6, 7].

$$\bar{\epsilon}_r = \frac{\bar{\sigma}_r}{E_r} - \frac{\mu_\phi}{E_\phi}\bar{\sigma}_\phi + \alpha_r \bar{T}; \quad \bar{\epsilon}_\phi = -\frac{\mu_r}{E_r}\bar{\sigma}_r + \frac{\bar{\sigma}_\phi}{E_\phi} + \alpha_\phi \bar{T};$$
$$E_r \mu_\phi = E_\phi \mu_r, \tag{1}$$

where $\bar{\epsilon}_{r,\phi}$, $\bar{\sigma}_{r,\phi}$ are the components of deformations and stresses, respectively, in any layer at a distance z from the middle plane, which is at the same time also the neutral one. $E_{r,\phi}$, $\mu_{r,\phi}$, and $\alpha_{r,\phi}$ are Young's moduli, Poisson's ratios, and the thermal coefficients in radial and circumferential directions, respectively, when the centers of orthotropy and of the middle plane coincide. These parameters are taken to be independent of the stationary temperature $T(\rho,z)$, which can be a function of z and of the dimensionless radial coordinate $\rho = r/a$, and a is the radius of the plate.

For an arbitrary anisotropic body, the following inequality holds [6]:

$$\mu_r + \mu_\phi + \mu_z \leq 3/2 \tag{2}$$

If we take into account the Kirchhoff-Love hypothesis of incompressibility of the plate in the z direction, inequality (2) reduces to (3).

$$\mu_r + \mu_\phi \leq 1 \tag{3}$$

The corresponding basic equations of the Karman-

Marguerre type may be written in this case in the form (4) and (5) [7, 8].

$$L_k[(\omega(\rho,\tau)] = k^2\left\{-\frac{\theta^2(\rho,\tau)}{2\rho} + \frac{a^2}{h^2}\left[\frac{1-\psi^2}{\rho}N_T(\rho) - \psi^2 N'_T(\rho)\right]\right\}. \quad (4)$$

$$L_k[\theta(\rho,\tau)] = -\frac{m_a}{\rho}\left\{\int_0^\rho \rho[q(\rho,\tau) - s\,\ddot{W}(\rho,\tau)]d\rho + P - \omega(\rho,\tau)\theta(\rho,\tau)\right\} + \frac{12a^2}{h^2}\left[\frac{(1-k^2\mu_r) - k^2\psi^2(1-\mu_r)}{\rho}M_T(\rho) + (1+\mu_r k^2\psi^2)M'_T(\rho)\right]; \quad (0 \le \rho \le 1); \quad (0 \le \tau < \infty). \quad (5)$$

$\tau = t(g/h)^{1/2}$ is the dimensionless time and t is the real one. h = const. is the thickness of the plate, and the gravitational acceleration is denoted by g. Derivatives with respect to ρ and τ are denoted by ($'$) and ($\dot{}$), respectively. The dimensionless specific mass on one unit of the plate's area is $s = \gamma a^4/E_r h$; γ is the material's specific gravity.

$$k^2 = E_\phi/E_r; \quad \psi^2 = \alpha_\phi/\alpha_r; \quad m_a = 12(1-\mu_r\mu_\phi) = 12(1-\mu_r^2 k^2) \quad (6)$$

$$N_T = \alpha_r\int_{-1/2}^{1/2} T(\rho,\zeta)d\zeta; \quad M_T = \alpha_r\int_{-1/2}^{1/2}\zeta T(\rho,\zeta)d\zeta; \quad \zeta = z/h. \quad (7)$$

The operator L_K is

$$\rho^2 L_K(\ ) = \rho^2(\ )'' + \rho(\ )' - k^2(\ ) \quad (8)$$

The basic unknowns are the membrane stress function ω and the angle of revolution θ of the normal to the plate's midplane. All the parameters of this problem are expressed by these functions (see, for example, the relationships referred to in the following where the corresponding dimensional (physical) quantities are denoted by asterisks).

The membrane normal forces and bending moments are given by the formulas:

$$N_r(\rho,\tau) = N_r^*(\rho,\tau)a^2/E_r h^2 = \omega(\rho,\tau)/\rho;$$
$$N_\phi(\rho,\tau) = N_\phi^*(\rho,\tau)a^2/E_r h^2 = \omega'(\rho,\tau) \quad (9)$$
$$M_r(\rho,\tau) = M_r^*(\rho,\tau)a^2/D_r h = \theta'(\rho,\tau) + \mu_r k^2\theta(\rho,\tau)/\rho - 12a^2(1+k^2\psi^2\mu_r)M_T(\rho)/h^2;$$
$$M_\phi(\rho,\tau) = M_\phi^*(\rho,\tau)a^2/D_r h = k^2[\theta(\rho,\tau)/\rho + \mu_r\theta'(\rho,\tau)] - 12a^2k^2(\psi^2+\mu_r)M_T(\rho)/h^2;$$
$$D_r = E_r h^3/[12(1-\mu_r{}^2k^2)] = E_r h^3/m_a \quad (10)$$

The angle θ and bending moment M_r are positive when the convexity of the plate's deformed shape is directed downward as well as that of the positive direction of the z-axis.

