F. J. PIERCE
Assistant Professor of Mechanical Engineering, Cornell University, Ithaca, N. Y. Assoc. Mem. ASME

The Turbulent Flow at the Plane of Symmetry of a Collateral Three-Dimensional Boundary Layer

Momentum integral equations for the turbulent flow at the plane of symmetry of a three-dimensional boundary layer are rigorously derived. The use of orthogonol curvilinear coordinates allows a simple physical interpretation to be given to the terms of the resulting equations. Evaluation and comparison are made between the derived results and earlier works in Cartesian sets and ambiguities are discussed.

Results of an experimental program are reported for the case of a plane of symmetry flow in a collateral three-dimensional turbulent boundary layer wherein four different momentum integral equations are examined in predicting boundary-layer growth. As an aside, two common variations of shape parameter equations were also tested to determine their adequacy in application to this case.

Introduction

The three-dimensional turbulent boundary layer, while undoubtedly one of the most common boundary-layer flows encountered, is perhaps the least understood. For example, in the problem areas of external airflow over nonaxially symmetric airframes and the internal flow of turbomachinery and irregular shaped ducts, the problem of the three-dimensional turbulent boundary layer still is awaiting a satisfactory and hopefully a simple treatment. As in the case of two-dimensional turbulent boundary-layer flows, the initial works in the three-dimensional problem are based on studies of the phenomenological nature of turbulent flows.

The particular problem reported on herein is a special case of the more general and considerably more complicated three-dimensional turbulent boundary-layer flows. Yet the case considered is of importance not only as an introduction to the more complex general flow, but in its own right, the plane of symmetry flow studied appears in many two and three-dimensional diffusers.

As is well known, one of the most useful methods of analysis for turbulent boundary-layer flows lies in the use of momentum integral equations. Since a formal and rigorous analysis of the plane of symmetry flow is lacking in the literature, the following analysis is presented, with specialization to the collateral flow case made as a last step.

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Momentum Integral Equations

The problem concerns itself with flow over a flat surface and the equations of motion are written for a streamline coordinate system generated about the physical boundary and its parallel planes, the free-stream streamline directions as projected onto this family of planes, and the orthogonal trajectories of these projected free-stream streamline directions.

Formally, a triply orthogonal curvilinear coordinate set (ξ, η, ζ) is generated about the Lamé family [1][1] of parallel plane surfaces η = const defined by the physical boundary to the flow. The surfaces ξ = const and ζ = const are generated, respectively, by the projections of the free-stream streamlines and the orthogonal trajectories of these projections onto the η = const surfaces. It is necessary to speak of the projections of the free-stream streamline direction since the existence of the boundary layer implies that, in general, none of the free-stream velocity vectors will actually be contained in any of the η = const surfaces.

The coordinate system with its plane of symmetry is pictured in Fig. 1. The ξ-direction is approximately the free-stream direction. ζ is the lateral direction and η is the normal direction to the bounding surface. The metrics are, respectively, h_1, h_2, h_3 corresponding to the ξ, η, ζ directions with the additional simplification that if the value of unity is assigned to the constant metric h_2, the arc length $d\eta$ is the distance normal to η = const surfaces.

The boundary-layer equations are obtained following Howarth's [2] treatment for laminar incompressible flow. For turbulent flow, however, the laminar shear terms in the boundary-layer equations are replaced by the turbulent shears. There has been considerable discussion as to the adequacy of such a simple

[1] Numbers in brackets designate References at end of paper.

Nomenclature

$C_f = \dfrac{\tau_0}{\rho U^2/2}$ = skin-friction coefficient
d = tunnel width at entrance to test section
h_1, h_2, h_3 = metrics in the orthogonal curvilinear coordinate set
H = shape factor
K_1, K_2 = curvatures
p = pressure
$q = \dfrac{\rho U^2}{2}$ = free-stream dynamic pressure
$R_\theta = \dfrac{U\theta}{\nu}$ = Reynolds number
s = arc length
$\bar{u}, \bar{v}, \bar{w}$ / $\bar{U}, \bar{V}, \bar{W}$ = boundary layer and free-stream velocity components in orthogonal curvilinear set
u, v, w / U, V, W = boundary layer and free-stream velocity components in Cartesian set
$x\,y\,z$ = Cartesian set
α = flow angle with respect to plane of symmetry
β = included angle for diffuser
δ = boundary-layer thickness
ρ = density
ν = kinematic viscosity
τ_0 = wall shear in plane of symmetry
τ_ξ, τ_ζ = shear component
τ_s = defined in reference [22]
$\xi\,\eta\,\zeta$ = orthogonal curvilinear set

