On a modification of algebraic multilevel iteration method for finite element matrices*

M. Larin

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В настоящее время многосеточные и многоуровневые методы очень популярны для решения разреженных систем линейных алгебраических уравнений. Они обладают как широкой областью применения, так и эффективностью. В работе [6] был предложен алгебраический многоуровневый итерационный (AMLI) метод для решения конечно-элементных систем линейных алгебраических уравнений. Однако этот метод имеет два ограничения на свойства исходной матрицы, которые могут нарушаться на практике. С целью избежать их и улучшить качество AMLI-предбуславливателя предлагается и анализируется семейство итерационных параметров релаксации.

Ключевые слова: *алгебраический многоуровневый метод, метод сопряженных градиентов с предобуславливателем, системы линейных алгебраических уравнений, метод конечных элементов.*

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Today, multigrids and multilevel methods for solving a sparse linear system of equations are well known. They are both robust and efficient. In [6], the algebraic multilevel iteration (AMLI) method for finite element matrices has been proposed. However, this method has two restrictions on the properties of the original matrix, which can fail in practice. To avoid them and to improve the quality of the AMLI-preconditioner, a family of relaxation parameters is suggested and analyzed.

Key words: *algebraic multilevel iterative method, preconditioned conjugate gradient method, finite element matrices.*

Dedicated to Valery Il'in on the occasion of his 70th birthday

1. Introduction

In the last two decades, much interest has been attracted to the construction of optimal or nearly optimal iterative methods for solving a sparse linear system of equations resulting from the finite element approximation of elliptic self-adjoint second order boundary value problems. For such iterative methods, the rate of convergence independent of the number of unknowns n, and moreover, a computational cost per iteration, is proportional to n. In particular, the algebraic multilevel iteration (AMLI) methods allow us to construct preconditioning matrices with these properties, see [2] and references therein.

There are three different ways to define the AMLI methods. The first approach is based on the finite element approximation of the corresponding Schur complement to a sequence

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of nested meshes [8–11, 24]. The second one employs the iterative incomplete factorization method for constructing an approximation of the Schur complement as a new matrix on the lower (coarse) level [15, 16, 20–22]. In the third one, we use an approximation for the first pivoting block to ensure an a priori chosen sparsity pattern for its Schur complement on a coarse level [3, 5, 7]. However, in the latter case, the AMLI method of the third type has been analyzed only for Stieltjes matrices.

In [6], an attempt to extend the methods to a more general class of matrices (finite element matrices) has been made. However, to prove the optimal conditions for the method, two assumptions are made. These assumptions provide the existence of the positive definite approximations $\overline{A_{11}^{(k)}}$ for the first pivoting blocks $A_{11}^{(k)}$. If ever the first assumption caused inessential inconvenience (the method is not applicable for isosceles right triangular grids), whereas the second one (blocks $A_{21}^{(k)}$ has to be non-negative matrices), in practice, considerably reduced the class of finite element matrices to the Stieltjes ones. In [17], an improved version of the method, which is based on the error compensation principle, has been proposed. However, this improvement helps us to resolve only the first restriction on triangulation.

In the present paper, a modified version of the AMLI method for finite element matrices, which allow us to avoid both above assumptions, is proposed. Here, to improve the quality of the AMLI-preconditioner, a family of relaxation parameters (matrices $\Theta_{11}^{(k)}$), chosen according to the convergence and positive definite conditions, is suggested and analyzed.

The paper is organized as follows. In Section 2, a modified version of the AMLI method from [6] is presented. Classical results about the computational complexity and the rate of convergence are presented. The local analysis and the algorithm for definition of the approximation $\overline{A_{11}^{(k)}}$ for the first pivoting block $A_{11}^{(k)}$ are proposed and analyzed. In the final section, performance results on model problems are presented and discussed.

2. The AMLI method

For solving a linear system of equations with a sparse symmetric positive definite matrix

$$A\boldsymbol{x} = \mathbf{b}, \qquad A = A^T > 0, \qquad A \in \mathbf{R}^{n\times n}, \tag{1}$$

which arises from finite element approximations of elliptic self-adjoint second order boundary value problems, we use the preconditioned conjugate gradient method, see [2], for instance. To create the preconditioning matrix M, we first have to construct a sequence of the matrices $\{A^{(k)}\}$, $k = k_0, k_0+1, \ldots, L-1, L$ of an increasing order n_k starting from $A^{(L)} = A$. Second, we have to define a sequence of the preconditioning matrices $\{M^{(k)}\}$ from bottom to top, where $M = M^{(L)}$. Finally, we have to properly define all required parameters such as polynomial degrees ν_k, the sequence of nested grids $\{X_k\}$, the matrix approximation $\overline{A_{11}^{(k)}}$ and the relaxation parameters $\Theta_{11}^{(k)}$.

2.1. Construction of a sequence of matrices $\{A^{(k)}\}$

Following [6], we first define a sequence of the matrix graphs $\{G_k\}$ corresponding to the sequence of matrices $\{A^{(k)}\}$, i.e., $G_k = (X_k, E_k) = G(A^{(k)})$, where $X_k = X(A^{(k)})$ is a set of vertices and $E_k = E(A^{(k)})$ is a set of edges in the matrix graph for A^k, where $(i,j) \in E_k$ if and only if $a_{ij}^{(k)} \neq 0$. We assume that $\{X_k\}$ is a sequence of *nested* graphs, i.e., $X_{k_0} \subset \ldots \subset X_k \subset X_{k+1} \subset \ldots \subset X_L$, and moreover, we assume that the number of vertices n_k increases in geometric progression, i.e.,

$$\frac{n_{k+1}}{n_k} = \rho_k \geq \rho > 1, \qquad k = k_0, k_0+1, \ldots, L-1.$$

Now we define a sequence of matrices $\{A^{(k)}\}$ by recursion for decreasing the values of k: Each matrix $A^{(k+1)}$, $k > 0$, is partitioned in a certain manner into a two-by-two block matrix form

$$A^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & A_{12}^{(k+1)} \\ A_{21}^{(k+1)} & A_{22}^{(k+1)} \end{bmatrix},$$

where the first pivoting block corresponds to the vertices in X_{k+1}, which are not in X_k, and the second one corresponds to those in X_k. The selection of sets of the vertices X_k will be discussed in detail in Section 2.3. It is assumed that $A_{11}^{(k+1)}$ is diagonally dominant.

