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ZARISKI-LIKE SPACES OF CERTAIN MODULES

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Abstract. Let R be a commutative ring with identity and M be a unitary R-module. The primary-like spectrum $Spec_L(M)$ is the collection of all primary-like submodules Q such that M/Q is a primeful R-module. Here, M is defined to be RSP if $rad(Q)$ is a prime submodule for all $Q \in Spec_L(M)$. This class contains the family of multiplication modules properly. The purpose of this paper is to introduce and investigates a new Zariski space of an RSP module, called a Zariski-like space. In particular, we provide conditions under which the Zariski-like space of a multiplication module has a subtractive basis.

1. Introduction

This paper focuses on rings, which all are commutative with an identity and modules are unitary. Let M be an R-module and N be a submodule of M. The colon ideal of M into N is the ideal $(N : M) = \{r \in R \mid rM \subseteq N\}$ of R. A proper submodule P of M is called p-prime if for $p = (P : M)$, whenever $rm \in P$, $r \in R$ and $m \in M$, then $m \in P$ or $r \in p$. The collection of all prime submodules of M is denoted by $Spec(M)$. If N is a submodule of M, then the radical of N, denoted $rad(N)$, is the intersection of all prime submodules of M which contain N, unless no such primes exist, in which case $rad(N) = M$.
A proper submodule Q of M is said to be primary-like if $rm \in Q$ implies $r \in (Q : M)$ or $m \in rad(Q)$ [5]. We state that a submodule N of

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an R-module M satisfies the primeful property if for each prime ideal p of R with $(N : M) \subseteq p$, there exists a prime submodule P containing N such that $(P : M) = p$. In this case $\sqrt{(N : M)} = (rad(N) : M)$ [10, Proposition 5.3]. For example the zero submodule of the $\mathbb{Z}$-module $M = \prod_{p \in \Omega}(\frac{\mathbb{Z}}{p\mathbb{Z}})$ is not a primary-like submodule of M, but it satisfies the primeful property [7, Example 1.1(6)]. On the other hand although $M' = \oplus_{p \in \Omega}(\frac{\mathbb{Z}}{p\mathbb{Z}})$ is a primary-like submodule of M, it dose not satisfy the primeful property [10, Example 1(5) and (6)]. In [5, Lemma 2.1] it is shown that, if Q is a primary-like submodule satisfying the primeful property, then $p = \sqrt{(Q : M)}$ is a prime ideal of R and so in this case, is Q called a p-primary-like submodule.

The primary-like spectrum $Spec_L(M)$ is defined to be the set of all primary-like submodules of M satisfying the primeful property. For example if M is the $\mathbb{Z}$-module $\mathbb{Q} \bigoplus \mathbb{Z}_p$, where $\mathbb{Q}$ is the abelian group of rational numbers and $\mathbb{Z}_p$ is the cyclic group of order p, then $Spec(M) = \{\mathbb{Q} \bigoplus 0, 0 \bigoplus \mathbb{Z}_p\}$ by [15, Example 2.6] and $Spec_L(M) = \{\mathbb{Q} \bigoplus 0\}$ by [6, Example 3.1]. In [6, Lemma 2.1], it is shown that if $Spec(M) = \emptyset$, then $Spec_L(M) = \emptyset$. However for the $\mathbb{Z}$-module $\mathbb{Q}$, we have $Spec(\mathbb{Q}) = \{0\}$ and $Spec_L(\mathbb{Q}) = \emptyset$.

There are different module theoretic generalizations of the well-known Zariski topology on the spectrum of a ring R having $\{V(I) \mid I$ is an ideal of $R\}$ as the collection of closed sets, where $V(I) = \{p \in Spec(R) \mid I \subseteq p\}$ (see for example [1, 2, 3, 12]).

We set $\eta^*(M) = \{\nu^*(N) \mid N$ is a submodule of $M\}$, where $\nu^*(N) = \{Q \in Spec_L(M) \mid N \subseteq rad(Q)\}$. This collection of varieties of submodules is not closed under finite unions. An R-module M is called top-like if $\eta^*(M)$ satisfies the axioms of a Zariski-like topology $\mathcal{T}^*$ for closed sets [6].

A module M over a ring R is called a multiplication module if each submodule of M is of the form IM, where I is an ideal of R. In this case, we can take $I = (N : M)$ [4]. Multiplication modules are top-like [7, Theorem 2.2]. Also if R is an Artinian ring, then Bezout R-modules and distributive R-modules are top-like [6, Proposition 4.1].

From an algebraic point view, some Zariki spaces have been studied related to these topologies [14, 16]. It is easily seen that $\eta^*(M)$ with the binary operation $\nu^*(N) + \nu^*(N') = \nu^*(N + N') = \nu^*(N) \cap \nu^*(N')$ is a semigroup with zero. Moreover $\eta^*(R)$ with the similar addition and multiplication as $\nu^*(I) * \nu^*(J) = \nu^*(IJ) = \nu^*(I \cap J)$ is a semiring.

