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Pole Constraints of Reference Models in 2-DOF Servo System Design for Non-Minimum Phase Systems

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Abstract : This paper is concerned with the analysis of pole constraints in servo system design for non-minimum phase (NMP) systems. We first characterize the achievable closed-loop system for a SISO plant. For simplicity, we assume that the plant has only one NMP zero. Based on the characterization and the tracking condition, we show for some combinations of degree and relative degree of the closed-loop system that the admissible location of poles is restricted. For these cases, we provide a quantitative measure for the limitation. We also provide its concrete formula for some specific cases.

Key Words : non-minimum phase systems, servo systems, pole constraints.

1. Introduction

A problem of perpetual interest in the field of control is the difficulty to control non-minimum phase (NMP) systems. NMP systems are generally known to be difficult to control when compared to minimum phase systems. In the field of tracking control, this difficulty is evident in the step response. In this case, NMP zeros may cause severe transient like the output undershoots [1]. Hence, this inherent property imposes limitation to the transient performance of the system.

Over the years, the performance limitations due to NMP zeros have been revealed theoretically [2]. Recently, the analytic forms of the limitations on some performance criteria have also been given [3]–[6]. In servo system design, analysis on realizable transfer functions and its fundamental structures have been given [7],[8]. However, more details such as possible pole locations have not been explored.

This paper deals with the constraints of poles and zeros in servo system design for NMP systems. In order to achieve good transient performance, the locations of the closed-loop poles play an important role. In this paper, by considering the constraints caused by NMP zeros, we seek to analyze and clarify the limitations of the closed-loop poles in servo system design. This study is carried out for a single input single output (SISO) linear time invariant plant with one NMP zero in order to expose the essential features of pole and zero constraints.

The remainder of this paper is organized as follows. In Section 2, we establish the problem formulation of this paper. In Sections 3 and 4, we present the main result of our analysis. In Section 3, we show our results on the constraints of poles caused by NMP zeros and tracking conditions. More detailed analysis is given in Section 4 for some rather simple but practically important cases.

Notation is standard. Polynomial $m(s)$ is of degree μ, if $m(s)$ is written by

$$m(s) = a_\mu s^\mu + a_{\mu-1} s^{\mu-1} + \cdots + a_1 s + a_0$$

where $a_\mu \neq 0$. Moreover, $m(s)$ is Hurwitz, if all the roots of $m(s)$ lie in the open left half plane.

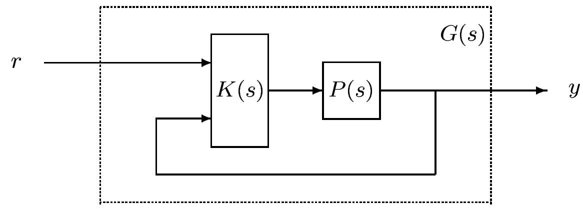

Fig. 1 Block diagram of SISO system.

2. Problem Formulation

We consider a SISO system as depicted in Fig. 1, where $P(s)$ is a given strictly proper plant and $K(s)$ is a controller to be designed. Note that $K(s)$ has two degree of freedom. u, r and y are the control input, the reference input and the control output respectively. $G(s)$ is the closed-loop system from r to y.

The purpose of our study is to analyze how NMP zeros of $P(s)$, i.e. the zeros in the closed right half plane, affects the possible locations of the closed-loop poles. To simplify the analysis, we focus on set of possible $G(s)$, instead of constructing or parameterizing $K(s)$. The set of $G(s)$, given by the set of proper and internally stabilizing controllers $K(s)$, is characterized by the following lemma:

Lemma 1 There exists $K(s)$ such that the feedback system in Fig. 1 is internally stable, iff $G(s)$ satisfies the following conditions:

(C1) : $G(s)$ has the same NMP zeros as $P(s)$

(C2) : RD of $G(s) \geq$ RD of $P(s)$

(C3) : $G(s)$ is stable

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where RD stands for the relative degree.