The radial and vertical displacements of the midplane's points are given by the formulas:

$$U = U^*/h = \rho a\epsilon_\phi/h = \rho h[\omega' - \mu_r k^2\omega/\rho]/ak^2 + a\psi^2\rho N_T/h \quad (11)$$

$$W = W^*/h = -\int_0^\rho \theta d\rho + \xi; \quad \xi = W(0); \quad \theta(r,\tau) = -W'(\rho,\tau) \quad (12)$$

Positive U is directed away from the plate's center; W and ξ are positive in the direction of the Z-axis, and are measured from the undeformed midplane.

The dimensionless distributed and concentrated loads, which are positive in the downward direction, are expressed by the relationships

$$q(\rho,\tau) = q^*(\rho,\tau)a^4/E_r h^4; \quad P(\tau) = P^*(\tau)a^2/2\pi E_r h^4. \quad (13)$$

The most frequently used boundary conditions can be written in the forms:

$$\omega(0,\tau) \equiv \theta(0,\tau) \equiv 0; \quad (0 \le \tau < \infty) \quad (14)$$

$$\alpha_1\omega'(1,\tau) + \beta_1\omega(1,\tau) = \gamma_1(\tau);$$
$$\alpha_2\theta'(1,\tau) + \beta_2\theta(1,\tau) = \gamma_2(\tau); \quad (15)$$

α_i, β_i, $\gamma_i(\tau)$ are given quantities. The conditions (14) ensure the continuity and boundness of the functions ω and θ in the vicinity of the center. The condition at $\rho = 1$ for ω characterizes the degree of mobility of the boundary supports in the plane of the plate. The second relationship (15) characterizes the type of the support affecting the bending conditions at the boundary. Nine basic combinations of the boundary conditions are given in Table 1.

If necessary, the initial conditions (for $\tau = 0$) may be written in the recognized standard form.

The preceding described state of the art of the problem in question is valid for a plate without a central hole and with only inertia due to the defections W taken into account. But there are no definable obstacles to adding the influence of other factors, if necessary. Similar state of the art is true for the different cases of constructively orthotropic plates, for example corrugated or reinforced ones.

Method and Algorithm

In this section are described the method and, based on it,

Table 1 Nine basic combinations of the boundary conditions

	Mobile support without boundary forces	Mobile support with active normal boundary forces $N(\tau)$	Immovable support
Conditions for ω	$\alpha_1 = 0$; $\beta_1 = 1$; $\gamma_1 \equiv 0$;	$\alpha_1 = 0$; $\beta_1 = 1$; $\gamma_1(\tau) = N(\tau)$	$\alpha_1 = 1$; $\beta_1 = -\mu_r k^2$; $\gamma_1 = -\frac{a^2}{h^2}k^2\psi^2 N_T(1)$
Conditions for θ	$\alpha_2 = 0$; $\beta_2 = 1$; $\gamma_2 \equiv 0$	$\alpha_2 = 1$; $\beta_2 = \mu_r k^2$; $\gamma_2 = \frac{12a^2}{h^2}(1+k^2\psi^2\mu_r)M_T(1)$	$\alpha_2 = 1$; $\beta_2 = \mu_r k^2$; $\gamma_2(\tau) = M(\tau) + \frac{12a^2}{h^2}(1+k^2\psi^2\mu_r)M_T(1)$
	Clamping	Hinged support without boundary bending moment	Hinged support with boundary bending moment $M(\tau)$

the numerical algorithm used to find the frequencies of the plate's small eigenvibrations around some of its equilibrium states. Let this equilibrium state be characterized by the functions $\omega_c(\rho)$ and $\theta_c(\rho) = -W'_c(\rho)$ (see (12)). Thus, the aforementioned small eigenvibrations can be expressed in forms (16)

$$\omega(\rho,\tau) = \omega_c(\rho) + \delta\omega(\rho,\tau); \quad W(\rho,\tau) = W_c(\rho) + \delta W(\rho,\tau)\cdot$$
$$\theta(\rho,\tau) = \theta_c - (\delta W)' = \theta_c(\rho) + \delta\theta(\rho,\tau), \quad (16)$$

where $\delta\omega$, δW, and $\delta\theta$ are small dynamic perturbations. Then, substituting in equations (4) and (5) the functions ω, W, and θ by their expressions (16) and linearizing afterward with respect to the aforementioned perturbations, we will obtain the following two linear equations for them:

$$L_K(\delta\omega) = -\frac{k^2}{\rho}\theta_c\delta\theta; \quad L_K(\delta\theta) = \frac{m_a}{\rho}\left[s\int_0^\rho \rho(\delta\ddot{W})d\rho\right.$$
$$\left.+\omega_c\delta\theta + \theta_c\delta\omega\right]; \quad (17)$$