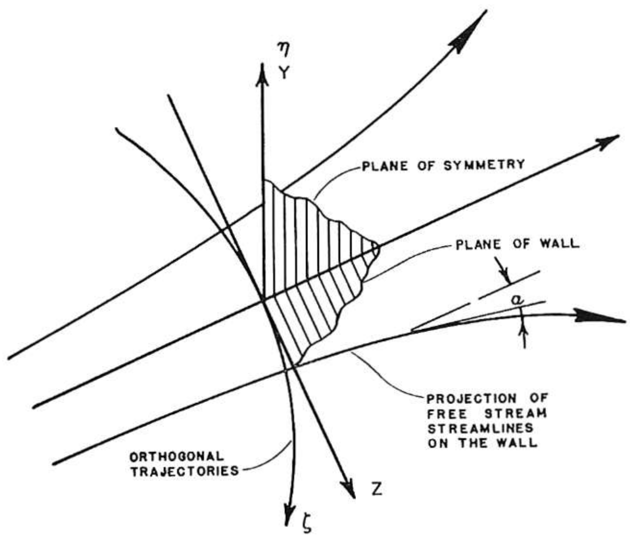

Fig. 1 Coordinate systems and the plane of symmetry

substitution. Several writers have shown that in the case of two-dimensional turbulent flows, the classically accepted momentum integral equation which is based on such a substitution is not adequate in predicting skin-friction coefficient values. In particular, as separation is approached, the integral equation yields increasing values rather than decreasing values. Newman [3], Hewson [4], Bidwell [5], Goldschmiedt [6], and Ross [7] all consider the omission of terms originating from pressure gradients through the boundary layer and/or turbulence. There is, however, a lack of any general agreement as to precisely which terms should be included. Clauser [8] indicates that weak migratory flows could create the differences in the predicted C_f values that are attributed to these omitted terms. However, the adequacy of simply replacing the laminar shear term by the turbulent shear term appears seriously questionable only in regions of separating flows. The equations below, and any results which follow, which are based on such a simple substitution, should therefore be used with caution in regions of separation.

The boundary-layer equations of motion in the free-stream and lateral directions and the continuity equations follow.

$$\frac{\bar{u}}{h_1}\frac{\partial \bar{u}}{\partial \xi} + \bar{v}\frac{\partial \bar{u}}{\partial \eta} + \frac{\bar{w}}{h_3}\frac{\partial \bar{u}}{\partial \zeta} - K_2\bar{u}\bar{w} + K_1\bar{w}^2 = -\frac{1}{\rho h_1}\frac{\partial p}{\partial \xi} + \frac{1}{\rho}\frac{\partial}{\partial \eta}(\tau_\xi) \quad (1)$$

$$\frac{\bar{u}}{h_1}\frac{\partial \bar{w}}{\partial \xi} + \bar{v}\frac{\partial \bar{w}}{\partial \eta} + \frac{\bar{w}}{h_3}\frac{\partial \bar{w}}{\partial \zeta} + K_2\bar{u}^2 - K_1\bar{u}\bar{w} = -\frac{1}{\rho h_3}\frac{\partial p}{\partial \zeta} + \frac{1}{\rho}\frac{\partial}{\partial \eta}(\tau_\zeta) \quad (2)$$

$$\frac{1}{h_1}\frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \bar{v}}{\partial \eta} + \frac{1}{h_3}\frac{\partial \bar{w}}{\partial \zeta} - K_1\bar{u} - K_2\bar{w} = 0 \quad (3)$$

where $K_1 = -\frac{1}{h_1h_3}\frac{\partial h_3}{\partial \xi}$ the nonzero principal curvature of the ξ = const surface in the direction of the parameter ζ and $K_2 = -\frac{1}{h_1h_3}\frac{\partial h_1}{\partial \zeta}$ the nonzero principal curvature of the ζ = const surface in the direction of the parameter ξ. A more readily available reference for these equations is Rouse [9].

It is worth noting that the boundary-layer equations are more complex in this form, not, however, due to the curvature of the physical bounding surface (whose first and second curvatures are both zero since it is by definition flat), but due to the nonzero curvature of the other two members of the triply orthogonal family of surfaces.

The preceding boundary-layer equations are valid over the entire flat plane. The plane of symmetry problem is simplified by the following conditions valid on this plane.

$$K_2 = 0 \quad (4)$$

$$\bar{w} = 0 \text{ hence } \frac{\partial \bar{w}}{\partial \xi} = \frac{\partial \bar{w}}{\partial \eta} = 0 \quad (5a)$$

$$\frac{1}{h_3}\frac{\partial \bar{u}}{\partial \zeta} = \frac{1}{h_3}\frac{\partial \bar{U}}{\partial \zeta} = \frac{1}{h_3}\frac{\partial p}{\partial \zeta} = 0 \quad (5b)$$

Additionally on this plane the metric h_1 may also be assigned the value unity, hence $d\xi$ is the distance normal to the ξ = const surfaces (but only on the plane of symmetry).

Hence on the plane of symmetry the motion and continuity equations in the boundary layer are

$$\bar{u}\frac{\partial \bar{u}}{\partial \xi} + \bar{v}\frac{\partial \bar{u}}{\partial \eta} + \frac{1}{\rho}\frac{\partial p}{\partial \xi} = \frac{1}{\rho}\frac{\partial \tau_\xi}{\partial \eta} \quad (6)$$

$$\frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \bar{v}}{\partial \eta} + \frac{1}{h_3}\frac{\partial \bar{w}}{\partial \zeta} - K_1\bar{u} = 0 \quad (7)$$

Additionally the equation of motion in the ξ-direction in the free stream is

$$\bar{U}\frac{\partial \bar{U}}{\partial \xi} + \frac{1}{\rho}\frac{\partial p}{\partial \xi} = 0 \quad (8)$$

On the plane of symmetry only one boundary-layer equation of motion remains and the lack of knowledge of the turbulent shear distribution through the boundary layer is overcome by turning to a momentum integral equation. It is convenient to express the curvature term K_1 in terms of a more easily recognizable physical quantity, namely, the projection onto the flat surface of the free-stream streamline direction and its orientation with respect to the plane of symmetry. The angle between the tangent to the projected free-stream streamline direction and the plane of symmetry is designated α.