Proof: (Necessity) We first write $K(s)$ as

$$K(s) = [\, K_{\rm FF}(s)\; K_{\rm FB}(s)\,]$$

according to its inputs. Then, the transfer function from r to y is given by

$$G(s) = \frac{P(s)}{1 + P(s)K_{\rm FB}(s)} K_{\rm FF}(s).$$

Since $K(s)$ is internally stabilizing, $G(s)$ must be stable and have the same NMP zeros as $P(s)$. Moreover, the relative degree of $G(s)$ is greater than or equal to that of $P(s)$.
(Sufficiency) Suppose $G(s)$ satisfies (C1), (C2) and (C3). Then, construct $K(s)$ as depicted in Fig. 2 [7], where $C_{\rm FB}(s)$ is an arbitrary controller that internally stabilizes the closed-loop system composed by $P(s)$ and $C_{\rm FB}(s)$. The transfer function from r to y is $G(s)$ regardless of $C_{\rm FB}(s)$. Moreover, due to (C1) and (C2), $P(s)^{-1}G(s)$ is proper and stable. Hence, the whole control system in Fig. 2 is internally stable. □

Lemma 1 clarifies the requirement for $G(s)$. If $G(s)$ satisfies (C1) to (C3), there always exists an internally stabilizing $K(s)$ that attains $G(s)$. Hence, instead of searching $K(s)$, we can design the control system by choosing an appropriate $G(s)$ under (C1), (C2) and (C3). In other words, $G(s)$ can be used as a reference model for the control system to be designed. As long as (C1) to (C3) are satisfied, $G(s)$ can be chosen arbitrarily. For example, the order of $G(s)$ can be much smaller than those of $P(s)$ and $K(s)$ to be implemented.

While attaining internal stability, we consider servo system design. In this paper, we shall consider a step and/or trigonometrical reference signals whose frequencies are denoted by $0 \le \omega_1 < \omega_2 < \cdots < \omega_N$. Therefore, the condition of tracking is given by

(C4) : $G(j\omega_i) = 1$ for any $i = 1, \ldots, N$.

(C4) introduces the additional constraint to $G(s)$. This may result in constraining pole locations. Hence, this paper aims to investigate the constraints of poles of $G(s)$ in the above tracking control problem.

In order to make the technical manipulations easier, we parametrize $G(s)$ such that (C3) and (C4) hold. The following lemma gives such a parameterization:

Lemma 2 Let $G(s)$ be a given real rational transfer function. Then, (C3) and (C4) hold, iff there exist a polynomial $n(s)$ and a Hurwitz polynomial $m(s)$ such that the following equation holds:

$$G(s) = \frac{n(s)\,\sigma(s)\prod_{i=2}^{N}(s^2+\omega_i^2) + m(s)}{m(s)} \qquad (1)$$

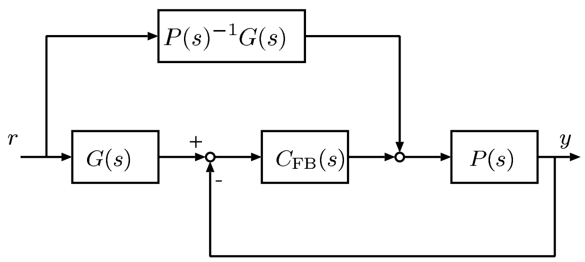

Fig. 2 2DOF control system.

where

$$\sigma(s) = \begin{cases} s & (\omega_1 = 0) \\ s^2 + \omega_1^2 & (\omega_1 > 0) \end{cases}.$$

Proof: Suppose that $G(s)$ is given by (1). Since $m(s)$ is Hurwitz, (C3) holds trivially. (C4) also holds, since, for any $i = 1, \ldots, N$, the followings hold:

$$n(j\omega_i)\,\sigma(j\omega_i)\prod_{k=2}^{N}((j\omega_i)^2+\omega_k^2) = 0, \quad m(j\omega_i) \neq 0$$

where the latter condition follows since $m(s)$ is assumed to be Hurwitz.