The corresponding boundary conditions resulting from (14), (15), and (16) are given by (18).

$$\delta\omega(0,\tau) \equiv \delta\theta(0,\tau) \equiv 0; \quad \alpha_1\delta\omega'(1,\tau) + \beta_1\delta\omega(1,\tau) \equiv 0;$$
$$\alpha_2\delta\theta'(1,\tau) + \beta_2\delta\theta(1,\tau) \equiv 0; \quad (18)$$

The homogeneity of equations (17) and (18) is a consequence of the invariability of loadings and temperature field when the crossing from the investigated equilibrium state to the corresponding eigenvibrations takes place.

It is convenient to attach to the second equation (17) another form which does not obviously contain δW. For this we can substitute δW by $\delta\theta$ (using (12)) and integrate by parts the integral of (17). Then, we get (19) instead of (17).

$$\rho L_K(\delta\omega) = -k^2\theta_c\delta\theta;$$
$$\rho L_K(\delta\theta) = m_a\left\{\frac{s}{2}\frac{\partial^2}{\partial\tau^2}\left[\rho^2\int_0^1\delta\theta d\rho - \rho^2\int_0^\rho\delta\theta d\rho\right.\right.$$
$$\left.\left.+\int_0^\rho \rho^2\delta\theta d\rho\right] + \omega_c\delta\theta + \theta_c\delta\omega\right\} \quad (19)$$

We shall look for the periodic solution of (19) in the form

$$\delta\omega(\rho,\tau) = \Omega(\rho)\sin p\tau; \quad \delta\theta(\rho,\tau) = Q(\rho)\sin p\tau. \quad (20)$$

Then, after separation of variables, we will obtain the corresponding equations for Ω and Q and the boundary conditions for them.

$$\rho L_K(\Omega) = -k^2\theta_c Q; \quad \rho L_K(Q) = m_a(\omega_c Q + \theta_c\Omega) -$$
$$-\frac{\lambda^2}{2}\left[A\rho^2 - \rho^2\int_0^\rho Qd\rho + \int_0^\rho \rho^2 Qd\rho\right] \quad (21)$$

where

$$\lambda^2 = m_a s p^2; \quad A = \int_0^1 Qd\rho. \quad (22)$$

$$\Omega(0) = Q(0) = 0; \quad \alpha_1\Omega'(1) + \beta_1\Omega(1) = 0;$$
$$\alpha_2 Q'(1) + \beta_2 Q(1) = 0. \quad (23)$$

Thus, we obtained naturally a linear boundary value problem for the eigenvalues λ^2. It is obvious that if the first (lowest) eigenvalue λ_1^2 will be positive (the corresponding frequency, p, will be real), the investigated equilibrium state is stable in the small (as regarding small perturbations). In the opposite case, the equilibrium state is unstable. When $\lambda_1^2 = 0$ we have a critical case from the point of view of stability in the small. It is very important to underline here that stability in the small does not preclude stability in the large (as regarding the large perturbations), but instability in the small means instability in general.

Before proceeding to the description of the algorithm of numerical solution of the aforementioned boundary value problem we must note the following two properties.

(a) The operator L_K contains a singularity at the point $\rho = 0$ (see (8)). Because of this, the solutions to (19) which satisfy the condtions (23) at $\rho = 0$ have in neighborhood of $\rho = +0$ the order ρ^k and then they may be represented there in the form (24) [9].

$$\Omega(\rho) \cong a\rho^K; \quad Q(\rho) \cong b\rho^k; \quad (0 \le \rho \le \Delta\rho << 1) \quad (24)$$

(a, b are some constants.)

(b) The preceding formulated boundary value problem is linear and homogeneous and, in consequence, the functions Ω and Q may be found only with exactness to an arbitrary factor.