Now define a symmetric and positive definite matrix $\overline{A_{11}^{(k+1)}}$ as approximation of $A_{11}^{(k+1)}$, which will be discussed in detail below (see (3)). Next, we consider an intermediate matrix

$$\widetilde{A}^{(k+1)} = \begin{bmatrix} \overline{A_{11}^{(k+1)}} & A_{12}^{(k+1)} \\ A_{21}^{(k+1)} & A_{22}^{(k+1)} \end{bmatrix},$$

which is also symmetric positive definite [13]. We now define the matrix $A^{(k)}$ as the Schur complement of $\widetilde{A}^{(k+1)}$, i.e.,

$$A^{(k)} = A_{22}^{(k+1)} - A_{21}^{(k+1)} \overline{A_{11}^{(k+1)}}^{-1} A_{12}^{(k+1)}. \tag{2}$$

Note that $A^{(k)}$ is a symmetric positive definite matrix being a Schur complement of the symmetric positive definite matrix [13]. In order to preserve some sparsity pattern in the matrix sequence, one must choose $\overline{A_{11}^{(k+1)}}$ properly. Typically, it is diagonal or block-diagonal.

Here we approximate $A_{11}^{(k+1)}$ by a diagonal and positive matrix $\overline{A_{11}^{(k+1)}}$ defined by

$$\overline{A_{11}^{(k+1)}}\mathbf{e}^{(k+1)} = A_{11}^{(k+1)}\mathbf{e}^{(k+1)} - \big(I_{11}^{(k+1)} - \Theta_{11}^{(k+1)}\big) \cdot \operatorname{off}\big(A_{11}^{(k+1)}\big)\mathbf{e}^{(k+1)}, \tag{3}$$

where $\mathbf{e}^{(k+1)} = (1, \ldots, 1)^T$ is a column-vector of order $n_{k+1} - n_k$, $I_{11}^{(k+1)}$ is an identity matrix, $\Theta_{11}^{(k+1)}$ is a diagonal matrix of prescribed iterative parameters and $\operatorname{off}(A_{11}^{(k+1)})$ is the off-diagonal part of the matrix $A_{11}^{(k+1)}$.

The main idea of this approximation is simple and evident. First of all, when $A_{11}^{(k+1)}$ is a Stieltjes matrix, we want to use a standard "unmodified" technique, i.e., $\Theta_{11}^{(k+1)}$ is an identity matrix. Otherwise, when $A_{11}^{(k+1)}$ is not a Stieltjes matrix, we have to use the "modified" technique to avoid the situation when $\overline{A_{11}^{(k+1)}}$ is a non-positive definite or singular matrix, which may arise if we try to use the original AMLI method from [6]. Note that a similar idea has been applied to incomplete factorization methods in [4, 14, 19]. The finite element algorithm for dynamic calculation of the matrices $\Theta_{11}^{(k+1)}$ will be discussed in detail in Section 3.

To proceed, we apply a similar operation to each matrix $A^{(k)}$ and repeat this process until the matrix corresponding to a coarse mesh is attained, where a linear system of equations can be solved by a direct method with computational cost sufficiently low in relation to n_L. For example, one can use the following criterion

$$n_{k_0} \approx (n_L)^{1/4}. \tag{4}$$

In this case, the total computational cost to solve a system with the matrix $A^{(k_0)}$ has order $O(n_L^{1/2})$ and the cost of the factor $A^{(k_0)}$ has order $O(n_L^{3/4})$. Note that computational costs for solving a coarse problem by an iterative method depends on the condition number of $A^{(k_0)}$, see [2], for instance.

2.2. Construction of the sequence of matrices $\{M^{(k)}\}$

Following [6], the preconditioning matrix M is recursively defined as follows:

$$M^{(k_0)} = A^{(k_0)} \quad \text{for} \quad k = k_0 + 1, \ldots, L,$$
$$M^{(k+1)} = \begin{bmatrix} \overline{A_{11}^{(k+1)}} & 0 \\ A_{21}^{(k+1)} & I \end{bmatrix} \begin{bmatrix} I & \overline{A_{11}^{(k+1)}}^{-1} A_{12}^{(k+1)} \\ 0 & S^{(k)} \end{bmatrix}, \tag{5}$$

where $S^{(k)}$ is approximation for the Schur complement of $\widetilde{A}^{(k+1)}$, defined as follows:

$$S^{(k)} = A^{(k)} \left[I - P_{\nu_k}(M^{(k)^{-1}} A^{(k)}) \right]^{-1}. \tag{6}$$

Here $P_{\nu_k}(x)$ is a polynomial of degree ν_k, normalized by $P_{\nu_k}(0) = 1$, and defined on the interval $I_k = [\underline{t}_k, \bar{t}_k]$ containing the eigenvalues of $M^{(k)^{-1}} A^{(k)}$ as follows:

$$P_{\nu_k}(t) = \frac{T_{\nu_k}\left(\frac{\bar{t}_k + \underline{t}_k - 2t}{\bar{t}_k - \underline{t}_k}\right) + 1}{T_{\nu_k}\left(\frac{\bar{t}_k + \underline{t}_k}{\bar{t}_k - \underline{t}_k}\right) + 1}, \tag{7}$$

where $T_m(t)$ are the Chebyshev polynomials of degree m,

$$T_0 = 1, \quad T_1 = t, \quad T_{\nu+1} = 2tT_\nu - T_{\nu-1}. \tag{8}$$

The choice of degrees ν_k will be discussed in detail in Section 2.4. Note that both $A^{(k)}$ and $M^{(k)}$ are symmetric and positive definite, so the preconditioned matrix $M^{(k)^{-1}} A^{(k)}$ has a positive spectrum.

Note that one can use an inner iterative process to solve the coarser problem with the matrix $A^{(k)}$ instead polynomial stabilization by Chebyshev polynomials, see [11] for details.

2.3. Description of sets X_k

Let a matrix $A^{(k+1)}$, which corresponds to the structure of a triangular finite element mesh/matrix, be given on each level, i.e., the graph $G_{k+1} = (X_{k+1}, E_{k+1})$ is defined.