Suppose (C3) and (C4) hold. Then, $G(s) - 1$ has zeros at $s = j\omega_i$ $(i = 1, \ldots, N)$. Since $G(s)$ is real rational and stable, there exist $n(s)$ and Hurwitz $m(s)$ such that the following equation holds:

$$G(s) - 1 = \frac{n(s)\,\sigma(s)\prod_{i=2}^{N}(s^2+\omega_i^2)}{m(s)}. \qquad (2)$$

(2) gives (1). □

Note that $n(s)$ in (1) must be a nonzero polynomial. Otherwise, $G(s) = 1$ and it contradicts to (C2) for strictly proper $P(s)$. Moreover, we can assume $m(s)$ monic without loss of generality.

Note that, even if the degree of $m(s)$ is μ, the order of $G(s)$ in (1) may no be μ, since there can be stable pole/zero cancellations between $m(s)$ and $n(s)$. However, the relative degree of $G(s)$ is independent of the existence of the pole/zero cancellations.

We carry out the analysis based on (1), (C1) and (C2). However, it looks difficult to obtain general results. Hence, we focus on some rather simple but practically important cases. Specifically, we consider the case that the following conditions hold:

- $P(s)$ has only one NMP zero at $s = z_1 > 0$.
- The number of the frequencies is one, i.e. $N = 1$.

Since $P(s)$ is strictly proper, the possible order μ of $G(s)$ is greater than or equal to two, and the relative degree of $G(s)$ can be between one and $\mu - 1$.

3. Analysis of Pole Constraints

In this section, we reveal the constraints on the pole locations for some cases of degree and relative degree. We first consider the case of $\omega_1 = 0$. In this case, there is no pole restrictions. In fact, the following fact is obtained:

Lemma 3 Suppose $N = 1$ and $\omega_1 = 0$. Let $\mu \ge 2$ be a given order and $z_1 > 0$ be a given NMP zero. Then, for any Hurwitz $m(s)$ of degree μ and any ρ where $1 \le \rho \le \mu - 1$, there exists $n(s)$ such that the relative degree of $G(s)$ in (1) is ρ and that (C1) holds.

Proof: Choose an arbitrary polynomial $d(s)$ of degree $\mu - \rho$ such that the following equations hold:

$$d(0) = m(0), \quad d(z_1) = 0.$$

Such $d(s)$ always exists, since $\mu - \rho \ge 1$ is assumed. Define $n(s)$ as

$$n(s) = \frac{d(s) - m(s)}{s}.$$

Note that the above $n(s)$ is a polynomial, since $d(0) - m(0) = 0$, i.e. s is a factor of $d(s) - m(s)$. Then, $G(s)$ in (1) is given by

$$G(s) = \frac{n(s)\, s + m(s)}{m(s)} = \frac{d(s)}{m(s)}.$$

The relative degree of $G(s)$ is $\mu - (\mu - \rho) = \rho$. We have assumed $d(z_1) = 0$. Moreover, $m(z_1) \neq 0$ since $m(s)$ is Hurwitz. Therefore, (C1) holds. □

Lemma 3 shows that there is no constraints on the poles of $G(s)$. Hence, we can assign the poles anywhere, unless other specifications are concerned.

We next consider the case of $\omega_1 > 0$. In this case, the situation depends on the relative degree of $G(s)$.

Lemma 4 Suppose $N = 1$ and $\omega_1 > 0$. Let $\mu \geq 3$ be a given order and $z_1 > 0$ be a given NMP zero. Then, for any Hurwitz $m(s)$ of degree μ and any ρ where $1 \leq \rho \leq \mu - 2$, there exists $n(s)$ such that the relative degree of $G(s)$ in (1) is ρ and that (C1) holds.