Taking into account the singularity of L_K we will begin the realization of the numerical solutions from the point $\rho = \Delta\rho << 1$ instead of $\rho = 0$. Then, on the basis of (24):

$$k\Omega(\Delta\rho) = \Omega'(\Delta\rho)\Delta\rho; \quad kQ(\Delta\rho) = Q'(\Delta\rho)\Delta\rho \quad (25)$$

On the basis of the second property, we can multiple Q by an arbitrary factor taking $A = 1$ (see second relation (22)). In this case, since the value of A is fixed beforehand, the system (19) is no more homogeneous and then the solutions of this system (where $A = 1$) may be represented as follows:

$$\Omega(\rho) = \Omega'(\Delta\rho)M_1(\rho) + Q'(\Delta\rho)N_1(\rho) + C_1(\rho);$$
$$Q(\rho) = \Omega'(\Delta\rho)M_2(\rho) + Q'(\Delta\rho)N_2(\rho) + C_2(\rho);$$
$$(0 < \Delta\rho \le \rho \le 1) \quad (26)$$

where $\Omega'(\Delta\rho)$ and $Q'(\Delta\rho)$ are arbitrary constants. The initial parameters $\Omega(\Delta\rho)$ and $Q(\Delta\rho)$ are considered known, since they may be found by (25) when $\Omega'(\Delta\rho)$ and $Q'(\Delta\rho)$ are given. The basic functions M_i, N_i, and C_i may be found by solving any three arbitrary independent Cauchy problems. We have used for numerical solutions of these Cauchy

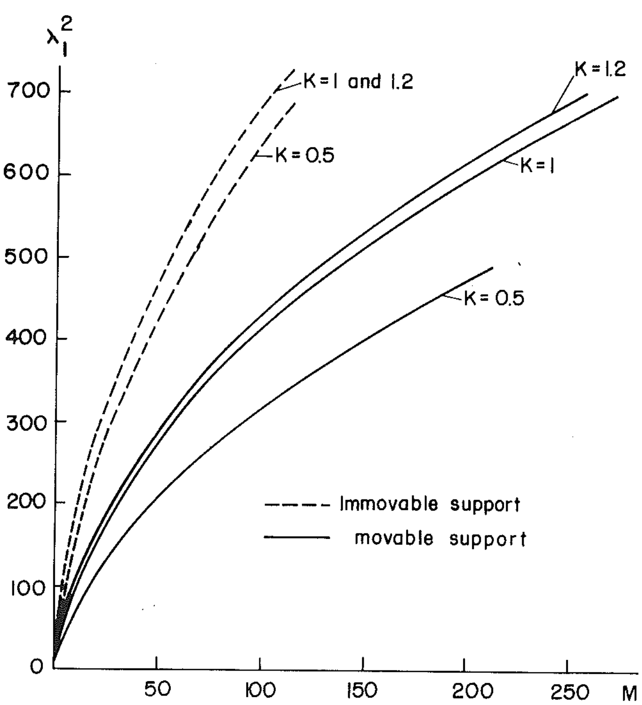

Fig. 1 First eigenvalue λ_1^2 versus loading (boundary moment M) for k = 0.5, 1, and 1.2 and hinged supports

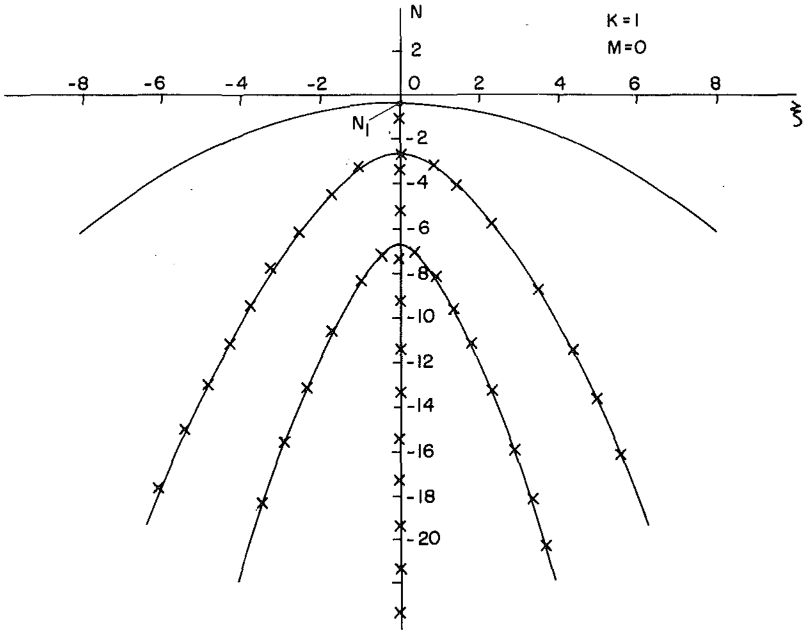

Fig. 2 Prebuckling and postbuckling behavior of the plate when only edge thrust N acts. Hinged support. $k = 1$.