The curvature K_1 may then be expressed in terms of the equation relating the rate of change of the unit tangent vector with arc length

$$K_1 = \frac{\partial \bar{T}}{\partial s} = \frac{\partial \alpha}{\partial s} = -\frac{1}{h_3}\frac{\partial \alpha}{\partial \zeta}$$

Substituting this expression into equation (7), together with equation (8), the motion equation for the boundary layer (6) is integrated to yield the momentum integral equation for the plan-of-symmetry flow.

$$\frac{\partial \theta_\xi}{\partial \xi} + (2\theta_\xi + \delta_\xi^*)\frac{1}{\bar{U}}\frac{\partial \bar{U}}{\partial \xi} + \frac{1}{h_3}\frac{\partial \theta_{\xi\zeta}}{\partial \zeta} + \frac{\theta_\xi}{h_3}\frac{\partial \alpha}{\partial \zeta} = \frac{C_f}{2} \quad (9)$$

with the usual definitions

$$\theta_\xi = \frac{1}{\bar{U}^2} = \int_0^\delta (\bar{U} - \bar{u})\bar{u}d\eta$$

$$\theta_{\xi\zeta} = \frac{1}{\bar{U}^2}\int_0^\delta (\bar{U} - \bar{u})\bar{w}d\eta$$

$$\delta_\xi^* = \frac{1}{\bar{U}}\int_0^\delta (\bar{U} - \bar{u})d\eta$$

The use of orthogonal curvilinear coordinates permits a simple physical interpretation of equation (9) following the suggestions of Johnston [10]. The term $\frac{1}{h_3}\frac{\theta_{\xi\zeta}}{\partial \zeta}$ represents the rate of change, with respect to the ζ-direction, of the transport of free-stream direction momentum in the lateral direction. The term $\frac{\theta_\xi}{h_3}\frac{\partial \alpha}{\partial \zeta}$ accounts for the free-stream direction momentum which, while being transported downstream, is also being distributed laterally, not by a lateral velocity but by the spreading of the flow. It should be emphasized here that a lateral velocity refers to a velocity normal to the free-stream streamline direction and not necessarily normal to the plane of symmetry.

If a boundary-layer flow has a velocity component lateral to the free-stream streamline direction, the flow is commonly called skewed as in reference [11]. It is clear then that if no lateral flows exist in the boundary layer with $\theta_{\xi\zeta} = 0$, only the $\frac{\theta_\xi}{h_3}\frac{\partial\alpha}{\partial\zeta}$ term discussed previously will remain. Such a nonskewed (or collateral flow) may occur on two and three-dimensional diffuser walls and in the case of some axisymmetric flows. The term remaining $\frac{\theta_\xi}{h_3}\frac{\partial\alpha}{\partial\zeta}$ is related to the spreading of the streamlines in such a wedge type flow.

This plane-of-symmetry momentum integral equation may also be written in a Cartesian set. The Cartesian set is oriented with respect to the curvilinear coordinates at the plane of symmetry as shown in Fig. 1. Near the plane of symmetry one may write

$$\bar{U} \approx U$$

$$\bar{u} \approx u + w\alpha$$

$$\bar{w} \approx w - u\alpha$$

From these it may be shown that

$$\delta_\xi^* = \delta_x^*$$

$$\theta_\xi = \theta_x$$

$$\frac{\partial\theta_{\xi\zeta}}{\partial\zeta} = h_3\left[\frac{\partial\theta_{xz}}{\partial z} - \theta_x\frac{\partial\alpha}{\partial z}\right]$$

and

$$\frac{\partial\theta_\xi}{\partial\xi} = \frac{\partial\theta_x}{\partial x}$$

Since on the plane of symmetry

$$\frac{\partial x}{\partial\zeta} = 0 \qquad \frac{\partial x}{\partial\xi} = 1$$

$$\frac{\partial z}{\partial\xi} = 0 \qquad \frac{\partial z}{\partial\zeta} = 1$$

and

$$\frac{\partial}{\partial\zeta} = h_3\frac{\partial}{\partial z} \qquad \frac{\partial}{\partial\xi} = \frac{\partial}{\partial x}$$

These relations are used to express equation (9) as follows,

$$\frac{\partial\theta_x}{\partial x} + (2\theta_x + \delta_x^*)\frac{1}{U}\frac{\partial U}{\partial x} + \frac{\partial\theta_{xz}}{\partial z} = \frac{C_f}{2} \qquad (10)$$

With the usual definitions

$$\theta_x = \frac{1}{U^2}\int_0^\delta (U - u)u\,dy$$

$$\theta_{xz} = \frac{1}{U^2}\int_0^\delta (U - u)w\,dy$$

$$\delta_x^* = \frac{1}{U}\int_0^\delta (U - u)dy$$

In this case, the physical meaning of the term $\frac{\partial\theta_{xz}}{\partial z}$ is not as clear as in the curvilinear streamline set since one may have a rate of change of the x-direction momentum transport in this z-direction due to either or both "skewing" and "spreading." This indistinguishability occurs since in a Cartesian set even the wedge-type flow has a boundary-layer velocity in the lateral or z-direction.