Proof: Choose an arbitrary polynomial $d(s)$ of degree $\mu - \rho$ such that the following equations hold:

$$d(\pm j\omega_1) = m(\pm j\omega_1), \quad d(z_1) = 0.$$

Such $d(s)$ always exists, since $\mu - \rho \geq 2$ is assumed. The rest of the proof is quite similar to Lemma 3 and it is omitted. □

Lemma 5 Suppose $N = 1$ and $\omega_1 > 0$. Let $z_1 > 0$ be a given NMP zero. Then, for any $\mu \geq 2$, there exists a Hurwitz $m(s)$ of degree μ that can not satisfy both of the following conditions for $G(s)$ in (1):

- (C1) holds.
- The relative degree of $G(s)$ is $\mu - 1$.

In other words, for any $n(s)$, one of the above conditions is violated.

Proof: Suppose that $\mu \geq 2$ is given. Then, $m(s)$ of degree μ can be written by

$$m(s) = s^{\mu} + a_{\mu-1} s^{\mu-1} + \cdots + a_1 s + a_0.$$

Note that $m(s)$ can be monic without loss of generality. On the other hand, in order to make $G(s)$ have the relative degree of $\mu - 1$, the degree of $n(s)(s^2 + \omega_1^2) + m(s)$ must be one. Hence, $n(s)$ must have degree $\mu - 2$, and it can be written by

$$n(s) = b_{\mu-2} s^{\mu-2} + b_{\mu-3} s^{\mu-3} + \cdots + b_1 s + b_0$$

where $b_{\mu-2} \neq 0$. If $G(s)$ has the relative degree of $\mu - 1$ and $G(z_1) = 0$, there exists $c \neq 0$ such that the following equation holds:

$$n(s)(s^2 + \omega_1^2) + m(s) = c(s - z_1)$$

Hence, the coefficients of the above equation satisfies

$$Mv = -u$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \omega_1^2 & 0 & 1 & \cdots & 0 & 0 \\ 0 & \omega_1^2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & \omega_1^2 & z_1 \end{bmatrix},$$

$$v = \begin{bmatrix} b_{\mu-2} \\ b_{\mu-3} \\ b_{\mu-4} \\ \vdots \\ b_0 \\ c \end{bmatrix}, \quad u = \begin{bmatrix} a_{\mu} \\ a_{\mu-1} \\ a_{\mu-2} \\ a_{\mu-3} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}.$$

Since the number of rows of M is strictly greater than the number of columns, there exists a nonzero vector η such that $\eta M = 0$. This implies $\eta u = 0$. Such η can be written by

$$\eta = \begin{bmatrix} * & \cdots & * & z_1 & 1 \end{bmatrix}$$

without loss of generality.

Now, suppose that $m(s)$ is given by $(s - p)^{\mu}$. Note that $m(s)$ is Hurwitz, if p is negative real. The coefficients of $m(s)$ are given by

$$a_0 = (-p)^{\mu},$$

$$a_i = \frac{\prod_{k=0}^{i-1}(\mu - k)}{i!} (-p)^{\mu - i}, \quad i = 1, \ldots, \mu - 1.$$

In this case, ηu is the polynomial of p and can be written by

$$\eta u = (-p)^{\mu} + z_1 \mu (-p)^{\mu-1} + \cdots.$$

Hence, if p is sufficiently large, ηu is nonzero. Thus, there exists a Hurwitz $m(s)$ such that $\eta u \neq 0$ holds. For this $m(s)$, no $n(s)$ satisfies the requirement. □

Lemma 4 implies that, if the relative degree of $G(s)$ is less than $\mu - 1$, the poles of $G(s)$ can be arbitrarily chosen. However, in this case, $G(s)$ must have another zero in addition to z_1. On the other hand, by Lemma 5, if $G(s)$ has only one zero at $s = z_1$, the possible poles of $G(s)$ is constrained.

4. Detailed Analysis of Pole Constraints

The results in the previous section clarify that the pole locations are restricted in the case of $\omega_1 > 0$ and $\rho = \mu - 1$. We will further investigate the pole constraints deeply in the cases $\mu = 2$ and $\mu = 3$. In these cases, there is no chance to have pole/zero cancellations between $m(s)$ and $n(s)$ in (1). Hence, the orders of the resultant $G(s)$ are also two and three, respectively.