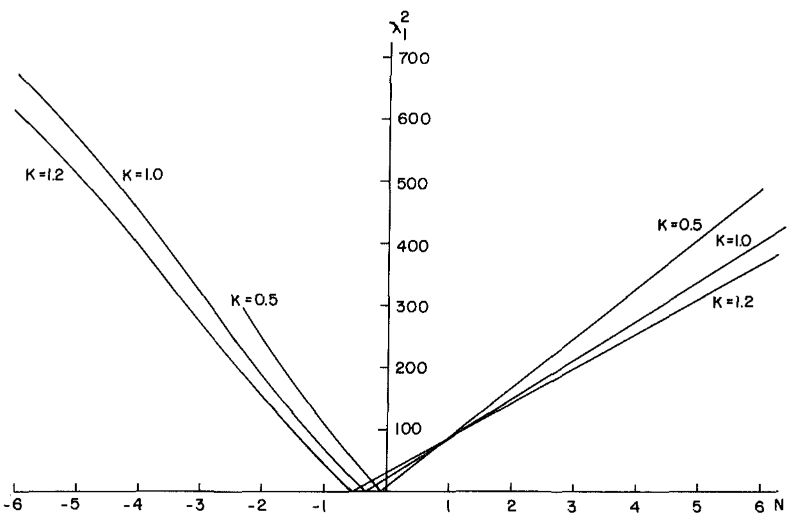

Fig. 3 First eigenvalue λ_1^2 versus edge thrust N for $k = 0.5$, 1, and 1.2

problems, the Euler's method of fourth order with corrections at each step, described in [10].

Knowing the functions M_i, N_i, and C_i, we can fulfill the conditions (23) at $\rho = 1$ and obtain two linear algebraic equations for $\Omega'(\Delta\rho)$ and $Q'(\Delta\rho)$. In this way we will get the unknown $\Omega(\rho)$ and $Q(\rho)$ which are dependent on λ^2. The eigenvalues λ^2 are finally the roots of the equation (27)

$$A = 1, \text{ i.e., } \int_0^1 Q(\rho)\,d\rho = 1 \tag{27}$$

The numerical determination of ω_c and θ_c are realized by the "shooting" method [4] or by the so-called "deformation map" technique [9].

Some Numerical Results

First, we will give the graphs of the dependence of the first eigenvalues λ_1^2 on loading which is the edge bending moment M (see Fig. 1). We took, for example, $k = 0.5$, 1, and 1.2, and $\mu_r = 0.3$. (For all specific cases examined in this paper Poisson's ratio is $\mu_r = 0.3$.) The supports are hinged, both movable (without boundary thrust N), and immovable. The thermal stresses are not considered here. These data demonstrate that the frequencies of the small eigenvalues depend on the value of the crossloading. This fact, typical for geometrically nonlinear objects, is caused by membrane stresses neglected in the usual linear theory of bending. All the eigenfrequencies are real; that is a natural consequence of the stability of the plate's equilibrium states in this case. The increase of k leads to the increase of λ_1^2 for fixed values of M. This phenomenon is explained by the rise of the plate's bending rigidity together witht the increase of k. The same effect of k on the behavior of the plate is also observed for static deformations. The graphs of $\lambda_1^2(\xi)$, not presented here, are also monotical by increasing curves, but are concave and not convex as the lines $\lambda_1^2(M)$. $\xi = W(0)$ is the displacement of the plate's center (see (12)).

We shall now examine the behavior of the plate under membrane edge thrust N when the thermal stresses are absent. The graph of $N(\xi)$ shown in Fig. 2 is typical of this case. Here $K = 1$ and the contour is movable hinged. All branches of the