An R-module M is called RSP if the radical of each element of $Spec_L(M)$ is prime. In Section 2, we introduce a Zariski-like space over RSP modules. In fact we show that for an RSP module M, the semigroup

$(\eta^*(M),+)$ with the scalar multiplication $\nu^*(I)*\nu^*(N)=\nu^*(IN)$ is an $\eta^*(R)$-semimodule (Theorem 2.4). In this case $(Spec_L(M),\eta^*(M))$ also means an $\eta^*(R)$-space, called the Zariski-like space. In this section we provide some background material and results regarding subtractive subsemimodules of $\eta^*(M)$.
The notion of Z*-radical of a submodule N of M, defined in Section 3 and denoted by $\sqrt[Z^*]{N}$, is the intersection of all elements of $\nu^*(N)$, unless $\nu^*(N)=\emptyset$, in which case $\sqrt[Z^*]{N}=M$. It is proved that for submodules N and N' of a multiplication module M, $\sqrt[Z^*]{N\cap N'}=\sqrt[Z^*]{N}\cap\sqrt[Z^*]{N'}$. Moreover, if $|\,Spec_L(M)\,|<\infty$, then $\sqrt[Z^*]{\sqrt[Z^*]{N}}=\sqrt[Z^*]{N}$ (Lemma 3.9).
Since these identities are frequently needed to examine the new notion of a subtractive basis for a Zariski-like space, in a main part of Section 3, we restrict ourselves on the class of multiplication modules as a subclass of RSP modules. Such bases provide a means of generating Zariski-like Spaces, which exploits both the algebraic and topological-type properties of these spaces.
It is shown that if M is a Z*-radical Noetherian multiplication R-module with $|\,Spec_L(M)\,|<\infty$ such that for every submodule N of M and $Q\in Spec_L(M)$, $N\subseteq\sqrt[Z^*]{N}$ and $rad(Q)\cap N=rad(Q\cap N)$, then $\eta^*(M)$ has a subtractive basis (Corollary 3.14).

2. The Zariski-like Space of RSP modules and $\eta^*(R)$-homomorphisms

The saturation of a submodule N of an R-module M with respect to a prime ideal p of R is the contraction of N_p in M and designated by $S_p(N)$. It is known that $S_p(N)=\{m\in M\mid rm\in N$ for some $r\in R\backslash p\}$ [11]. Hereafter we will use $\mathcal{X}$ to represent $Spec_L(M)$. Hence for any $Q\in\mathcal{X}$, the ideal $\sqrt{(Q:M)}=(rad(Q):M)$ is prime and so is $rad(Q)\neq M$.

Lemma 2.1. *Let M be an R-module and Q be a primary-like submodule of M. Then $S_p(Q)\subseteq rad(Q)$ for every $p\in V(Q:M)$. In particular, if $S_p(Q)$ is a prime submodule of M for some $p\in V(Q:M)$, then $S_p(Q)=rad(Q)$.*

Proof. Straightforward. □

Lemma 2.2. *Let M be an R-module and Q be a submodule of M. Consider the following statements.*

(1) *$rad(Q)$ is a p-prime submodule of M.*
(2) *$rad(Q)$ is a p-primary-like submodule of M.*
(3) *Q is a p-primary-like submodule of M*

Then $(1) \Leftrightarrow (2)$. *Furthermore, if* $Q \in \mathcal{X}$ *and* $(Q : M)$ *is a radical ideal of* R, *then* $(1)-(3)$ *are equivalent.*

Proof. $(1) \Leftrightarrow (2)$ is clear since $rad(rad(Q)) = rad(Q)$.
$(1) \Rightarrow (3)$ Clear. $(3) \Rightarrow (1)$ Since $S_p(Q) \subseteq rad(Q)$, then $S_p(Q) \neq M$. Thus by [11, Proposition 2.4] and Lemma 2.1 $rad(Q)$ is prime. The verification of the other implications is straightforward. □

Recall that an R-module M is called RSP if the radical of each element of $\mathcal{X}$ is prime. In the following we list some conditions under which an R-module M is RSP.

Theorem 2.3. *Let* M *be an* R*-module. Then* M *is RSP in each of the following cases.*

(1) R *is a zero-dimensional ring.*
(2) *For each* $Q \in \mathcal{X}$ *and* $p = \sqrt{(Q : M)}$, $(S_p(Q) : M)$ *is a radical ideal.*
(3) *For each* $Q \in \mathcal{X}$ *and* $p = (Q : M)$, $S_p(Q) \neq M$.
(4) M *is a multiplication module.*
(5) R *is a Noetherian domain and* $Q \in \mathcal{X}$ *is contained in only finitely many prime submodules of* M.

Proof. (1) Suppose $Q \in \mathcal{X}$. Since $\sqrt{(Q : M)} = (rad(Q) : M)$ is prime and hence maximal, $\sqrt{(Q : M)} = (P : M)$ for all prime submodules P containing Q. Now if $rm \in rad(Q)$ and $m \notin rad(Q)$, there is a prime submodule P containing Q such that $rm \in P$ and $m \notin P$ and so $r \in (P : M) = \sqrt{(Q : M)} = (rad(Q) : M)$. Thus $rad(Q)$ is prime.
(2) $p = \sqrt{(Q : M)} \subseteq \sqrt{(S_p(Q) : M)} \subseteq (rad(Q) : M) = \sqrt{(Q : M)} = p$. It follows that $\sqrt{(S_p(Q) : M)} = p$. Now since $(S_p(Q) : M)$ is a radical ideal, we have $(S_p(Q) : M) = p$. It follows from [11, Theorem 2.3] and Lemma 2.1, $rad(Q)$ is a prime submodule of M.
(3) Suppose $S_p(Q) \neq M$. By [11, Proposition 2.4], $S_p(Q)$ is a prime submodule of M. It follows from Lemma 2.1 $rad(Q)$ is a prime submodule of M.
(4) Since $(rad(Q) : M)$ is a prime ideal of R for every $Q \in \mathcal{X}$, $rad(Q)$ is a prime submodule of M by [4, Corollary 2.11].
(5) By Lemma 2.2 we may assume that $(Q : M) \neq 0$. If P is a prime submodule containing Q, then $0 \subset \sqrt{(Q : M)} \subseteq (P : M)$ is a chain of prime ideals of R. If $\sqrt{(Q : M)} \subseteq (P : M)$ is a proper containment, then by [9, P.144] there are infinitely many prime ideals p with $(Q : M) \subset p \subset (P : M)$ and so we have infinitely prime submodules P containing Q, a contradiction. Hence we have $\sqrt{(Q : M)} = (P : M)$, for all prime submodules P containing N. Now if $rm \in rad(Q)$ and