Note that simplicity of $G(s)$ may not restrict possible applications severely. As is explained in Section 2, $G(s)$ can be seen as a reference model of the control system to be designed. Hence, even if $G(s)$ is quite simple, complex high order $P(s)$ can be dealt with, provided (C1) to (C4) are satisfied.

In some cases, simple $G(s)$ is even desirable. If we synthesize the control system by choosing an appropriate reference model, we need to design $G(s)$ first, i.e. we need to determine the poles and the zeros of $G(s)$. To make the design easy to deal with, the number of the poles and the zeros are desired to be small. In practical situations, complex high order systems are not appropriate for reference models.

For constrained poles, we evaluate the limitation based on the following criterion:

$$\beta_* = \sup\left\{\beta : \begin{array}{l}\exists G(s) \text{ s.t. } G(s-\beta) \text{ is stable} \\ \text{and } G(s) \text{ satisfies (C1) to (C4)}\end{array}\right\}.$$

β_* is the maximum value such that all the poles can be assigned on the shaded region in Fig. 3 when $G(s)$ is carefully chosen.

β_* gives a measure of the best achievable stability margin and also performance. If β_* is small, the achievable transient performance would be poor. If there is no constraint on pole locations, $\beta_* = \infty$. In the sequel, for $\mu = 2, 3$, β_* will be denoted as $\beta_*^{(2)}$ and $\beta_*^{(3)}$, respectively. Concrete formula of $\beta_*^{(2)}$ and $\beta_*^{(3)}$ are given as follows:

Theorem 1 $\beta_*^{(2)}$ is given by

$$\beta_*^{(2)} = -z_1 + \sqrt{z_1^2 + \omega_1^2} = \frac{\omega_1^2}{z_1 + \sqrt{z_1^2 + \omega_1^2}} \quad (3)$$

Moreover, $\beta_*^{(2)}$ is attained when the poles p_1 and p_2 of $G(s)$ are given by $p_1 = p_2 = -\beta_*^{(2)}$.

Proof: Since the degree is two, $m(s)$ and $n(s)$ can be written as follows:

$$m(s) = (s - p_1)(s - p_2), \quad (4)$$
$$n(s) = \alpha \quad (5)$$

where $\alpha \neq 0$ is a real constant. p_1 and p_2 are negative reals or a complex conjugate pair whose real parts are negative.

Since the relative degree is $\mu - 1 = 1$, we must choose $\alpha = -1$. In order to achieve (C1), the numerator $N(s)$ of $G(s)$ given by

$$N(s) = \omega_1^2 + (p_1 + p_2)\, s - p_1\, p_2 \quad (6)$$

must satisfy $N(z_1) = 0$. Therefore, the following equation can be deduced:

$$p_1\, p_2 - z_1\,(p_1 + p_2) - \omega_1^2 = 0. \quad (7)$$

Applying Lemma 6 in Appendix to (7), (3) is proven. As is given in the proof of Lemma 6, $\beta_*^{(2)}$ is attained by the poles $p_1 = p_2 = -\beta_*^{(2)}$. □

Theorem 2 $\beta_*^{(3)}$ is given by

$$\begin{aligned}\beta_*^{(3)} &= -z_1 - 2\sqrt{z_1^2 + \omega_1^2}\cos\left(\frac{\theta + 2\pi}{3}\right) \\ &= -z_1 + \sqrt{z_1^2 + \omega_1^2} \times \\ &\quad \left(\cos\frac{\theta}{3} + \sqrt{3}\sin\frac{\theta}{3}\right)\end{aligned} \quad (8)$$

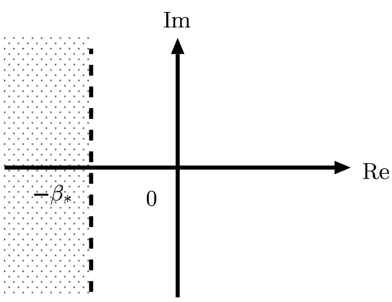

Fig. 3 β_* on the left-hand plane.

where $\theta = \angle(z_1 + \omega_1 j)$. Moreover, $\beta_*^{(3)}$ is attained, when the poles p_1, p_2 and p_3 of $G(s)$ are given by $p_1 = p_2 = p_3 = -\beta_*^{(3)}$.