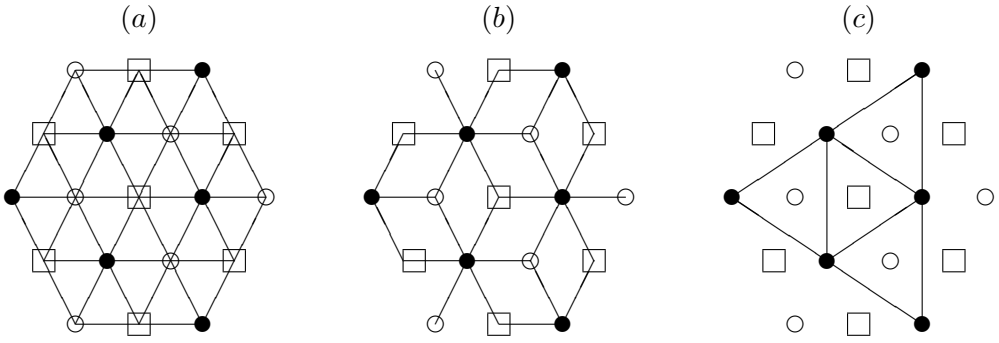

Figure 1. Transformation of the graph G_{k+1} into the graph G_k: (a) — example of coloring a triangular mesh; (b) — the triangular mesh after deleting certain couplings; (c) — a new triangular mesh

(a) *Partitioning the set of vertices* X_{k+1}: Color the nodes of the graph G_{k+1} three colors: red ($\circ$), blue ($\square$) (the set $X_{k+1}\backslash X_k$) and green ($\bullet$) (the set X_k) by the rule: nodes, which have the same color, are not connected (Figure 1(a)).

(b) *Approximation of the submatrix* $A_{11}^{(k+1)}$: By definition of the approximation process, we have to delete all couplings between the red and the blue nodes. This leads to the mesh corresponding to the intermediate matrix $\widetilde{A}^{(k+1)}$, which is shown in Figure 1(b).

(c) *Calculation of the new matrix* $A^{(k)}$: Make a graph reconstruction, which corresponds to the Gaussian elimination of all the red and the blue nodes. This generates a new triangular finite element mesh G_k for the green nodes, which has the same structure and much fewer number of nodes (Figure 1(c)).

As the corresponding graph G_k has again the same finite element structure, the process can be repeated until a sufficiently coarse graph G_0 satisfying (4) is reached.

2.4. The choice of degrees ν_k

Following [6], we define the degrees of polynomials as follows:

$$\begin{aligned} &\nu_L = 1, \quad \nu_{L-1} = 1, \quad \dots, \quad \nu_{L-\mu+1} = 1, \quad \nu_{L-\mu} = \nu, \\ &\nu_{L-\mu-1} = 1, \quad \nu_{L-\mu-2} = 1, \quad \dots, \quad \nu_{L-2\mu} = 1, \quad \nu_{L-2\mu-1} = \nu, \\ &\nu_{L-2\mu-2} = 1, \quad \nu_{L-2\mu-3} = 1, \quad \dots, \quad \nu_{k_0} = 1, \end{aligned} \tag{9}$$

where ν, $0 \le \nu \le L$ and μ are integer parameters. Here ν is the number of iterations of the above polynomial type iterative process, which solves the system with the matrix $M^{(k)^{-1}}A^{(k)}$, and is used for obtaining a spectrally equivalent preconditioner, and μ is the number of levels, after which we repeat this process.

It is well known that the upper and the lower bounds for polynomial degrees are defined from the optimal order conditions for the computational complexity and the convergence rate, respectively. Since the modification put forward here, refers to the rate of convergence, the lower bound on ν_k will be modified, whereas the upper bound is not changed.

Introduce a positive value δ by

$$\delta \le n_{k_0}^{-1/(L-k_0)}. \tag{10}$$

It has been shown in [6] that the inequality

$$\nu < (\delta\rho)^{\mu+1} \tag{11}$$

defines the upper bound of the degrees of polynomials, under which the total computational complexity for one application of preconditioner is proportional to the number of nodes on a fine mesh.

Now we define the new lower bound on ν_k. Since $\overline{A_{11}^{(k+1)}}$ is defined by the diagonal compensation of all off-diagonal entries of $A_{11}^{(k+1)}$ (see (3) and Section 2.3), there are positive constants $\alpha_{k+1} \le 1$ and $\beta_{k+1} \ge 1$ such that

$$0 < \alpha_{k+1}\, \boldsymbol{x}^T \widetilde{A}^{(k+1)}\boldsymbol{x} \le \boldsymbol{x}^T A^{(k+1)}\boldsymbol{x} \le \beta_{k+1}\, \boldsymbol{x}^T \widetilde{A}^{(k+1)}\boldsymbol{x} \tag{12}$$

for all $\boldsymbol{x} \in R^{n_{k+1}}$. Now, using the definition of polynomials (7) and (8), we obtain the boundary points of the interval I_{k+1}

$$\underline{t}_{k+1} = \alpha_{k+1}(1 - P_{\nu_k}(\underline{t}_k)), \quad \bar{t}_{k+1} = \beta_{k+1}.$$

Now, using a standard technique, we obtain the final condition on the lower bound of polynomial degrees [7]. Hence, the optimal order condition of the rate of convergence is

$$\nu > \left(\max_{k=1,2,\ldots,L/\mu} \prod_{s=L-k\mu}^{L-(k-1)\mu} \frac{\beta_s}{\alpha_s} \right)^{1/2}, \tag{13}$$

where ν_k is chosen as in (9). Finally, note that the derived condition is equal to the old one from [6] when $\alpha_s = 1$, and hence, it is its generalization for a more general case $\alpha_s \in (0; 1]$. Note that this result is similar to the earlier one from [16].

3. Theoretical estimates for α_s and β_s

To ensure the lower bound of polynomial degrees (13), we have to define the constants α_k and β_k from (12), i.e., we have to estimate maximal and minimal eigenvalues of the following generalized eigenvalue problem on each level

$$A^{(k)}\boldsymbol{v} = \lambda \widetilde{A}^{(k)}\boldsymbol{v}. \tag{14}$$

Lemma 1 [6]. *Let $\{A_i\}_{i=1}^n$ and $\{M_i\}_{i=1}^n$ be sequences of symmetric positive semidefinite matrices, $A = \sum_{i=1}^n A_i$, $M = \sum_{i=1}^n M_i$. Then, if for some positive constants α_i and $\beta_i \geq \alpha_i$ and for all $\boldsymbol{x} \in R^n$*

$$\alpha_i \boldsymbol{x}^T M_i \boldsymbol{x} \leq \boldsymbol{x}^T A_i \boldsymbol{x} \leq \beta_i \boldsymbol{x}^T M_i \boldsymbol{x}$$

holds, then

$$\alpha \boldsymbol{x}^T M \boldsymbol{x} \leq \boldsymbol{x}^T A \boldsymbol{x} \leq \beta \boldsymbol{x}^T M \boldsymbol{x},$$

where $\alpha = \min \alpha_i$, $\beta = \max \beta_i$.

Lemma 1 shows that the analysis of the global generalized eigenvalue problem (14) can be reduced to the analysis of a local one

$$A^{(E)}\boldsymbol{v} = \lambda \widetilde{A}^{(E)}\boldsymbol{v}, \tag{15}$$

where E is a non-empty union of neighboring elements of the graph G_k.