$m \notin rad(Q)$, there is a prime submodule P containing Q such that $rm \in P$ and $m \notin P$ and so that $r \in (P : M) = \sqrt{(Q : M)} = (rad(Q) : M)$. □

For the remainder of this section, we assume that M and M' are RSP R-modules.

Let (X, Ω) be a topological space, and let Γ be a collection of subsets of a set Y such that $Y \in \Gamma$ and Γ is closed with respect to finite intersections. Further suppose that there exists a mapping $* : \Omega * \Gamma \to \Gamma$ such that $(\Gamma, \cap)$ is an Ω-semimodule. That is to say, for all $\tau, \tau' \in \Omega$ and for all $\gamma, \gamma' \in \Gamma$, the following properties hold.

(1) $\tau * (\gamma \cap \gamma') = (\tau * \gamma) \cap (\tau * \gamma')$;
(2) $(\tau \cap \tau') * \gamma = (\tau * \gamma) \cap (\tau' * \gamma)$;
(3) $(\tau \cup \tau') * \gamma = \tau * (\tau' * \gamma)$;
(4) $\emptyset * \gamma = \gamma$;
(5) $\tau * Y = Y = X * \gamma$.

Then (Y, Γ) is called an Ω-space [14].

Theorem 2.4. *Let M be an R-module and let the $\eta^*(R)$-action on $\eta^*(M)$ be given by $\nu^*(I) * \nu^*(N) = \nu^*(IN)$, where I is an ideal of R and N is a submodule of M. Then $(\mathcal{X}, \eta^*(M))$ is an $\eta^*(R)$-space.*

Proof. It is easy to see that $(\eta^*(M), \cap)$ is a commutative monoid with identity $\mathcal{X} = \nu^*(0)$. Now assume that $\nu^*(I) = \nu^*(J)$ and $\nu^*(N) = \nu^*(N')$, where I, J are ideals of R and N, N' are submodules of M. We must show that $\nu^*(IN) = \nu^*(JN')$. Suppose $Q \in \nu^*(IN)$. Therefore $IN \subseteq rad(Q)$. Since $rad(Q)$ is prime, $N \subseteq rad(Q)$ or $I \subseteq (rad(Q) : M)$ by [15, Lemma 1.1]. Hence $JN' \subseteq rad(Q)$ or $JN' \subseteq (rad(Q) : M)N' \subseteq rad(Q)$. By symmetry we have $\nu^*(IN) = \nu^*(JN')$. Hence the operation $(*)$ is well-defined. Now we check the condition (3) of the above definition. $\nu^*(I) * (\nu^*(J) * \nu^*(N)) = \nu^*(I) * \nu^*(JN) = \nu^*(I(JN)) = \nu^*(IJ) * \nu^*(N) = (\nu^*(I) \cup \nu^*(J)) * \nu^*(N)$. The other properties follow similarly. □

The $\eta^*(R)$-space $(\mathcal{X}, \eta^*(M))$ is called a Zariski-like space. As mentioned in the introduction, from another point view, $(\eta^*(M), +)$ may be considered as an semimodule over a semiring $\eta^*(R)$ with addition and multiplication defined as:

$$\nu^*(N) + \nu^*(N') = \nu^*(N + N') = \nu^*(N) \cap \nu^*(N'),$$
$$\nu^*(I) * \nu^*(N) = \nu^*(IN) = \nu^*(IM) \cup \nu^*(N).$$

Let $\mathcal{R}$ be a semiring. By a $\mathcal{R}$-semimodule homomorphism, we mean a map $f : \mathcal{M} \to \mathcal{M}'$ of $\mathcal{R}$-semimodules $\mathcal{M}$ and $\mathcal{M}'$ which is $\mathcal{R}$-linear. Also subsemimodules and subspaces are defined naturally (For further reading about semirings, semimodules, and Zariski spaces, see for example [8, 14, 13]).