Proof: Since the degree is three, $m(s)$ and $n(s)$ can be chosen, without loss of generality, as follows:

$$m(s) = (s - p_1)(s - p_2)(s - p_3) \quad (9)$$
$$n(s) = \alpha(s + a) \quad (10)$$

where $\alpha \neq 0$ and a are constants. p_1, p_2 and p_3 are complex numbers such that $m(s)$ is a real polynomial. All the real parts of p_1, p_2 and p_3 are assumed negative.

For the relative degree to be $\mu - 1 = 2$, we need to choose α and a as

$$\alpha = -1, \quad a = -(p_1 + p_2 + p_3)$$

In order to achieve (C1), the numerator for $G(s)$ given by

$$\begin{aligned}N(s) = {}& (\omega_1^2 - (p_1p_2 + p_2p_3 + p_3p_1))\, s \\ & - (p_1 + p_2 + p_3)\omega_1^2 + p_1p_2p_3\end{aligned} \quad (11)$$

must satisfy $N(z_1) = 0$. Therefore, the following equation follows:

$$\begin{aligned}&(\omega_1^2 - (p_1p_2 + p_2p_3 + p_3p_1))\, z_1 \\ &\quad - (p_1 + p_2 + p_3)\,\omega_1^2 + p_1p_2p_3 = 0.\end{aligned} \quad (12)$$

Since $m(s)$ is real polynomial, one of p_1, p_2 and p_3 is real. We here assume that p_3 is real without loss of generality. Then, (12) can be written by

$$p_1\, p_2 + \phi\,(p_1 + p_2) - \omega_1^2 = 0 \quad (13)$$

where

$$\phi = -\frac{p_3 z_1 + \omega_1^2}{p_3 - z_1} = -z - \frac{\omega_1^2 + z_1^2}{p_3 - z_1}.$$

For $p_3 < 0$, ϕ takes the value in $\left(-z_1, \frac{\omega_1^2}{z_1}\right)$. Since ϕ is monotonically increasing in the interval, p_3 and ϕ have one-to-one correspondence. In fact, p_3 can be represented with respect to ϕ as follows:

$$p_3(\phi) = \frac{\phi z_1 - \omega_1^2}{\phi + z_1} = z_1 - \frac{z_1^2 + \omega_1^2}{\phi + z_1}$$

which is monotonically increasing over its domain.

By Lemma 6, $\gamma = \min_{p_1,p_2} \max\{\mathrm{Re}(p_1), \mathrm{Re}(p_2)\}$ is given by

$$\gamma(\phi) = -\phi - \sqrt{\phi^2 + \omega_1^2}.$$

Hence, $\beta_*^{(3)}$ is given by

$$\beta_*^{(3)} = -\inf_{\phi} \max\{p_3(\phi), \gamma(\phi)\}.$$

Since the derivative of $\gamma(\phi)$ is negative over $\left(-z_1, \frac{\omega_1^2}{z_1}\right)$, γ is monotonically decreasing. Moreover, the following relations hold between $\gamma(\phi)$ and $p_3(\phi)$:

$$\gamma(-z_1) = z_1 - \sqrt{z_1^2 + \omega_1^2} \;>\; \lim_{\phi \downarrow -z_1} p_3(\phi) = -\infty,$$

$$\gamma\left(\frac{\omega_1^2}{z_1}\right) = -\frac{\omega_1^2}{z_1}\left(1 + \sqrt{\omega_1^2 + z_1^2}\right) \;<\; p_3\left(\frac{\omega_1^2}{z_1}\right) = 0.$$