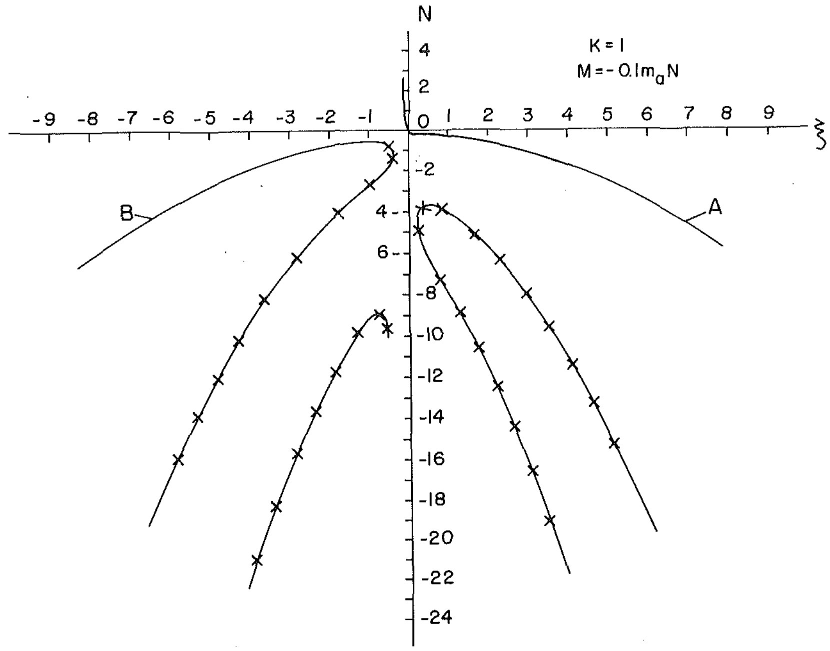

Fig. 4 Prebuckling and postbuckling behavior of the plate when the edge thrust N has an eccentricity $e = -0.1m_a$. Hinged support. $k = 1$.

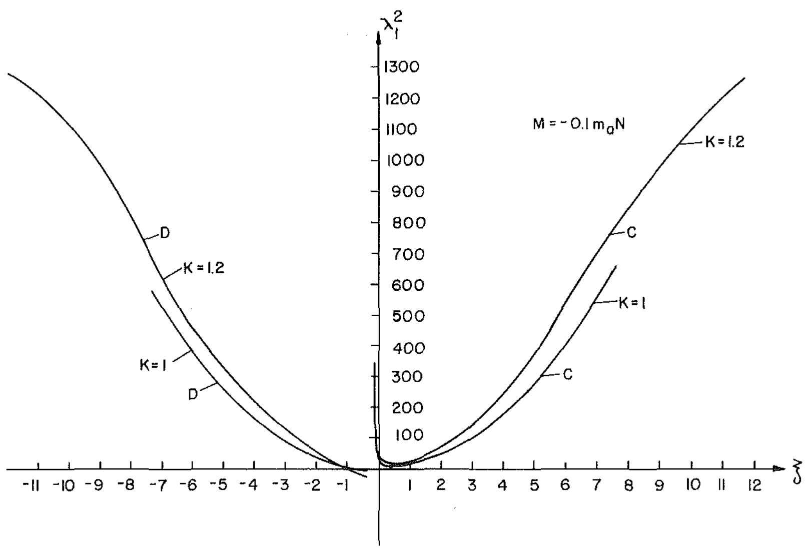

Fig. 5 First eigenvalue λ_1^2 versus $\xi = W(0)$ for the case shown on Fig. 4. $k = 1$ and 1.2.

graph of $N(\xi)$ of the parabolic type represent the postbuckling deformations. The intersections between these branches and the axis N determine the classical critical values of N, when the stability loss by transition from the compressed (flat) forms of the plate to axisymmetric bending ones occurs. (In [9] are given the first three critical values of N for different k and movable hinge.)

The investigations of the stability (by the aforementioned algorithm) of the equilibrium states in this case demonstrate that all the equilibrium states denoted on Fig. 2 by "x" are unstable. For them all the eigenvalues λ_1^2 are negative. For the other states the lowest eigenvalues $\lambda_1^2 > 0$ with the exception of first critical (bifurcation) states (when $N = N_1$) where $\lambda_1^2 = 0$. These effects are given by the graphs on Fig. 3. Each curve on this figure is formed by two branches which start from the corresponding bifurcation point ($\lambda_1^2 = 0$). The right-hand branches correspond to the flat prebuckling equilibrium states, and the left-hand ones characterize the postcritical stable bent states. For the precritical states the rise of N leads to the increase of λ_1^2 linearly. In the postcritical region these relationships are, naturally, nonlinear. Thus, after bifurcation the plate's route may be only one of the two branches of the graph of $N(\xi)$ (see Fig. 2) starting from N_1. This assertion is valid only for axisymmetric deformations which are the subject of this paper. In general, secondary bifurcation phenomena may occur; these are linked with the transition from the axisymmetrical bent (postcritical) state to the asymmetrical (wrinkled) one. The foregoing is partly investigated in [12] for isotropic plates.