It is important to observe the differences between equations (9) and (10) and to note that the $\frac{\partial\theta_{xz}}{\partial z}$ term of the Cartesian system is the equivalent of the two terms $\frac{1}{h_3}\frac{\partial\theta_{\xi\zeta}}{\partial\zeta}$ and $\frac{\theta_\xi}{h_3}\frac{\partial\alpha}{\partial\zeta}$ of the curvilinear set. In particular, the distinction between derivatives in the z and ζ-directions must be carefully made as these derivatives are not equal even on the plane of symmetry. These distinctions are not clear in references [10 and 12].

The Plane of Symmetry for Collateral Boundary-Layer Flow. For the particular problem investigated experimentally, the boundary-layer flow is assumed collateral. With this assumption equation (9) is slightly simplified.

$$\frac{\partial\theta_\xi}{\partial\xi} + (2\theta_\xi + \delta_\xi^*)\frac{1}{\bar{U}}\frac{\partial\bar{U}}{\partial\xi} + \frac{\theta_\xi}{h_3}\frac{\partial\alpha}{\partial\zeta} = \frac{C_f}{2} \qquad (11)$$

Using the previously stated plane-of-symmetry conditions

$$\frac{\theta_\xi}{h_3}\frac{\partial\alpha}{\partial\zeta} = \theta_x\frac{\partial\alpha}{\partial z} = \frac{\partial\theta_{xz}}{\partial z}$$

equation (10) may be written as

$$\frac{\partial\theta_x}{\partial x} + (2\theta_x + \delta_x^*)\frac{1}{U}\frac{\partial U}{\partial x} + \theta_x\frac{\partial\alpha}{\partial z} = \frac{C_f}{2} \qquad (12)$$

and this is the most convenient form of the momentum integral equation to work with. The momentum integral equation for a two-dimensional flow is written

$$\frac{d\theta}{dx} + (2\theta + \delta^*)\frac{1}{U}\frac{dU}{dx} = \frac{C_f}{2} \qquad (13)$$

The similarity between the two-dimensional case and the plane of symmetry in collateral flow is clear. The additional term $\partial\alpha/\partial z$ may be evaluated from either the assumption of an ideal wedge flow or from a continuity consideration of the free-stream flow, the former giving

$$\frac{\partial\alpha}{\partial z} = \frac{1}{x_0}$$

where x_0 is the distance from the point source of the wedge flow, and the latter

$$\frac{\partial\alpha}{\partial z} = -\frac{1}{U}\frac{\partial U}{\partial x}$$

These together with equation (12) yield the momentum integral equations for the plane of symmetry in the collateral flow in their most useful forms as

$$\frac{\partial\theta_x}{\partial x} + \frac{\theta_x}{U}\frac{\partial U}{\partial x} + \frac{\delta_x^*}{U}\frac{\partial U}{\partial x} = \frac{C_f}{2} \qquad (14)$$

$$\frac{\partial\theta_x}{\partial x} + \theta_x\left[\frac{2}{U}\frac{\partial U}{\partial x} + \frac{1}{x_0}\right] + \frac{\delta_x^*}{U}\frac{\partial U}{\partial x} = \frac{C_f}{2} \qquad (15)$$

The latter of these was used by Kehl [13] and Norbury [14] while the former, which has stronger physical basis for use, was used by this writer.

In addition to equations (14) and (15) which were derived for the special case of plane-of-symmetry flow for the collateral boundary layers, the momentum integral equation of Johnston [15, 16] for the plane of symmetry in a skewed boundary layer was considered.

In the notation of the coordinate system of Fig. 1, Johnston writes

$$\frac{1}{h_3}\frac{\partial\theta_{\xi\zeta}}{\partial\zeta} = \frac{1}{h_3}\frac{\partial\alpha}{\partial\zeta}(\delta_\xi^* - \theta_\xi) = \frac{1}{-\bar{U}}\frac{\partial\bar{U}}{\partial\xi}(\delta_\xi^* - \theta_\xi)$$

This result, based on calculations and some experimental evidence, together with other plane-of-symmetry conditions already stated, allows equation (9) to be written as

$$\frac{\partial\theta_x}{\partial x} + \frac{2\theta_x}{U}\frac{\partial U}{\partial x} = \frac{C_f}{2} \quad (16)$$

Equation (16), which has no dependence on shape parameters, is for plane-of-symmetry flows in skewed boundary layers. The result, however, appears independent of the amount or strength of skewing, and the relation was examined experimentally in this work on the assumption that this relation would still be valid for a weakly skewed (or nearly collateral) flow. While agreement would be perhaps fortuitous, the simplicity in the absence of the shape parameter is clearly desirable.