Therefore, $\beta_*^{(3)}$ is attained at the point $p_3 = \gamma$, i.e. the point such that p_3 is a negative solution of the following equation:

$$p_3^4 - 4z_1 p_3^3 + 3(z_1^2 - \omega_1^2)p_3^2 + 4z_1\omega_1 p_3 - z_1^2\omega_1^2$$
$$= (p_3 - z)(p_3^3 - 3z_1 p_3^2 - 3\omega_1^2 p_3 + z_1\omega_1^2) = 0$$

Since $p_3 - z < 0$ by the assumptions, p_3 is a solution of the following equation:

$$f(p_3) = p_3^3 - 3z_1 p_3^2 - 3\omega_1^2 p_3 + z_1\omega_1^2 = 0.$$

It is straightforward to confirm

$$f'\left(z_1 \pm \sqrt{z_1^2 + \omega_1^2}\right) = 0,$$
$$f\left(z_1 + \sqrt{z_1^2 + \omega_1^2}\right)$$
$$= -2(z_1^2 + \omega_1^2)\left(z_1 + \sqrt{z_1^2 + \omega^2}\right) < 0,$$
$$f\left(z_1 - \sqrt{z_1^2 + \omega_1^2}\right)$$
$$= 2(z_1^2 + \omega_1^2)\left(\sqrt{z_1^2 + \omega_1^2} - z_1\right) > 0$$

where f' is the derivative of f. Hence, we can see by elementary calculus that all the roots of f are real. Moreover, since $z_1 + \sqrt{z_1^2 + \omega_1^2} > 0$, $z_1 - \sqrt{z_1^2 + \omega_1^2} < 0$ and $f(0) = z_1\omega_1^2 > 0$, there exists a unique negative root $p_3 < 0$. Thus, by applying the formulae for the roots of cubic polynomials, we can find the negative root as follows:

$$p_* = z_1 + 2\sqrt{z_1^2 + \omega_1^2}\cos\left(\frac{\theta + 2\pi}{3}\right).$$

$\beta_*^{(3)}$ is given by $-p_*$. □

By Theorem 1, $\beta_*^{(2)}$ must be small, if ω_1 is small and/or z_1 is large. Such a small $\beta_*^{(2)}$ will contribute to poor transient performance and stability margin.

On the other hand, since $0 \le \theta \le \frac{\pi}{2}$ in Theorem 2, we can deduce

$$-\frac{\sqrt{3}}{2} \le \cos\left(\frac{\theta + 2\pi}{3}\right) \le -\frac{1}{2}.$$

Hence, $\beta_*^{(3)}$ satisfies

$$\beta_*^{(3)} \ge -z + \sqrt{z_1^2 + \omega_1^2} = \beta_*^{(2)}.$$

In other words, if we increase the order of $G(s)$ from two to three, the maximal stability margin is not degraded. Moreover, if $z_1 = \omega_1$, $\beta_*^{(3)}$ is given by $\beta_*^{(3)} = z_1 = \omega_1$.

Figure 4 shows $\beta_*^{(2)}$ and $\beta_*^{(3)}$ for each z_1, where ω_1 is fixed to $\omega_1 = 1$. We can see that $\beta_*^{(3)}$ is strictly greater than $\beta_*^{(2)}$. Notice that $\beta_*^{(3)} = 1 = \omega_1$ holds, when $z_1 = 1$. Figure 5 shows $\beta_*^{(2)}$ and $\beta_*^{(3)}$ for each w_1, where z_1 is fixed to $z_1 = 1$. Again, $\beta_*^{(3)}$ is strictly greater than $\beta_*^{(2)}$.

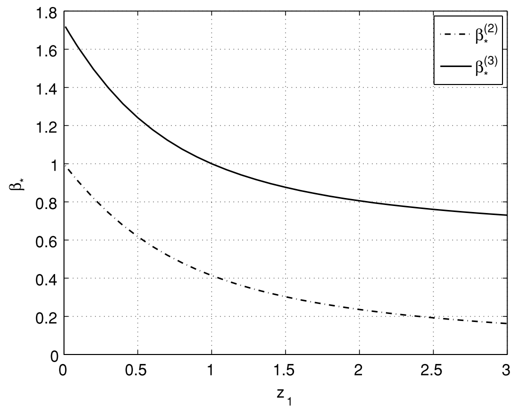

Fig. 4 β_* for $\omega_1 = 1$.