Lemma 2.5. *Let M, M' be R-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-homomorphism. If N, N' are submodules of M such that $\nu^*(N) \subseteq \nu^*(N')$, then $f(\nu^*(N)) \subseteq f(\nu^*(N'))$.*

Proof. Since $\nu^*(N) \subseteq \nu^*(N')$, we have $\nu^*(N) = \nu^*(N) \cap \nu^*(N') = \nu^*(N) + \nu^*(N')$. Hence $f(\nu^*(N)) = f(\nu^*(N) + \nu^*(N')) = f(\nu^*(N)) + f(\nu^*(N')) = f(\nu^*(N)) \cap f(\nu^*(N')) \subseteq f(\nu^*(N'))$. □

Lemma 2.6. *Let M, M' be R-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-surjective homomorphism. Then $f(\nu^*(M)) = \nu^*(M')$.*

Proof. Since f is surjective, there exists a submodule N of M such that $f(\nu^*(N)) = \nu^*(M')$. Hence $f(\nu^*(M)) = f(\nu^*(M + N)) = f(\nu^*(M) + \nu^*(N)) = f(\nu^*(M)) + f(\nu^*(N)) = f(\nu^*(M)) + \nu^*(M') = \nu^*(M')$. □

Lemma 2.7. *Let M, M' be R-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-injective homomorphism. If N, N' are submodule of M such that $f(\nu^*(N)) \subseteq f(\nu^*(N'))$, then $\nu^*(N)) \subseteq \nu^*(N')$.*

Proof. Since $f(\nu^*(N)) \subseteq f(\nu^*(N'))$, we have $f(\nu^*(N)) = f(\nu^*(N)) \cap f(\nu^*(N')) = f(\nu^*(N) \cap \nu^*(N'))$. Hence $\nu^*(N) = \nu^*(N) \cap \nu^*(N')$ because f is injective. Thus $\nu^*(N)) \subseteq \nu^*(N')$. □

A subsemimodule Δ is a subtractive subsemimodule of $\eta^*(M)$ if for submodules N, N' of M the conditions $\nu^*(N) \in \Delta$ and $\nu^*(N) + \nu^*(N') \in \Delta$ implies that $\nu^*(N') \in \Delta$. In this paper, we use Bourne factor semimodule of a semimodule Γ over a semiring Ω (that is, the elements of $\frac{\Gamma}{\Delta}$ are the equivalency classes $[\gamma]$ $(\gamma \in \Gamma)$ of the congruence $\gamma \sim \gamma' \Leftrightarrow \exists \delta, \delta' \in \Delta$: $\gamma + \delta = \gamma' + \delta'$. Also addition and scalar multiplication is defined naturally; $[\gamma] + [\gamma'] = [\gamma + \gamma']$ and $\omega * [\gamma] = [\omega * \gamma]$).

Lemma 2.8. *Let $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-homomorphism. Then $Kerf$ is a subtractive subsemimodule of $\eta^*(M)$. Conversely, if Δ is a subtractive subsemimodule of $\eta^*(M)$, then $\pi : \eta^*(M) \to \frac{\eta^*(M)}{\Delta}$ which is defined by $\pi(\nu^*(N)) = [\nu^*(N)]$ is an $\eta^*(R)$-surjective homomorphism with $Ker\pi = [0]$.*

Proof. It is clear that $Kerf$ is a subtractive subsemimodule of $\eta^*(M)$ by [8]. Conversely, it is easy to see that π is a surjective homomorphism. Now we have $\pi(\nu^*(N) + \nu^*(N')) = [\nu^*(N) + \nu^*(N')] = [\nu^*(N)] + [\nu^*(N'] = \pi(\nu^*(N)) + \pi(\nu^*(N'))$ and $\pi(\nu^*(I) * \nu^*(N)) = [\nu^*(I) * \nu^*(N)] = \nu^*(I) * [\nu^*(N)] = \nu^*(I) * \pi(\nu^*(N))$. Thus π is an $\eta^*(R)$-homomorphism. Also $Ker\pi = \{\nu^*(N) \in \eta^*(M) \mid [\nu^*(N)] = [0]\} = \{\nu^*(N) \in \eta^*(M) \mid \nu^*(N) \in [0]\} = [0]$. □

Lemma 2.9. *Let Δ be a subspace of $\eta^*(M)$. Then the following are equivalent.*

(1) *Δ is a subtractive subsemimodule of $\eta^*(M)$;*
(2) *For submodules N, N' of M the conditions $\nu^*(N) \in \Delta$ and $\nu^*(N) \subseteq \nu^*(N')$ implies that $\nu^*(N') \in \Delta$.*

Proof. (1)$\Rightarrow$(2) By Lemma 2.8, Δ is the kernel of the $\eta^*(R)$-surjective homomorphism $\pi : \eta^*(M) \to \frac{\eta^*(M)}{\Delta}$. Suppose N, N' are submodules of M. Assume $\nu^*(N) \in \Delta$ and $\nu^*(N) \subseteq \nu^*(N')$. Hence $\pi(\nu^*(N)) + \pi(\nu^*(N')) = \nu^*(0)$. Thus $\pi(\nu^*(N')) = \nu^*(0)$ and so $\nu^*(N') \in \Delta$.
(2)$\Rightarrow$(1) Assume N, N' are submodules of M. Suppose $\nu^*(N) \in \Delta$ and $\nu^*(N) \cap \nu^*(N') \in \Delta$. Since $\nu^*(N) \cap \nu^*(N') \subseteq \nu^*(N')$, then $\nu^*(N') \in \Delta$. Thus Δ is a subtractive subsemimodule of $\eta^*(M)$. □

Proposition 2.10. *Every proper subtractive subspace of $\eta^*(M)$ is contained in a maximal subtractive subspace.*

Proof. Suppose Δ is a proper subtractive subspace of $\eta^*(M)$. Put $\mathcal{A} = \{\Phi \mid \Delta \subseteq \Phi\}$. Assume $\mathcal{C} = \{\Phi_i \mid i \in I\}$ is a chain of elements of $\mathcal{A}$. It is easy to see that $\Delta \in \mathcal{A}$ and $\cup_{i \in I}\Phi_i \in \mathcal{A}$. Thus the assertion holds by Zorn's lemma. □