Wall Friction. In the evaluation of boundary-layer parameter growth (i.e., momentum thickness, shape factor), it is necessary to predict the local skin-friction coefficient, generally with a closed form expression of the type of Ludwieg and Tillmann [17].

$$C_f = 0.246[\exp(-1.56H)]R_\theta^{-0.268} \quad (17)$$

There appear to be, however, no measured data for wall shear in any three-dimensional, turbulent boundary-layer flows, and since the flow model under consideration was analogous to an axisymmetric flow, the precedence established for the axisymmetric case was followed and, in fact, equation (17) was used in all calculations reported herein. Since the flows were of variable Euler number, it is clear from Fig. 2 that a constant value or a simple expression such as the Squire and Young or Schultz-Grunow law would be in fact inadequate.

Solution of Plane-of-Symmetry Flow. Solution of a plane-of-symmetry flow would require simultaneous solution of a momentum integral equation such as (14) or (15), a wall shear coefficient law as equation (17), and a variation of shape parameter equation. It is not implied that these equations would predict a separation point since, while it may be argued that in a three-dimensional diffuser a separation point will be first expected at the plane of symmetry, the three-dimensional separation problem has not been sufficiently investigated, and criteria for separation points and lines in three-dimensional boundary-layer problems are not clear. In particular, the notion that shape factor alone should predict separation on the plane of symmetry is not established, although Johnston [10] does show evidence which supports the two-dimensional rule for a singular separation point encountered with a plane-of-symmetry flow. Additionally, as discussed earlier, the adequacy of the momentum equation used here in regions of separation is questionable.

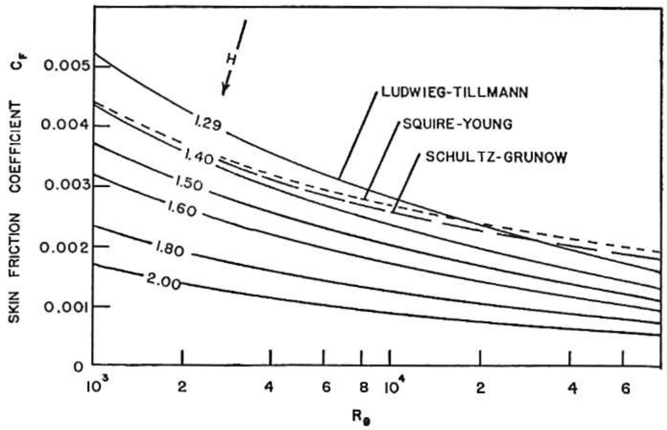

Fig. 2 Comparison of one and two-parameter skin-friction coefficient laws

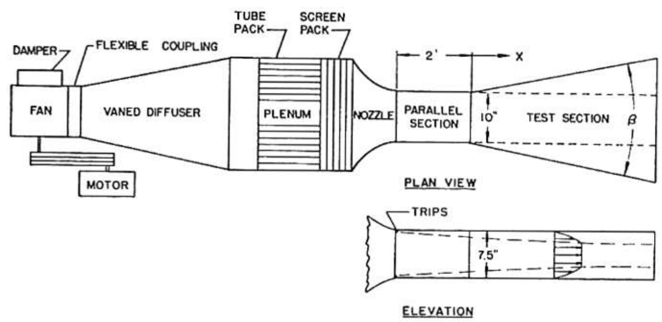

Fig. 3 Schematic view of experimental apparatus

Experimental Apparatus and Program

The plane-of-symmetry flow for a collateral three-dimensional turbulent boundary layer was examined experimentally in the apparatus shown schematically in Fig. 3. Room air was forced through a plenum 35-in. square filled with flow straighteners and several screens ranging from 14 × 18 mesh to 40 mesh. Following a flow nozzle with contraction ratio of approximately 15 to 1, a two-ft long two-dimensional 7.5 × 10 in. section led directly into the test section. The test section was a two-dimensional diffuser with a variable included angle β. That is, the floor and roof were parallel and the two sides could be adjusted from 0° to 14°, included angle. While the actual diffuser was approximately five feet long, only the leading 48 in. were used as the test section proper, the last 12 in. being present to accommodate effects of the free discharge which preliminary probing indicated were influencing the boundary-layer development just prior to the free discharge.

The floor of the test section was aluminum and the roof and side walls were acrylic so that tufts placed on the walls could be easily observed. In particular, flow separation from the constant width side walls could be detected easily. Static pressure orifices were located along the floor of the tunnel and 24 access ports for probing the floor were located in the roof, ten of these on the plane of symmetry and 7 on either side. These latter were probed at random to detect any skewing of the boundary layer. Wire trips of 0.0625-in. dia were used all around the nozzle exit to insure an early turbulent boundary-layer flow.

The principal boundary-layer probe was circular with a tip OD of 0.018 in. stepped up to a 0.125-in. stem. The traverse used a dial indicator to position the probe; wall contact being established ±0.001 in. by electrical contact. The random probings made off the plane of symmetry were with a United Sensor Corporation claw probe No. CA-120-12-CD which allowed probing of the flow to within 1/64 in. of the wall. All probes used were calibrated against two United Sensor Corporation standard Kiel type probes. Pressure readings were made on a 10-in. Meriam Instrument Company Micromanometer with a least count unit of 0.001 in. of water and a Statham pressure transducer with a strain bridge having an equivalent least count unit of approximately 0.010 in. of water. Error analysis indicated the tunnel free-stream velocity was measured to $\pm 1/2$ percent and the lowest velocity read on the plane of symmetry was read to ±2.0 percent.