It is well known that on the original finite element grid, the element matrix $A^{(e)}$ for linear basis functions is

$$A^{(e)} = \frac{1}{2} \begin{bmatrix} \beta+\gamma & -\gamma & -\beta \\ -\gamma & \alpha+\gamma & -\alpha \\ -\beta & -\alpha & \beta+\alpha \end{bmatrix},$$

where α, β, γ are cotangents of the angles of triangles. However, it is not true for the rest grids. Thus, for each triangular element of the graph G_k, $0 < k < L$, we define a similar element matrix, where α, β, γ are values of off-diagonal entries of $A^{(k)}$. For example,

$$\alpha = -\frac{1}{2}\, a^{(k)}_{p_2p_3}, \qquad \beta = -\frac{1}{2}\, a^{(k)}_{p_1p_3}, \qquad \gamma = -\frac{1}{2}\, a^{(k)}_{p_1p_2},$$

where p_1, p_2, p_3 are numbers of vertices of a triangular element. In what follows, we impose two standard restrictions on all elements of triangulation [1]:

- *maximal angle condition*: There is a constant $\phi_* < \pi$ (independent of the meshsize h) such that the maximal interior angle ϕ of any element e is bounded by ϕ_*, i.e., $\phi \le \phi_*$.
- *coordinate system condition*: The angle ψ between the longest side E of the triangle e and x-axis is bounded by $|\sin\psi| \le h_2/h_1$, where h_1 is the length of E and h_2 is the thickness of e perpendicular to E, i.e., there are no triangles with *two* very sharp angles.

Then the following natural relations are valid

$$\alpha_1 + \alpha_2 \neq 0, \quad \beta_1 + \beta_2 \neq 0. \tag{16}$$

Now let us consider the superelement E consisting of the two triangular elements e_1 and e_2, which have a common edge between the red (1) and the blue (2) vertices (see Figure 2). After having assembled elementary matrices, we derive the following superelement matrix

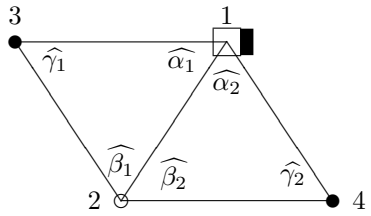

Figure 2. The superelement $E = e_1 \cup e_2$

$$A^{(E)} = \frac{1}{2}\begin{bmatrix} \beta_1+\gamma_1+\beta_2+\gamma_2 & -\gamma_1-\gamma_2 & -\beta_1 & -\beta_2 \\ -\gamma_1-\gamma_2 & \alpha_1+\gamma_1+\alpha_2+\gamma_2 & -\alpha_1 & -\alpha_2 \\ -\beta_1 & -\alpha_1 & \beta_1+\alpha_1 & 0 \\ -\beta_2 & -\alpha_2 & 0 & \beta_2+\alpha_2 \end{bmatrix}.$$

The corresponding superelement preconditioned matrix $\widetilde{A}^{(E)}$ is

$$\widetilde{A}^{(E)} = \frac{1}{2}\begin{bmatrix} \beta_1+\beta_2+(1-\theta)(\gamma_1+\gamma_2) & 0 & -\beta_1 & -\beta_2 \\ 0 & \alpha_1+\alpha_2+(1-\theta)(\gamma_1+\gamma_2) & -\alpha_1 & -\alpha_2 \\ -\beta_1 & -\alpha_1 & \beta_1+\alpha_1 & 0 \\ -\beta_2 & -\alpha_2 & 0 & \beta_2+\alpha_2 \end{bmatrix},$$

for some $-1 \le \theta \le 1$. Further, without loss of generality, we assume that

$$\gamma_1 + \gamma_2 \neq 0, \tag{17}$$

since if $\gamma_1 + \gamma_2 = 0$, then for all θ we obtain $\widetilde{A}^{(E)} = A^{(E)}$, which is the best approximation to $A^{(E)}$.

Lemma 2 [6]. *Assume that no entries α_i, $i = 1, 2$, and β_i, $i = 1, 2$, of the matrix B are equal to zero. Then all the entries of its Schur's complement are nonzero, too.*

Lemma 2 shows that if the matrix graph of $A^{(k)}$ is identical to the triangular finite element grid, then the matrix graphs on the remaining levels are also identical to the triangular finite element grids.

Now to solve the local generalized eigenvalue problem (15), we rewrite it in the following two-by-two block form:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} = \lambda \begin{bmatrix} D_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix},$$

the solution of which is as follows:

1. $\lambda_1 = 1$, $\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;
2. $\lambda_2 = 1$, $\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;
3. Let $\lambda \neq 1$, then $\boldsymbol{v}_2 = -A_{22}^{-1} A_{21} \boldsymbol{v}_1$, and we obtain the following problem:

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})\boldsymbol{v}_1 = \lambda(D_{11} - A_{12}A_{22}^{-1}A_{21})\boldsymbol{v}_1,$$

which is reduced to

$$(\gamma + \eta)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\boldsymbol{v}_1 = \lambda\left((1-\theta)\gamma\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \eta\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\boldsymbol{v}_1, \tag{18}$$

where

$$\gamma = \gamma_1 + \gamma_2 \quad \text{and} \quad \eta = \frac{\alpha_1\beta_1}{\beta_1 + \alpha_1} + \frac{\alpha_2\beta_2}{\beta_2 + \alpha_2}. \tag{19}$$

Problem (18) has the solution if

$$\det \begin{bmatrix} \gamma + \eta(1-\lambda) - \lambda(1-\theta)\gamma & -(\gamma + \eta(1-\lambda)) \\ -(\gamma + \eta(1-\lambda)) & \gamma + \eta(1-\lambda) - \lambda(1-\theta)\gamma \end{bmatrix} = 0,$$

or, the same,

$$\lambda(1-\theta)\gamma\left[-2(\gamma + \eta(1-\lambda)) + \lambda(1-\theta)\gamma\right] = 0,$$

from which we obtain

$$\lambda_3 = \begin{cases} 1, & \text{if } \theta = 1 \\ 0, & \text{in other cases} \end{cases} \quad \text{and} \quad \lambda_4 = \frac{2(\gamma + \eta)}{(1-\theta)\gamma + 2\eta}. \tag{20}$$