Proposition 2.11. *Let M, M' be R-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-homomorphism. If Δ is a subtractive subspace of $\eta^*(M')$, then the following hold.*

(1) *$f^{-1}(\Delta)$ is a subtractive subspace of $\eta^*(M)$ containing $Kerf$.*
(2) *f induces an $\eta^*(R)$-homomorphism $\phi : \frac{\eta^*(M)}{f^{-1}(\Delta)} \to \frac{\eta^*(M')}{\Delta}$ having kernel $f^{-1}(\Delta)$.*

Proof. (1) Suppose N, N' are submodules of M. Assume $\nu^*(N) \in f^{-1}(\Delta)$ and $\nu^*(N) \cap \nu^*(N') \in f^{-1}(\Delta)$. Hence $f(\nu^*(N)) \in \Delta$ and $f(\nu^*(N)) \cap f(\nu^*(N')) \in \Delta$. Since Δ is a subtractive subspace of $\eta^*(M')$, then $f(\nu^*(N')) \in \Delta$. Thus $\nu^*(N') \in f^{-1}(\Delta)$ and so $f^{-1}(\Delta)$ is a subtractive subspace of $\eta^*(M)$. It is easy to see that $Kerf \subseteq f^{-1}(\Delta)$.
(2) Use [8, Corollary 13.48]. □

It is common that if $\{\Delta_\lambda\}_{\lambda \in \Lambda}$ be a family of subtractive subspaces of $\eta^*(M)$, then $\cap_{\lambda \in \Lambda}\Delta_\lambda$ is subtractive. Let Υ be a subset of $\eta^*(M)$. The subtractive closure of Υ, denoted $\gamma(\Upsilon)$, is the smallest subtractive subspace of $\eta^*(M)$ which contains Υ. It is clear that if $\Upsilon \subseteq \Upsilon'$ be subsemimodules of $\eta^*(M)$, then $\gamma(\Upsilon) \subseteq \gamma(\Upsilon')$.

Lemma 2.12. *Let N be a submodule of an R-module M and Δ be a subsemimodule of $\eta^*(M)$. Then the following hold.*

(1) $\gamma(\Delta) = \{\nu^*(N') \mid \nu^*(N'') \subseteq \nu^*(N')$ *for some* $\nu^*(N'') \in \Delta\}$.

(2) $\gamma(\eta^*(R) * \nu^*(N)) = \{\nu^*(N') \mid \nu^*(N) \subseteq \nu^*(N')\}$.

Proof. (1) Suppose $A = \{\nu^*(N') \mid \nu^*(N'') \subseteq \nu^*(N')$ for some $\nu^*(N'') \in \Delta\}$ and $\nu^*(N') \in A$. Therefore there exists $\nu^*(N'') \in \Delta$ such that $\nu^*(N'') \subseteq \nu^*(N')$. Since $\gamma(\Delta)$ is the smallest subtractive subspace of $\eta^*(M)$ which contains Δ, then $\nu^*(N') \cap \nu^*(N'') = \nu^*(N'') \in \Delta \subseteq \gamma(\Delta)$. Thus $\nu^*(N') \in \gamma(\Delta)$ and so $A \subseteq \gamma(\Delta)$. For the reverse inclusion we show that A is a subtractive subspace of $\eta^*(M)$ which contains Δ. It is clear that $\Delta \subseteq A$. Now assume $\nu^*(N_1)$, $\nu^*(N_2) \in A$ and $\nu^*(N_1'')$, $\nu^*(N_2'') \in \Delta$ such that $\nu^*(N_1'') \subseteq \nu^*(N_1)$ and $\nu^*(N_2'') \subseteq \nu^*(N_2)$. Hence $\nu^*(N_1'') \cap \nu^*(N_2'') \subseteq \nu^*(N_1) \cap \nu^*(N_2)$. Thus $\nu^*(N_1) \cap \nu^*(N_2) \in A$. Suppose $\nu^*(I) \in \eta^*(R)$. So $\nu^*(I) * \nu^*(N_1'') \subseteq \nu^*(I) * \nu^*(N_1)$. Hence $\nu^*(I) * \nu^*(N_1) \in A$. Thus A is a subspace of $\eta^*(M)$. Now suppose $\nu^*(N) \in \eta^*(M)$ and $\nu^*(N')$, $\nu^*(N) \cap \nu^*(N') \in A$. Then there exists $\nu^*(N'') \in \Delta$ such that $\nu^*(N'') \subseteq \nu^*(N) \cap \nu^*(N')$. Thus $\nu^*(N'') \subseteq \nu^*(N)$ and so $\nu^*(N) \in A$. Therefore A is a subtractive subspace of $\eta^*(M)$ containing Δ. Thus $A = \gamma(\Delta)$.
(2) We have $\eta^*(R) * \nu^*(N) = \{\nu^*(IN) \mid I$ is an ideal of $R\}$. Therefore $\eta^*(R) * \nu^*(N)$ is a subspace of $\eta^*(M)$. Hence $\gamma(\eta^*(R)\nu^*(N)) = \{\nu^*(N') \mid \nu^*(N'') \subseteq \nu^*(N')$ for some $\nu^*(N'') \in \eta^*(R) * \nu^*(N)\} = \{\nu^*(N') \mid \nu^*(IN) \subseteq \nu^*(N')$ for some ideal I of $R\} \subseteq \{\nu^*(N') \mid \nu^*(N) \subseteq \nu^*(N')\}$ by (1). By the similar argument we have $\{\nu^*(N') \mid \nu^*(N) \subseteq \nu^*(N')\} \subseteq \gamma(\eta^*(R) * \nu^*(N))$. □