For purposes of comparison, all data were obtained with a tunnel inlet parameter U/ν fixed at 7.50×10^5 (ft^{-1}) ±1 percent. This resulted in a range of Reynolds number based on momentum thickness for the program of approximately 4000 to 9000. Turbulence intensity at tunnel inlet averaged at 0.8 percent, measurements being made with an Air Flow Corp., Model HWB2 hot wire anemometer. A complete tabulation of all the data on which the following results are based is contained in reference [18].

Experimental Results and Discussion. A series of five runs was made, at included angles of 0° through 12° at 3° intervals. Skin-friction coefficients obtained from the Ludwieg and Tillmann shear law were compared to those obtained from the Law of the Wall after the method suggested by Clauser [8] with the constants reported in that reference. While the plane-of-sym-

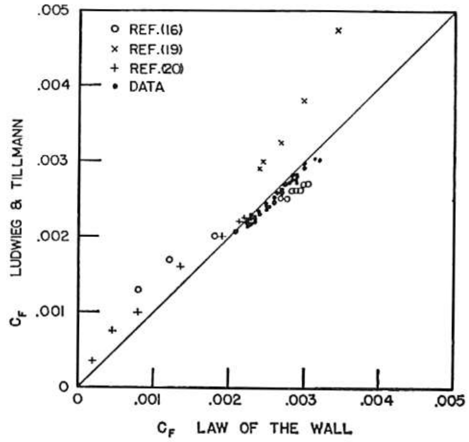

Fig. 4 Comparison of skin-friction coefficients as predicted by the two-dimensional Law of the Wall and the Ludwieg-Tillmann law

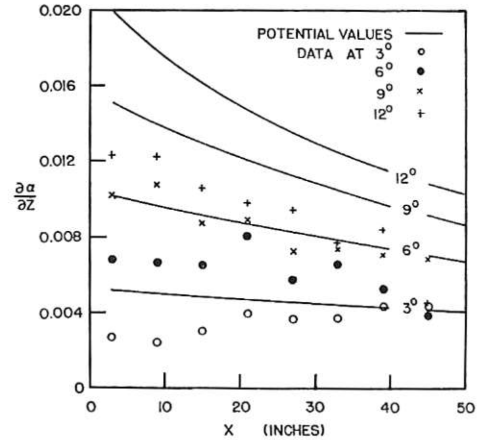

Fig. 6 Potential flow and free-stream continuity equation values of $\dfrac{\partial \alpha}{\partial z}$

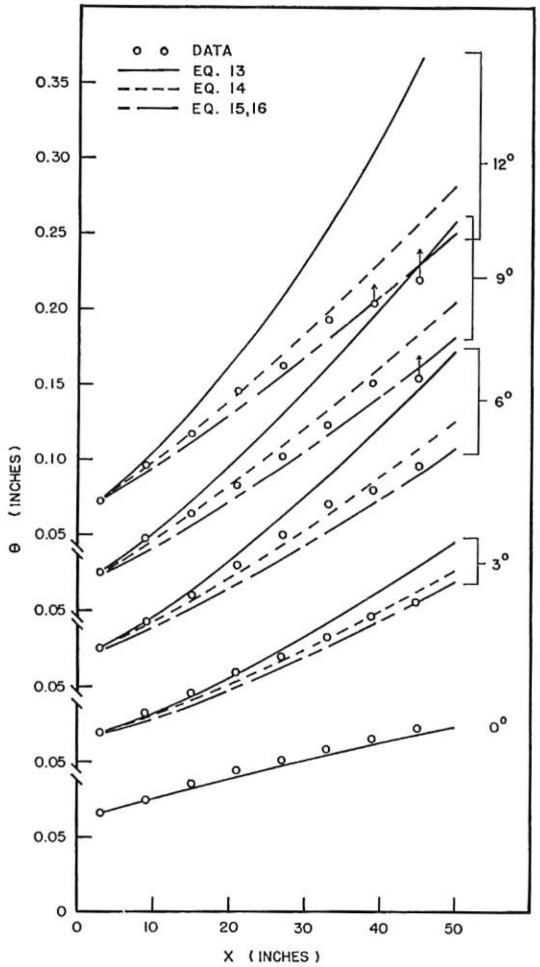

Fig. 5 Momentum thickness growth

metry flow may be described as a "locally" two-dimensional flow, one cannot assert, a priori, the validity of the two-dimensional Law of the Wall. The use of such a law here is only for purposes of comparison. Fig. 4 is a comparison of such skin friction coefficients for the plane-of-symmetry flow reported here, the plane-of-symmetry flows of Johnston [16], and Norbury [19] and, for comparison purposes, the two-dimensional flow of Schubauer and Klebanoff [20]. In the range of C_f values encountered in this work the Ludwieg and Tillmann formula gave lower values than the Law of the Wall. Johnston [16], using the same Law of Wall constants, shows this same trend in this C_f range. A reversal of this trend is seen in the data of references [16, 20] at the lower C_f values indicative of separation and at the higher C_f values recorded in [19]. The two sets of C_f values in this study were, with one exception, within 5 percent. This difference could have been altered in either direction by choosing slightly different constants for the Law of the Wall expression as there is some indecision in the literature regarding these constants. While the agreement shown between the two sets of values is perhaps fortuitous, it is reassuring in that the Law of the Wall results are based on flow only very near the wall while the Ludwieg and Tillmann results are based on integrated boundary-layer parameters. These results were used as a posteriori support for using the two-dimensional shear laws for the plane-of-symmetry flow which, as stated earlier, may be argued as "locally" two-dimensional. Clearly this matter should be verified experimentally.