Note that if $\gamma = 0$, then $\lambda_3 = \lambda_4 \equiv 1$, and moreover, by straightforward calculations one can verify that the eigenvalues estimates derived for the AMLI method in [6] ($\theta = 1$) follow directly from (20). However, as is readily seen from the definitions of γ and η, if we now try to use this approximation ($\theta = 1$) everywhere, it may happen that the generalized eigenvalue problem (15) has negative ($\eta < 0$) or even unbounded ($\eta = 0$) eigenvalues. In Figure 3, one can see the dependence of λ_4 with respect to the angles α_2 and β_2 of the second triangle. Here we consider four typical types of triangles for the first triangle, which one can meet in practice: (1) — equilateral triangle ($\alpha_1 = \beta_1 = 60°$); (2) — isosceles right-angled triangle ($\alpha_1 = 90°, \beta_1 = 45°$); (3) — practical case ($\alpha_1 = 30°, \beta_1 = 30°$); (4) — degenerative case ($\alpha_1 = 175°, \beta_1 = 4°$). Analyzing Figure 3, we can see that even in practical cases (1)–(3), there are situations when λ_4 considerably grows or falls below zero. To avoid these difficulties, the corresponding restrictions on matrix entries were imposed in [6]. However, those restrictions are very strong and exclude some practical triangulations. For example, if $\alpha_1 = \beta_1 = 60°$ and $\alpha_2 = 120°$, $\beta_2 = 45°$, then the maximal angle and the coordinate system conditions [1] are satisfied, but $\lambda_4 \approx -0.333 < 0$ since $\eta \approx -1.077 < 0$ and $\gamma \approx 4.309 > 0$. In this paper, we will take both restrictions from [6] away.

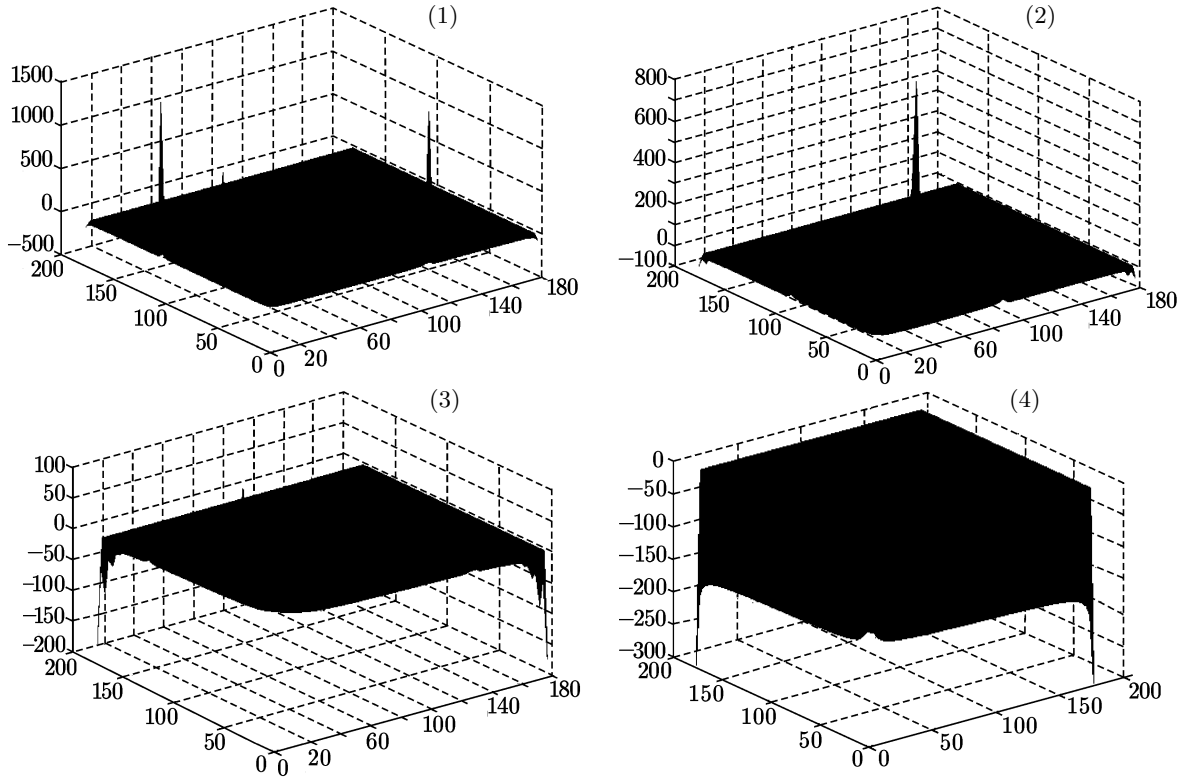

Figure 3. The behavior of λ_4 vs. α_2 and β_2 for the AMLI method

Lemma 3. *Let γ and η be defined as described above. Then*

$$\gamma + \eta \geq 0.$$

Proof. Since $A^{(E)}$ is a symmetric positive semidefinite matrix as an elementary submatrix of $A^{(k)}$, then its Schur's complement, defined as

$$S^{(E)} = A_{11} - A_{12}A_{22}^{-1}A_{21} = (\gamma + \eta)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

is also symmetric and positive semidefinite. On the other hand, by direct computations, we can find the explicit form for its eigenvalues

$$\lambda_1(S^{(E)}) = 0, \quad \lambda_2(S^{(E)}) = 2(\gamma + \eta),$$

from which follows

$$\gamma + \eta \geq 0. \qquad \square$$

Note that if $\gamma+\eta = 0$, then $\lambda_4 = 0$. Unfortunately, we could not use the results of the local analysis to estimate the lowest eigenvalues, since the above results show their non-negativity only. Nevertheless, we can use a local analysis to estimate the upper bounds and to ensure the positive definiteness of the preconditioned matrix.

To estimate the dependence of the lower bound on λ with respect to a variation of θ, we use the following estimate, which was suggested in [12, 18],

$$\lambda_{\min}(\widetilde{A}^{(k)^{-1}}A^{(k)}) \geq \frac{1}{1+\xi}, \tag{21}$$

where

$$\xi \le \frac{\left((I_{11}^{(k)} - \Theta_{11}^{(k)}) \cdot \mathrm{off}(A_{11}^{(k)})\mathbf{e}_1, \mathbf{e}_1\right)}{(D_A^{(k)}\mathbf{e}, \mathbf{e}) \cdot \lambda_{\min}(D_A^{(k)^{-1}} A^{(k)})}, \tag{22}$$

where $D_A^{(k)}$ is a diagonal part of $A^{(k)}$ and $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2)^T$. It is known that in practice $\lambda_{\min}(D_A^{(k)^{-1}} A)$ is of order $O(n_k^{-1})$ and the order of $(D_A^{(k)}\mathbf{e}, \mathbf{e})$ is $O(n_k)$. Hence, the order of $\lambda_{\min}(\widetilde{A}^{(k)^{-1}} A^{(k)})$ is proportional to the sum of all perturbations, i.e., if $\theta \neq 1$, then the lower bound goes away from 1 to 0.