Proposition 2.13. *Let N, N' be submodules of an R-module M and $\nu^*(N') \in \gamma(\nu^*(N))$. Then $\nu^*(N) \subseteq \nu^*(N')$.*

Proof. It is clear by Lemma 2.12. □

Proposition 2.14. *Let N, N' be submodules of an R-module M and $N' \subseteq rad(N)$. Then $\nu^*(N') \in \gamma(\nu^*(N))$.*

Proof. Suppose $N' \subseteq rad(N)$. Since $\nu^*(N) = \nu^*(rad(N))$, then $\nu^*(N) \subseteq \nu^*(N')$. Thus $\nu^*(N') \in \gamma(\nu^*(N))$ by Lemma 2.12. □

Theorem 2.15. *Let radical submodules of an R-module M satisfy ACC. Then every subtractive subspace of $\eta^*(M)$ is of the form $\gamma(\nu^*(N))$ for some submodule N of M.*

Proof. Suppose Δ is a subtractive subspace of $\eta^*(M)$. If $\nu^*(M) \in \Delta$, then $\Delta = \eta^*(M) = \gamma(\nu^*(M))$. So assume that $\nu^*(M) \notin \Delta$. Let A be the collection of all radical submodules N of M such that $\nu^*(N) \in \Delta$, and note that $A \neq \emptyset$ since $\nu^*(N) = \nu^*(rad(N))$ for every submodule N of M. Now choose N' to be a maximal element of A. To see that $\Delta = \gamma(\nu^*(N'))$, let $\nu^*(N'') \in \Delta$, where N'' is a submodule of M. If $S = rad(N' + N'')$, then $\nu^*(S) = \nu^*(N' + N'') = \nu^*(N') \cap \nu^*(N'') \in \Delta$.

Since S is a radical submodule of M, then $N'' \subseteq S = N' = rad(N')$. Hence $\nu^*(N'') \in \gamma(\nu^*(N'))$ by Lemma 2.12. Thus $\Delta \subseteq \gamma(\nu^*(N'))$. Since $\nu^*(N') \in \Delta$, then $\gamma(\nu^*(N')) \subseteq \Delta$. Thus $\Delta = \gamma(\nu^*(N'))$. □

Lemma 2.16. *Let M be an R-module and $\{N_i\}_{i\in I}$ be submodules of M. Then $\nu^*(\sum_{i\in I} N_i) = \sum_{i\in I} \nu^*(N_i)$.*

Proof. For $Q \in \mathcal{X}$ we have $Q \in \sum_{i\in I} \nu^*(N_i)$ if and only if $Q \in \nu^*(N_i)$ for every $i \in I$ iff $N_i \subseteq rad(Q)$ for each $i \in I$ iff $\sum_{i\in I} N_i \subseteq rad(Q)$ iff $Q \in \nu^*(\sum_{i\in I} N_i)$. □

Theorem 2.17. *Let M be an R-module and $\{N_i\}_{i=1}^n$ be submodules of M. Then $\gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i)) = \gamma(\nu^*(\sum_{i=1}^n N_i))$.*

Proof. Assume $\nu^*(N') \in \gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i))$. Hence by Lemma 2.12, $\nu^*(\sum_{i=1}^n J_i N_i) \subseteq \nu^*(N')$ for some ideal J_i of R. Since $\nu^*(\sum_{i=1}^n N_i) \subseteq \nu^*(\sum_{i=1}^n J_i N_i)$, then $\nu^*(\sum_{i=1}^n N_i) \subseteq \nu^*(N')$. So $\nu^*(N') \in \gamma(\nu^*(\sum_{i=1}^n N_i))$. Thus $\gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i)) \subseteq \gamma(\nu^*(\sum_{i=1}^n N_i))$. Now, we let $\nu^*(N') \in \gamma(\nu^*(\sum_{i=1}^n N_i))$. Then $\nu^*(\sum_{i=1}^n N_i) \subseteq \nu^*(N')$. By Lemma 2.16 we have $\nu^*(\sum_{i=1}^n N_i) = \sum_{i=1}^n \nu^*(N_i) \in \sum_{i=1}^n \eta^*(R) * \nu^*(N_i)$. Hence $\nu^*(N') \in \gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i))$. Thus $\gamma(\nu^*(\sum_{i=1}^n N_i)) \subseteq \gamma(i = 1^n \eta^*(R) * \nu^*(N_i))$. □

3. Subtractive Closure and Subtractive Bases

We define the Z*-radical of a submodule N of M, denoted by $\sqrt[Z^*]{N}$, to be the intersection of all members of $\nu^*(N)$. A submodule N of M is a Z*-radical submodule if $\sqrt[Z^*]{N} = N$. An R-module M is called Z*-radical if $\sqrt[Z^*]{0_M} = 0$. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$. The closure of $\mathcal{Y}$ in $\mathcal{X}$, denoted by $\overline{\mathcal{Y}}$, is the intersection of all closed subset of $\mathcal{X}$ containing $\mathcal{Y}$. Also $\xi(\mathcal{Y})$ is the intersection of all elements in $\mathcal{Y}$ (note that if $\mathcal{Y} = \emptyset$, then $\xi(\mathcal{Y}) = M$). It is easy to verify that, if $\mathcal{Y}_1, \mathcal{Y}_2 \subseteq \mathcal{X}$, then $\xi(\mathcal{Y}_1 \cup \mathcal{Y}_2) = \xi(\mathcal{Y}_1) \cap \xi(\mathcal{Y}_2)$.