Using experimentally determined values of H, U, $\Delta U/\Delta X$, the plane-of-symmetry momentum integral equations (14), (15), and (16) and the two-dimensional momentum integral equation (13) were used to compute momentum thickness development. Fig. 5 shows the agreement between measured and computed values obtained with the apparatus for included angles of 0°, 3°, 6°, 9°, and 12°. Equation (16), which was used here on the hypothesis that weak skewing did exist, and equation (15) gave very similar results, sufficiently close to warrant only a single curve to represent both sets of results for the purpose of clarity. As the included angle increases, the inadequacy of the two-dimensional equation becomes clear. The use of the potential approximation for the $\partial\alpha/\partial z$ term generally rerults in a lower growth rate than measured. By probing the boundary-layer flow off the plane of symmetry, it was found that all flows generated were collateral within $\pm 2°$, and hence any attempt to compute lateral transport of free-stream direction momentum was clearly impractical. For all intents and purposes, the flows were justifiably called collateral and equation (16), which was developed for strongly skewed flows, yielded values which were generally lower than measured values.

Fig. 6 shows results of the two means used in this study for evaluating the $\partial\alpha/\partial z$ term of the momentum integral equations. The higher values for the potential evaluation of $\partial\alpha/\partial z$ results directly in the lower momentum growth rate of Fig. 5. This potential solution for $\partial\alpha/\partial z$ is based on a wedge type source flow where $\partial\alpha/\partial z = 1/x_0$. In terms of the tunnel geometry of Fig. 3,

$$x_0 = a + x$$

where $$a = d/2 \tan(\beta/2)$$

Thus the comparison of equations (14) and (15) is made biasing the latter since measured data were used to evaluate $\partial\alpha/\partial z$ in (14). Accurate inclusion of sidewall boundary-layer growth in the potential evaluation of $\partial\alpha/\partial z$ should result in much better agreement between equations (14) and (15) in Fig. 5.

Fig. 5 may be discussed in light of previously reported work on spreading three-dimensional flows. Kehl's run K-3 of reference [13] is for a two-dimensional diffuser with included angle of 9°39′, and Norbury [14] reports on a two-dimensional diffuser with a continuously varying included angle.

Kehl shows excellent agreement between computed and measured values of momentum thickness where calculations were based on a potential evaluation of the term $\partial\alpha/\partial z$ in equation (12) neglecting sidewall boundary-layer growth and where a constant wall shear coefficient of 0.0034 was used. Using data reported in the reference and assuming that the lower and upper values of R_θ reported for the run occurred at the first and last stations at which boundary-layer parameters were measured, use of equation (17) gives wall shear coefficients of approximately 0.0032 and 0.00173. Assuming that it is reasonable to expect a monotone decreasing wall shear coefficient in adverse pressure gradient plane-of-symmetry flow, then the constant value used in the reference ranges from a value that is initially essentially correct, and then increases in error by approximately a factor of two. It was this apparent inconsistency which promoted in part this further investigation. The use of a constant average shear coefficient value was no doubt justified in that two-parameter shear laws were lacking at the time of Kehl's work, and in fact only since 1949 has the more renown Ludwieg and Tillmann law been available. Use of the higher shear coefficient, however, would have tended to raise the computed values of momentum thickness which, based on a potential evaluation of $\partial\alpha/\partial z$, would have been deficient. Hence an apparent explanation for the fortuitous agreement is available.

Although boundary-layer trips were used in the apparatus reported on here, their small size was such that comparison is made only with Norbury's Series A data. Norbury shows a computed momentum growth based on a potential evaluation of $\partial\alpha/\partial z$, which is slightly higher than the measured values for the plane-of-symmetry flow. From the results presented in this paper one would expect lower computed momentum thickness values when using the potential evaluation of $\partial\alpha/\partial z$. However, it should be noted that Norbury used experimentally determined values of $1/x$ for the $\partial\alpha/\partial z$ term and, as pointed out earlier, this then compares equations (14) and (15) without bias, and hence Norbury's Series A work is in general agreement with the results presented here. Since the shape factor reported was of the order of 1.35 and essentially invariant with distance, Norbury's use of the Squire and Young wall shear law is reasonable except at higher R_θ values as seen from Fig. 2. It should be pointed out that neither Kehl nor Norbury made any mention of examining their boundary layers for skewing.