On the other hand, as was mentioned above, it may happen that for some values of γ and η, the eigenvalue λ_4 is negative. Likely, we can control it by the corresponding choice of θ. However, to minimize possible perturbations, we have to choose θ equal to 1 where possible.

Now we let us find the value of θ with respect to γ and η, for which λ_4 is positive and smaller than a given bound ε^{-1} for an apriori chosen value $\varepsilon \in (0, 1]$. To do this, we consider all possible cases:

A. If $\gamma > 0$ and $\eta > 0$, then

$$1 \le \lambda_4 \le \begin{cases} \dfrac{\gamma + \eta}{\eta} & \text{if } \theta = 1, \\ \dfrac{2}{1 - \theta} & \text{in other cases.} \end{cases}$$

Thus, the main problem of the "unmodified" method, which sometimes is in the unlimited growth of its largest eigenvalues, can be removed by using the corresponding perturbation θ in the "modified" method. However, for the latter, there is an opposite problem: the value of the lowest eigenvalues essentially depends on the value of the perturbation used, see (21). Thus, in this case, to ensure an a-priori given bounds on the maximal eigenvalue

$$1 \le \lambda_4 \le \varepsilon^{-1} \tag{23}$$

for $0 < \varepsilon \le 1$, we assign

$$\frac{2}{1 - \theta} = \varepsilon^{-1} = \frac{\gamma + \eta}{\eta},$$

from which we derive

$$\theta = 1 - 2\varepsilon \quad \text{when} \quad \eta < \frac{\varepsilon\gamma}{1 - \varepsilon}. \tag{24}$$

B. If $\gamma > 0$ and $\eta < 0$, then we rewrite (20) as follows:

$$\lambda_4 = \left(1 - \frac{(1 + \theta)\gamma}{2(\gamma + \eta)}\right)^{-1},$$

from which, using the condition $\lambda_4 \ge 0$, we see that the following inequality must be valid

$$1 - \frac{(1 + \theta)\gamma}{2(\gamma + \eta)} \ge 0, \qquad \text{or} \qquad \theta \le 1 + 2\frac{\eta}{\gamma} = 1 - 2\frac{|\eta|}{\gamma}.$$

However, due to the result of Lemma 3, we have $\gamma > |\eta|$, and hence, the value of θ may be negative and very close to -1. Hence, choosing $\theta = -1$ we can always ensure that the preconditioner becomes positive definite, and moreover, $\lambda_4 = 1$.

C. If $\gamma < 0$ and $\eta > 0$, then using a simple inequality

$$0 \le (1 - \theta)|\gamma| \le 2|\gamma|$$

and the result of Lemma 3, we obtain

$$1 \geq \lambda_4 = \frac{2(\gamma+\eta)}{(1-\theta)\gamma+2\eta} = \frac{2\eta-2|\gamma|}{2\eta-(1-\theta)|\gamma|} \geq \frac{\eta-|\gamma|}{\eta} = \frac{\gamma+\eta}{\eta} > 0. \tag{25}$$

Here, to minimize possible perturbations we also use $\theta = 1$.

D. If $\eta = 0$, then $\gamma > 0$, and hence, we have $\lambda_4 = 2/(1-\theta)$. Similar to the case **A**, we use $\theta = 1 - 2\varepsilon$ to ensure

$$\lambda_4 \leq \varepsilon^{-1}. \tag{26}$$

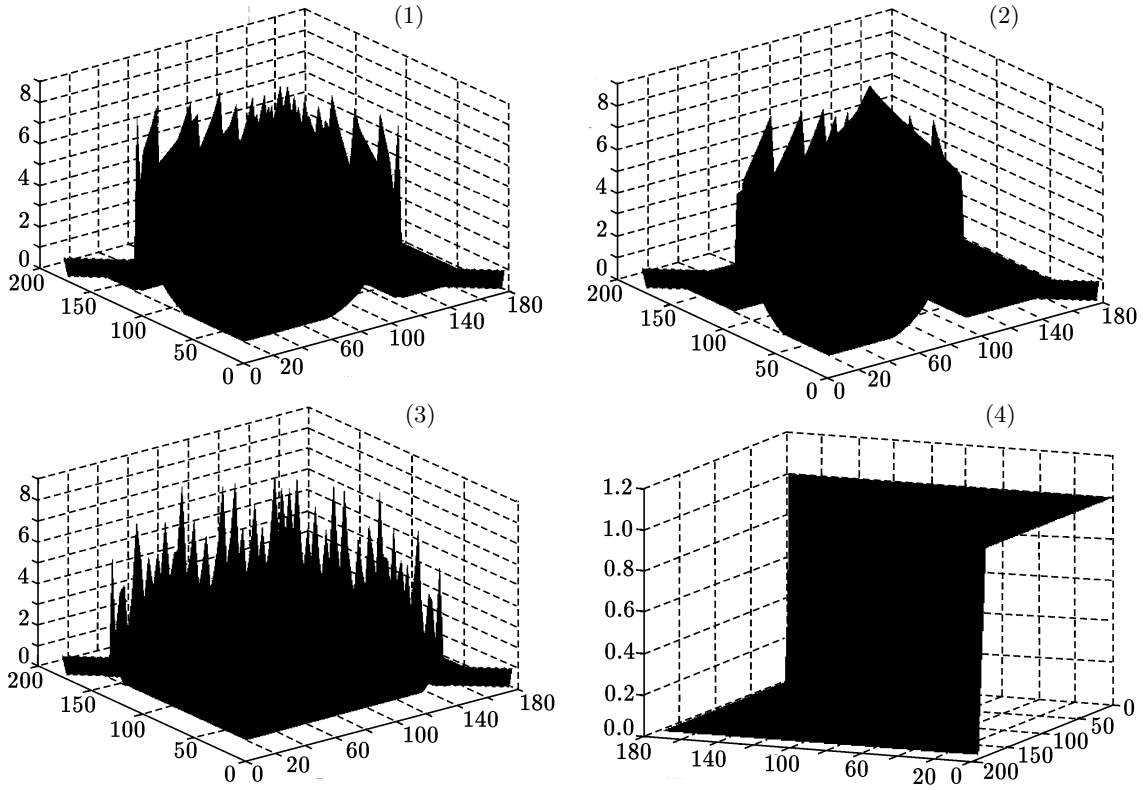

Figure 4. The behavior of λ_4 vs. α_2 and β_2 for the modified AMLI method: (1) — $\alpha_1 = 60°$, $\beta_1 = 60°$; (2) — $\alpha_1 = 60°$, $\beta_1 = 60°$; (3) — $\alpha_1 = 30°$, $\beta_1 = 30°$; (4) — $\alpha_1 = 175°$, $\beta_1 = 4°$

Thus, choosing the corresponding value of the iterative parameter θ, we can always ensure the positive definiteness of the preconditioned matrix $M^{(k)^{-1}}A^{(k)}$ on the local level, and hence, by Lemma 1 on the global one, too. Moreover, we can obtain an a priori chosen upper bound ε^{-1} on the maximal eigenvalue of $\widetilde{A}^{(k)^{-1}}A^{(k)}$, whereas the minimal eigenvalue depends on the sum of all perturbations. Note that these results are similar to the earlier ones [16]. Finally, in Figure 4, we present the dependence of λ_4 of the modified AMLI method ($\varepsilon = 10^{-1}$) from α_2 and β_2 for a similar choice of the first triangle as for Figure 3. One can see a considerable improvement in the eigenvalue behavior with respect to the unmodified AMLI method.