Lemma 3.1. *Let M be an R-module and N, N' be submodules of M. If $\nu^*(N) \subseteq \nu^*(N')$, then $\sqrt[Z^*]{N'} \subseteq \sqrt[Z^*]{N}$. The converse is true if $N' \subseteq \sqrt[Z^*]{N'}$.*

Proof. Suppose $\nu^*(N) \subseteq \nu^*(N')$. Hence $\xi(\nu^*(N')) \subseteq \xi(\nu^*(N))$ and so $\sqrt[Z^*]{N'} \subseteq \sqrt[Z^*]{N}$. Conversely, Suppose $Q \in \nu^*(N)$. Hence $\sqrt[Z^*]{N'} \subseteq \sqrt[Z^*]{N} \subseteq Q$. Thus $N' \subseteq rad(Q)$ and so $\nu^*(N) \subseteq \nu^*(N')$. □

Lemma 3.2. *Let M be a finitely generated R-module. Then $\sqrt[Z^*]{N} \neq M$ if and only if $\nu^*(N) \neq \emptyset$ if and only if $N \neq M$.*

Proof. Suppose $\sqrt[z^*]{N} \neq M$. Hence $N \neq M$. Now assume $N \neq M$. Then $(N : M) \neq R$ and so $(N : M) \subseteq p$ for some prime ideal p of R. Since M is finitely generated, M is primeful by [10, Theorem 2.2]. So there exists $Q \in Spec(M) \subseteq \mathcal{X}$ such that $N \subseteq rad(Q)$. Hence $Q \in \nu^*(N)$. Thus $\nu^*(N) \neq \emptyset$. If $\nu^*(N) \neq \emptyset$ and $Q \in \nu^*(N)$. Hence $N \subseteq rad(Q)$. Thus $\sqrt[z^*]{N} \subseteq Q \neq M$. □

Lemma 3.3. *Let M be an R-module. If $Q \in \mathcal{X}$ and N is a submodule of M such that $rad(Q) \cap N = rad(Q \cap N)$, then $N \subseteq Q$ or $Q \cap N$ is a primary-like submodule of N.*

Proof. Suppose $N \nsubseteq Q$, $n \in N$ and $rn \in Q \cap N$ such that $r \notin (Q \cap N : N)$. Then $rn \in Q$ and $r \notin (Q : M)$. Since Q is primary-like, we have $n \in rad(Q)$. Thus $n \in rad(Q \cap N)$. □

Lemma 3.4. *Let M be a Z^*-radical R-module such that every submodule N of M is finitely generated and $N \subseteq \sqrt[z^*]{N}$. If for every $Q \in \mathcal{X}$, $rad(Q) \cap N = rad(Q \cap N)$, then every direct summand of M is a Z^*-radical submodule of M.*

Proof. Suppose that N is a direct summand of M and $N \subset \sqrt[z^*]{N}$. Hence $M = N \bigoplus N'$ for some submodule N' of M. Therefore there exists $m = (n, n') \in \sqrt[z^*]{N} \backslash N$. So $0 \neq (0, n') \in \sqrt[z^*]{N}$. Since $M/N \cong N'$, there is a one-to-one correspondence between the primary-like submodules of N' which satisfy the primeful property and the primary-like submodules of M/N satisfying the primeful property. Since $(0, n') \in \sqrt[z^*]{N}$, $(0, n')$ belongs to every primary-like submodule of the module N' which satisfies the primeful property. Let $Q \in \mathcal{X}$. Then we show that $(0, n') \in Q$. If $N' \subseteq Q$, then $(0, n') \in Q$ because $(0, n') \in N'$. Suppose $N' \nsubseteq Q$. Hence by Lemma 3.3 and [10, Theorem 2.2], $Q \cap N' \in Spec_L(N')$. Thus $(0, n') \in Q \cap N' \subseteq Q$ and so $n' \in \sqrt[z^*]{0_M} = 0$, a contradiction. □

Let M be an R-module and $\{N_i\}_{i=1}^n$ be submodules of M. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ we recall the following definitions.

(1) Δ is a subtractive generating set of $\eta^*(M)$ if $\eta^*(M) = \gamma(\sum_{i \in I} \eta^*(R) * \nu^*(N_i))$.
(2) Δ is a subtractive linearly independent set of $\eta^*(M)$ if $\nu^*(0) \notin \Delta$ and $\gamma(\nu^*(N_i)) \cap \gamma(\sum_{j \neq i} \eta^*(R) * \nu^*(N_j)) = \{\nu^*(0)\}$ for each i, $(1 \leq i \leq n)$.
(3) Δ is a subtractive linearly independent generating set of $\eta^*(M)$ if Δ satisfies both conditions (1) and (2).