In order to predict the displacement thickness of the boundary layer, an auxiliary equation of some sort is required. Two such relations were examined here, both involving the shape factor H and both developed for two-dimensional flows. The first was that of Tetervin and von Doenhoff [21], chosen because of its early origin and simplicity in use.

$$\theta \frac{dH}{dx} = e^{4.86(H - 2.975)} \left[-\frac{\theta}{q}\frac{dq}{dx}\frac{2q}{\tau_0} - 2.035(H - 1.286) \right] \tag{18}$$

The second was that of Rubert and Persh [22]

$$\theta \frac{dH}{dx} = \frac{\tau_0}{2q} H(3H - 1) - \frac{\tau_s}{2q} \frac{(3H-1)^2}{2} - \frac{\theta}{q}\frac{dq}{dx}\frac{H(3H-1)(H-1)}{2} \tag{19}$$

This latter relation is quoted by the authors as applicable to axisymmetric flows and was used because of the similarity between the plane of symmetry in a collateral flow and any meridian plane in a true axisymmetric flow which has no circumferential flows. In using (18) this writer chose to evaluate wall shear stress τ_0 with the Ludwieg and Tillmann formula rather than the Squire and Young relationship. In reference [21] it is pointed out that the inclusion of a wall shear stress was needed to correlate certain two-dimensional data. The Squire and Young relationship was "tentatively assumed" because it gave generally good agreement between calculated and measured drag on air foils. The Ludwieg and Tillmann formula, being dependent on two parameters, would presumably give a more accurate wall shear stress value over a wider range of variables; hence its use here. While the range of shape factor encountered in this present work was very small, Fig. 7 shows that either of the two equations cited appear to be satisfactory for the early development of the plane-of-symmetry flow. There is, however, an increasing difference in the computed and measured values at the larger included angles, particularly at the discharge end of the tunnel. Maximum differences of the order of 15 percent occur at the 12° run. Similar though fewer differences appear in the momentum thickness parameter of Fig. 5. These differences (both in θ and H), which become progressively more pronounced as the included angle increased, were assessed as due to free discharge effects communicated through the outer portions of the boundary layer as the discharge was approached. Preliminary additional work has shown that the good agreement in predicting both θ and H by the relations cited here over the early length of the test section can be extended over the entire length of the existing unit. This is done by extending the existing test section in length. The differences in predicted and measured values in θ and H are still detected, having been apparently only moved to the more remote discharge end. Extensive probing rules out existence of any strong secondary flows at exit and the ability to extend the good agreement between measured and predicted momentum thickness and shape parameter over the entire length of the test section suggests that streamline curvature induced by and occurring at the discharge of the tunnel are being communicated to the latter part of the flow and such three-dimensional effects are not considered in the analysis presented earlier.

The results of Fig. 7 support the observation of Johnston [10] who, for a plane of symmetry in a strongly skewed flow, also used a two-dimensional shape factor equation and obtained good results. In the case of reference [10], Rotta's method based on energy considerations was used, and good agreement, up to separation, in the prediction of shape factor was obtained. Johnston's flow was, however, well contained and guided by an

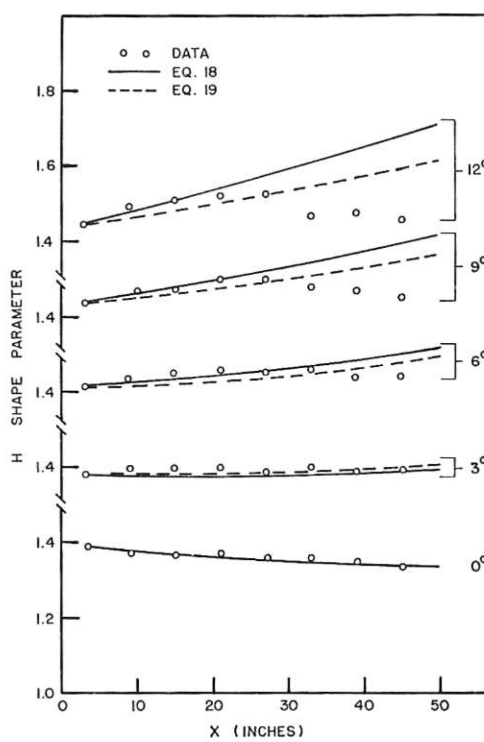

Fig. 7 Shape parameter growth

extensive roof-floor arrangement, and streamline curvature that might be induced by discharge from his test section would be far removed from this region of measurements. Only the work of Norbury is available for direct comparison, however, and the work reported herein agrees with his general results. Both works indicate that shape parameter tends to a constant value along the plane of symmetry, and both works show a drop in this parameter as the exit to the diffuser is approached with the effect more pronounced in this work.

Conclusions

Following a rigorous development of a momentum integral equation for a plane-of-symmetry flow in a three-dimensional turbulent boundary layer in a streamline coordinate set, it is clear that two fundamentally different cases result: the case of skewed boundary-layer flow and the case of collateral boundary-layer flow. For the latter case, the difference is a single term which can be evaluated from potential flow assumptions or from free-stream continuity considerations. In the experimental work reported on herein, the continuity considerations appears to agree better with the experimental results. This is due principally to the lack of consideration given to sidewall boundary-layer growth in the potential flow.

Prediction of momentum growth by two-dimensional equation was poor, giving very high rates of development, while an equation based on strongly skewed flow adjacent to the plane of symmetry predicted values lower, in general, to those measured.

The limited investigation into the use of ordinary two-dimensional variation of shape parameter equations and their applicability to plane-of-symmetry flow in collateral boundary layers does not show as good agreement as that of Johnston [10], who examined a plane of symmetry in a skewed boundary layer up to separation.

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