Theorem 1. *The upper and the lower bounds of eigenvalues of the generalized eigenvalue problem* (14) *at the level* k *under the above-defined choice of* θ *are the following*:

$$\alpha_k > \frac{1}{1+\xi} \quad \textit{and} \quad \beta_k < \varepsilon^{-1},$$

where ε *is a given value and* ξ *is defined by* (22).

Proof. The proof follows directly from the definitions of (19), (21), (22) and inequalities (23)–(26). □

Remark 1. A weak point of the AMLI method in [6] was an isosceles right triangular mesh. Indeed, for some superelements, we have $\widehat{\alpha_1} = \widehat{\beta_2} = \frac{\pi}{2}$, i.e., two isosceles right-angled triangles, then $\alpha_1 = \beta_2 = 0$, and hence, $\eta = 0$. Note that in this case, the coupling between nodes 1 and 2 is equal to zero and therefore the matrix graph is not triangular. This caused the condition on the original grid. For the new method, this situation is successfully resolved by using some perturbation θ, see case **D**.

Remark 2. Another impressive example for the original AMLI method, which is considered in this section, is $\alpha_1 = \beta_1 = 60^\circ$ and $\alpha_2 = 120^\circ, \beta_2 = 45^\circ$. Indeed, in this case, it may happen that the preconditioned system is singular or indefinite since $\lambda_4 \approx -0.333 < 0$. On the other hand, the new method solves this problem by choosing $\theta = -1$, see case **B**.

Finally, we collect the above results from Sections 2 and 3 in the following theorem.

Theorem 2. *The modified algebraic multilevel iteration method for the finite element matrices, based on a sequence of the matrices* $\{A^{(k)}\}$ *defined with* (2), (3), *and* (4), *and a sequence of their preconditioners* $\{M^{(k)}\}$, *recursively defined with* (5), (6), *and* (7), *has an optimal order of computational complexity iff*

$$(\delta\rho)^{\mu+1} > \nu > \left(\max_{k=1,2,\ldots,L/\mu} \prod_{s=L-k\mu}^{L-(k-1)\mu} \frac{1+\xi_s}{\varepsilon_s} \right)^{\frac{1}{2}},$$

where ξ_s *and* ε_s *are the above-defined constants by Theorem 1,* ρ *is a coefficient of geometric progression,* δ *is a parameter, which depends on the number of nodes on the coarse mesh and defined by* (10), ν *is the degree of the matrix polynomials used in* (9), *and* $\nu > 1$ *at every* $(\mu+1)$*th step.*

Unfortunately, we could not find the lower bound analytically, and hence, the optimal value of ε, which minimizes this ratio, is also unattainable. To verify Theorem 2, we have carried out the numerical experiments.

Remark 3. Since linear system of equations (1) arises as finite element approximation of a boundary value problem, we can use the following formula for definition of ε

$$\varepsilon \approx h \quad \Leftrightarrow \quad \varepsilon^{-1} = \mathbf{c}h^{-1}.$$

Indeed, one knows from the multigrid theory that finite element approximation of the original boundary value problem on coarser levels is usually a good approximation to the corresponding Schur complement. Clearly, all properties of those matrices depend on the discretization parameter h, and hence, the upper bound on eigenvalues is also dependent on the order of the system to be solved ($n \approx h^{-d}$).

4. Numerical results

Due to the fact that the AMLI method, developed in [6], has shown a good performance for non-uniform triangular meshes and does not work on the isosceles right triangular mesh,

we will test the new AMLI method on that mesh for model boundary value problems. Consider the Poisson equation

$$-\Delta u = f$$

in the domain $\Omega = [0,1]^2$ with the homogeneous Dirichlet boundary conditions

$$u\Big|_{\Gamma} = 1, \quad \text{where } \Gamma = \partial\Omega.$$

The coefficient matrix on the finest level was derived using standard piecewise linear finite elements on the triangular mesh. The right-hand side in the system of equations was chosen so that the solution has the form

$$u = x(1-x)y(1-y)e^{xy}.$$

The solution technique is the preconditioned conjugate gradient method with a precondi-

Table 1. The number of iterations for the modified AMLI method

	$\mu=0$			$\mu=1$			$\mu=2$		
N	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=1$	$\nu=2$	$\nu=3$
$\varepsilon^{-1}=h^{-1}=N+1$									
15	42	19	15	42	33	25	–	–	–
31	61	25	16	61	45	26	61	60	59
63	91	32	17	91	78	27	91	89	88
127	135	39	17	135	109	27	135	133	127
$\varepsilon^{-1}=2h^{-1}=2(N+1)$									
15	39	18	15	39	32	24	–	–	–
31	58	24	15	58	44	25	58	55	55
63	86	30	16	86	75	26	86	82	81
127	129	36	16	129	108	26	129	124	122
$\varepsilon^{-1}=4h^{-1}=4(N+1)$									
15	40	19	16	40	33	25	–	–	–
31	60	26	17	60	46	26	60	60	59
63	91	33	17	91	80	27	91	90	90
127	139	41	18	139	111	28	139	135	134

tioner M defined by (5), (6) and (7). The initial approximation was always taken as zero vector. The following stopping criterion was used

$$\frac{\mathbf{r}_i^T M^{-1}\mathbf{r}_i}{\mathbf{r}_1^T M^{-1}\mathbf{r}_1} < \epsilon = 10^{-12},$$

where $\mathbf{r}_1$ and $\mathbf{r}_i$ are the initial and the current residuals, respectively.