Now we will examine the influence of the eccentricity e of the edge thrust N on the phenomena studied in the previous case. The eccentricity may be interpreted as an imperfection of the loading. Let us say that the edge support is movable hinged. Then, to the thrust N is added a boundary moment $M = eN$, where $e > 0$ when N is below the middle plane of the plate and $e < 0$ in the contrary case. In Fig. 4 is given an

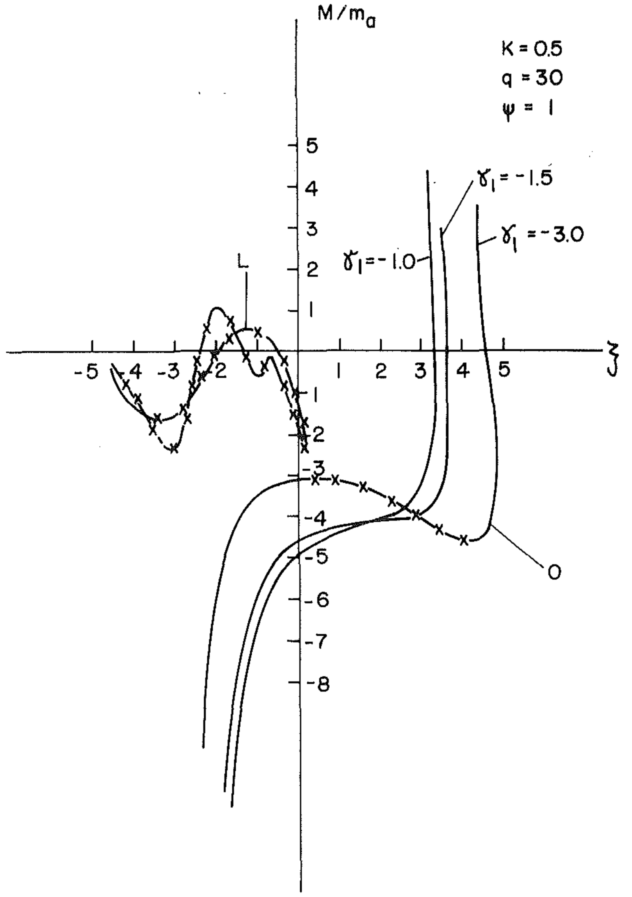

Fig. 6 Plot of $M(\xi)$ for the thermoelastic case when the thermal influence is characterized by parameter γ_1. The crossloading is $q = 30$ and $k = 0.5$.

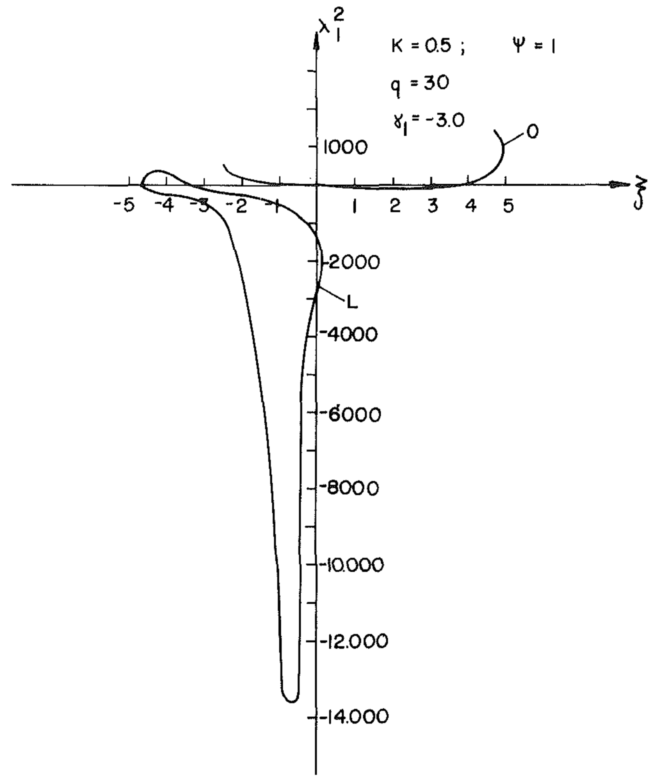

Fig. 7 Plot of λ_1^2 versus ξ for the case shown in Fig. 6 and $\gamma_1 = -3$

example of the graph of $N(\xi)$ when $e = -0.1\, m_a$ and $k = 1$. Here are the typical phenomena generated by the forementioned imperfection. The bifurcation points disappear, converting into limit ones, which is in complete concordance with Koiter's theory of postcritical behavior of structures with imperfections (see, for example, [13]). Our investigations of the stability of the equilibrium states demonstrate that all states noted by "x" on Fig. 4 are unstable. The graphs $\lambda_1^2(\xi)$ for stable states and two values of k are drawn on Fig. 5. Each graph consists of parts C and D. The C parts correspond to the curves of the type A shown on Fig. 4 and branches D correspond to the lines of the type B (see Fig. 4). There is a minimum point on all C lines (see Fig. 5). It may be shown that this minimum point corresponds to the inflection point that exists on each curve of the type A (Fig. 4).

It may be shown that similar behavior of the plate takes place also in another case of imperfection of the loading, when aside from N, the moment M = const•acts independently from N.