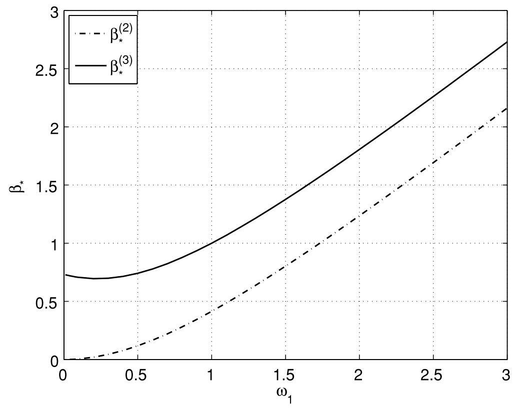

Fig. 5 β_* for $z_1 = 1$.

5. Conclusion

In this paper, we have shown that the poles of $G(s)$ can be constrained in servo system design for NMP systems. In some constrained cases, we have also shown that the limitation of pole assignments can be measured by using β_*.

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Appendix

Lemma 6 Suppose that the following equation is given:

$$p_1 p_2 + b(p_1 + p_2) - c = 0 \quad \text{(A. 1)}$$

where b and $c > 0$ are given real constants. (p_1, p_2) are variables that are either a pair of negative real numbers or of complex conjugates whose real parts are negative. Then, the following equation holds:

$$\min_{p_1,p_2} \max\{\mathrm{Re}(p_1), \mathrm{Re}(p_2)\} = -b - \sqrt{b^2 + c} \qquad \text{(A. 2)}$$

where $\min_{p_1,p_2}$ is taken over all (p_1, p_2) satisfying (A. 1).

Proof: Suppose that p_1 and p_2 are negative reals. Fix $p_1 < 0$ arbitrarily. Since (A. 1) holds, p_2 is a function of p_1. Indeed, $p_2(p_1)$ is given by

$$p_2(p_1) = \frac{c - b\,p_1}{p_1 + b} = -b + \frac{b^2 + c}{p_1 + b}.$$

In the case of $b \geq 0$, p_1 must satisfy $p_1 < -b$ so that p_2 is negative. $p_2(p_1)$ is monotonically decreasing in terms of p_1 over $(-\infty, -b)$, while p_1 is monotonically increasing with respect to p_1 itself. Moreover, we have

$$\lim_{p_1 \to -\infty} p_2(p_1) = -b \quad > \quad \lim_{p_1 \to -\infty} p_1 = -\infty,$$

$$\lim_{p_1 \uparrow -b} p_2(p_1) = -\infty \quad < \quad \lim_{p_1 \uparrow -b} p_1 = -b.$$

In this case, $\min_{p_1} \max\{p_1, p_2\}$ is attained at the point $p_1 = p_2$. Hence, the following equation holds:

$$\min_{p_1} \max\{p_1, p_2\} = -b - \sqrt{b^2 + c}. \qquad \text{(A. 3)}$$

In the case of $b < 0$, p_1 must satisfy $\frac{c}{b} < p_1 < 0$. Again, $\min_{p_1} \max\{p_1, p_2\}$ is attained at the point $p_1 = p_2$ due to the reason similar to the case of $b \geq 0$, and (A. 3) holds.

On the other hand, suppose that (p_1, p_2) is a complex conjugate pair. Then, they can be written by

$$p_1 = x + jy, \quad p_2 = x - jy$$

where $x < 0$. Therefore, (A. 1) leads to

$$x = -b - \sqrt{b^2 + c - y^2}. \qquad \text{(A. 4)}$$

(A. 4) implies that $\min_y x$ is given by

$$\min_{y} x = -b - \sqrt{b^2 + c}. \qquad \text{(A. 5)}$$

Thus, (A. 2) is proven from the above derivations. □

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