In Table 1, the results of the experiments for the preconditioned conjugate gradient method for different choices of polynomial degrees (μ,ν), for increasing the size of $(N\times N)$-grids and for various choices of ε are given. In all the tables of the present section, ν is the degree of the matrix polynomials used and $\nu > 1$ at every $(\mu+1)$th level. For example,

Table 2. The maximal and minimal eigenvalues of matrices $M^{(k)^{-1}}A^{(k)}$ on various levels for $(\mu,\nu)=(0,3)$

N	Lvl	$\lambda_{\max}^{(\mathrm{Lvl})}$	$\lambda_{\min}^{(\mathrm{Lvl})}$	$\kappa^{(\mathrm{Lvl})}$
15	1	2.2851	0.9342	2.4460
	2	2.8513	0.8301	3.4348
31	1	2.2962	0.9442	2.4319
	2	2.9122	0.8311	3.5040
	3	3.2110	0.7834	4.0988
63	1	2.6926	0.9221	2.9201
	2	3.1025	0.8289	3.7429
	3	3.4193	0.7738	4.4188
	4	3.7197	0.7012	5.3037
127	1	2.8836	0.9045	3.1881
	2	3.3012	0.8139	4.0560
	3	3.7215	0.7511	4.9547
	4	3.8812	0.6572	5.9057
	5	3.9117	0.6225	6.2838

in the case $(\mu,\nu)=(0,1)$, we have a V-cycle method, whereas $(\mu,\nu)=(0,2)$ leads to a W-cycle.

Based on the results of experiments, it can be recommended to use the W-cycle version $(\mu,\nu)=(0,3)$ of the suggested method with $\varepsilon^{-1}=2h^{-1}$, as it gives a smallest number of iterations, and the total computational cost is proportional to the number of unknowns on a fine mesh. Indeed, in this case, we have $\rho=3$, and hence, the following inequalities for the upper bound on polynomial degrees are valid for all μ

$$\nu=3\le 3^{\mu+1}=\rho^{\mu+1}.$$

Now, using the results from Table 2, we can verify the lower bound on the polynomial degrees from Theorem 2. For example, for $N=127$, we have

$$\left(\max_{k=1,2,\ldots,L/\mu}\prod_{s=L-k\mu}^{L-(k-1)\mu}\frac{1+\xi_s}{\varepsilon_s}\right)^{1/2}=\left(\max_{k=1,2,\ldots,L}\frac{\lambda_{\max}^{(k)}}{\lambda_{\min}^{(k)}}\right)^{1/2}=\left(\max_{k=1,2,\ldots,L}\kappa^{(k)}\right)^{1/2}$$

$$=\ (\max\{3.1881,4.0560,4.9547,5.9057,6.2838\})^{1/2}=(6.2838)^{1/2}=2.5065<3=\nu.$$

Analyzing the square root of the corresponding condition numbers for another N, one can see that it is always less than the corresponding value of $\nu=3$.

Now, to illustrate the efficiency of the techniques used and to compare the method presented here with the earlier ones, the corresponding experiments were performed (see Table 3). Here AMLI denotes the original AMLI method from [7] with optimal parameters $(\mu,\nu)=(1,2)$, AMLI-IF denotes the algebraic multilevel incomplete factorization method suggested in [15] also with optimal parameters $(\mu,\nu)=(1,3)$ and AMLI-FE denotes the new method used with $(\mu,\nu)=(0,3)$. Since all the methods were implemented in different programs and computer environments, we present only the number of iterations. Here we use the similar stopping criteria with $\epsilon=10^{-6}$.

Table 3. The number of iterations for different multilevel methods, the Laplace problem

N	AMLI	AMLI-IF	AMLI-FE
15	6	5	5
31	5	6	5
63	4	7	6
127	4	7	6

Based on the results of experiments, one can see that the new method required a similar number of iterations to reach the desired accuracy, and hence, due to the good results for arbitrary triangular grids, the new method is a good competitor to those methods for problems arising in finite element applications.

The final experiments are made for the elliptic boundary value problem with anisotropic coefficients

$$-\nabla a\nabla u=f,\qquad a=\begin{bmatrix}1&0\\0&\delta\end{bmatrix},$$

where the positive value δ is an *anisotropy ratio* in the domain $\Omega=[0,1]^2$ with the homogeneous Dirichlet boundary conditions on the boundary Γ. Similar to the Poisson problem, the right-hand side in the system of equations was chosen so that the solution has the form

$$u = x(1-x)y(1-y)e^{xy}.$$

Note that the solution of the anisotropic problem is smooth and does not depend on the anisotropy ratio δ.

Table 4. The number of iterations for the modified AMLI method

	$\mu = 0$			$\mu = 1$			$\mu = 2$		
δ	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 1$	$\nu = 2$	$\nu = 3$
$N = 63$									
1	86	30	16	86	75	26	86	82	81
10^{-2}	110	41	19	110	98	42	110	109	107
10^{-4}	151	54	24	151	118	56	151	151	150
10^{-6}	193	68	30	193	143	71	193	192	192
$N = 127$									
1	129	36	16	129	108	26	129	124	122
10^{-2}	178	49	20	178	143	44	178	178	178
10^{-4}	234	65	24	234	171	56	234	234	233
10^{-6}	307	83	32	307	207	73	307	308	307

Analogously to the isotropic case we first found the optimal values of iterative parameters for μ, ν and ε. They are $(\mu, \nu) = (0, 3)$ and $\varepsilon = 0.5h$. The corresponding results for the number of iterations with respect to the anisotropy ratio are given in Table 4. As is readily seen, the rate of convergence depends on the anisotropy ratio δ due to the non-optimality of the uniform refinement process. Indeed, it is well known that the best possible refinement algorithm for anisotropic problems is the so-called semi-coarsening, i.e., we make the coarsening along the strongest anisotropy direction. The coarsening algorithm, presented in Section 2.3, does not allow for a possible anisotropy, and hence, we lose the valuable information about the original problem during the coarsening process. Unfortunately, it is impossible to correct it later with the help of polynomial stabilization.

As a final conclusion, we have found that the modified algebraic multilevel iteration method, applied to the finite element matrices, leads to iterative methods of nearly optimal order of the computational complexity and rate of convergence. However, to extend its field of applicability, the present method required a special coarsening technique, which will use the physical information rather than the geometric one only. Recently, a very robust method for this type of a problem has been proposed by Notay [23]. In spite of the theoretical lack the coarsening technique suggested in this work, this method has shown its efficiency and robustness. Moreover, the method works as a "black box", i.e., it is a parameter free method. Nevertheless, the proposed in the present work modification extends the range of application for the old AMLI method and covers the existing gap in the matrix approach for the AMLI methods.

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Institute für Geometrie und Praktische Mathematik,
RWTH Aachen, Germany
E-mail: larin@igpm.rwth-aachen.de

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