In conclusion we will examine a group of thermoelastic problems when in the basic equations (4) and (5) all the thermal terms are absent. Then the influence of the temperature field is transferred by the boundary conditions (see Table 1). This may occur particularly in the following two cases:

(a) An isotropic plate ($k = \psi = 1$) where the temperature is a function only of z. Then N_T and M_T are constant.

(b) An orthotropic plate that is thermally isotropic ($k \neq 1$, $\psi = 1$) and T is an even function of z and does not depend on ρ. Thus N_T = const. and $M_T \equiv 0$.

We took for example the following specific case from the abovementioned second group of thermoelastic problems, $k = 0.5$. Acting here are edge moment M and uniform pressure $q = 30$. The support is an immovable hinge. The boundary conditions at $\rho = 1$ are: (see Table 1)

$$\theta'(1) + \mu_r k^2 \theta(1) = M; \quad \omega'(1) - \mu_r k^2 \omega(1) = \gamma_1 = -a^2 k^2 \psi^2 N_T / h^2 \tag{28}$$

From the three graphs of $M(\xi)$ drawn in Fig. 6 the dynamics of the alteration in the behavior of the plate when the value of $\gamma_1 < 0$ decreases (fixed $q = 30$) can be seen. Even $\gamma_1 > -1.5$ all the equilibrium states are stable because from each value of M only one equilibrium state exists. For $\gamma_1 < -1.5$ this uniqueness is disturbed which generates a stability loss of some of the equilibrium states. It is characteristic of the examined case that there are two branches in each graph $M(\xi)$ when $\gamma_1 < -1.5$. (See the graph $M(\xi)$ for $\gamma_1 = -3.0$.) One of these has a classical "open" form 0 with one maximum and one minimum, and the other branch is a complicated closed loop L. Our investigations of the stability of the equilibrium states demonstrated that all the states denoted by "x" on Fig. 6 are unstable. These conclusions are confirmed by the data for λ_1^2, when $\gamma_1 = -3.0$ given on Fig. 7. Curves 0 and L on Fig. 7 correspond to the ones in Fig. 6. Thus, there exist snapping processes of two different types. Snapping from one part of the open branch 0 to another part on the same curve, or snapping to a "stable" part of the loop L can occur. If the first type of snapping can be realized by continuous change of M, the second type of snapping can occur only by means of intervention of a certain external factor. The last makes it possible for the plate to overcome the corresponding energetical barrier by transitions from a part of the "open" line to the "stable" one of the loop.

In some of our earlier works we investigated cases for isotropic spherical shells with a clamped support loaded by a constant pressure (with no thermal stresses), where similar loops were obtained. We found that all the equilibrium states corresponding to the points of the loops were unstable [4, 5] contrary to the aforementioned case.

Now we can easily analyze the case when all the data of the previous example remain the same, except q whose sign is reversed.

Let $\gamma_1 = -3.0$ (as previously) and $q = -30$. It may be proven by qualitative methods (see [9, 11]) that the graph of $M(\xi)$ for this case ($q = -30$) may be obtained from the graph drawn in Fig. 6 (where $q = 30$) by reflection of the latter relative to the origin of the coordinate system. In other words, both the graphs of $M(\xi)$ (for $q = 30$ and -30) are situated symmetrically relative to the points $\xi = 0$, $M = 0$. Denoting the parameters of the equilibium states with $q \pm 30$ by additional indices "1" and "2", respectively, we may affirm the following. For each equilibrium state ω_1, θ_1 there exists a corresponding state ω_2, θ_2, linked with the first by relation (29)

$$\omega_1(\rho,\tau) \equiv \omega_2(\rho,\tau); \quad \theta_1(\rho,\tau) \equiv -\theta_2(\rho,\tau) \tag{29}$$

and then $\lambda_{i1} = \lambda_{i2}$. Because $\xi_1 = -\xi_2$ and $M_1 = -M_2$, the corresponding points of the plane (ξ, M) are situated symmetrically relative to the origin $\xi = 0$, $M = 0$.

Concluding Remarks

A numerical algorithm for investigations of the stability phenomena in the small of the equilibrium states of orthotropic plates was developed. This algorithm is based on the well-known dynamic method for studying the stability of equilibrium states. Thanks to its general character, this algorithm may be used for different cases, as for example, investigations of the stability of corrugated or constructively orthotropic axisymmetrically deformed plates and shells of revolution. The numerical solutions of a series of specific problems was preceded by some qualitative investigations of the properties of the studied solutions, similar to the investigations described in [9, 11]. These properties, which have an independent interest, also serve as the means of a qualitative control of the numerical results. They will be discussed in a separate paper.

The numerical solutions presented for a number of characteristic specific problems have demonstrated that the plates can lose stability by snapping of different kinds when they are subjected to multiparameter loading including the influence of the temperature field.

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