Lemma 3.5. *Let M be an R-module and $\{N_i\}_{i=1}^n$ be submodules of M. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$, then the following hold.*

(1) *Δ is a subtractive generating set of $\eta^*(M)$ iff* $\sqrt[z^*]{\sum_{i=1}^n N_i} = M$.
(2) *If M is a finitely generated, then Δ is a subtractive generating set of $\eta^*(M)$ iff $\sum_{i=1}^n N_i = M$.*

Proof. (1) By Theorem 2.17, $\gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i)) = \gamma(\nu^*(\sum_{i=1}^n N_i))$. So Δ is a subtractive generating set of $\eta^*(M)$ if and only if $\eta^*(M) = \gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i)) = \gamma(\nu^*(\sum_{i=1}^n N_i))$ iff $\nu^*(\sum_{i=1}^n N_i) \subseteq \nu^*(M) = \emptyset$ iff $\nu^*(\sum_{i=1}^n N_i) = \emptyset = \nu^*(M)$ iff $\sqrt[z^*]{\sum_{i=1}^n N_i} = M$.
(2) By (1) Δ is a subtractive generating set of $\eta^*(M)$ iff $\sqrt[z^*]{\sum_{i=1}^n N_i} = M$. Since M is finitely generated, by Lemma 3.2 Δ is a subtractive generating set of $\eta^*(M)$ iff $\sum_{i=1}^n N_i = M$. □

Theorem 3.6. *Let M, M' be R-modules and $f : \eta^*(M) \to \eta^*(M')$ be an $\eta^*(R)$-isomorphism. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ is a subtractive linearly independent set of $\eta^*(M)$, then $\{f(\nu^*(N_1)), \cdots, f(\nu^*(N_n))\}$ is a subtractive linearly independent set of $\eta^*(M')$.*

Proof. Since f is an isomorphism, $f(\nu^*(0)) = \nu^*(0)$. Hence $\nu^*(0) \notin \{f(\nu^*(N_1)), \cdots, f(\nu^*(N_n))\}$ because $\nu^*(0) \notin \Delta$. Now, suppose that there exists $1 \leq i \leq n$ such that

$$\nu^*(N') \in \gamma(f(\nu^*(N_i))) \cap \gamma(\textstyle\sum_{j \neq i} \eta^*(R)\nu^*(N_j)).$$

Since f is surjective, $\nu^*(N') = f(\nu^*(N))$ for some submodule N of M. Hence $f(\nu^*(N_i)) \subseteq f(\nu^*(N))$ and $f(\nu^*(\sum_{j \neq i} I_j N_j)) \subseteq f(\nu^*(N))$. By Lemma 2.7, $\nu^*(N_i) \subseteq \nu^*(N)$ and $\sum_{j \neq i} \nu^*(I_j N_j) \subseteq \nu^*(N)$. Thus $\nu^*(N) \in \gamma(\nu^*(N_i)) \cap \gamma(\sum_{j \neq i} \eta^*(R) * \nu^*(N_j))$. This implies that $\nu^*(N) = \nu^*(0)$. Therefore $f(\nu^*(N)) = \nu^*(0)$ and so $\nu^*(N') = \nu^*(0)$. Thus $\{f(\nu^*(N_1)), \cdots, f(\nu^*(N_n))\}$ is a subtractive linearly independent set of $\eta^*(M')$. □

For the remainder of this section, we assume that all modules are multiplication. So that $\nu^*(N) = \{Q \in \mathcal{X} \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$ for every submodule N of an R-module M.

Lemma 3.7. *Let M be an R-module and $\mathcal{Y} \subseteq \mathcal{X}$. If $\mid \mathcal{X} \mid < \infty$, then $\nu^*(\xi(\mathcal{Y})) = \overline{\mathcal{Y}}$. In particular, $\mathcal{Y}$ is closed if and only if $\nu^*(\xi(\mathcal{Y})) = \mathcal{Y}$.*

Proof. Suppose $Q \in \mathcal{Y}$. Hence $\xi(\mathcal{Y}) \subseteq Q$. Therefore $\sqrt{(Q : M)} \supseteq \sqrt{(\xi(\mathcal{Y}) : M)}$. Since M is multiplication, $Q \in \nu^*(\xi(\mathcal{Y}))$ and so $\mathcal{Y} \subseteq \nu^*(\xi(\mathcal{Y}))$. Next, let $\nu^*(N)$ be any closed subset of $\mathcal{X}$ containing $\mathcal{Y}$. Then $\sqrt{(Q : M)} \supseteq \sqrt{(N : M)}$ for every $Q \in \mathcal{Y}$ so that $\sqrt{(\xi(\mathcal{Y}) : M)} \supseteq \sqrt{(N : M)}$ since $\mid \mathcal{X} \mid < \infty$. Hence, for every $Q' \in \nu^*(\xi(\mathcal{Y}))$ we have $\sqrt{(Q' : M)} \supseteq \sqrt{(\xi(\mathcal{Y}) : M)} \supseteq \sqrt{(N : M)}$. Then $\nu^*(\xi(\mathcal{Y})) \subseteq \nu^*(N)$.

Thus $\nu^*(\xi(\mathcal{Y}))$ is the smallest closed subset of $\mathcal{X}$ containing $\mathcal{Y}$, hence $\nu^*(\xi(\mathcal{Y})) = \overline{\mathcal{Y}}$. □

Lemma 3.8. *Let M be an R-module and N be a submodule of M. If $|\ \mathcal{X}\ | < \infty$, then $\nu^*(\xi(\nu^*(N))) = \nu^*(\sqrt[z^*]{N}) = \nu^*(N)$.*

Proof. It is clear by Lemma 3.7. □

Lemma 3.9. *Let M be an R-module and N, N' be submodules of M. Then the following hold.*

(1) $\nu^*(N) \cup \nu^*(N') = \nu^*(N \cap N')$.
(2) *If* $|\ \mathcal{X}\ | < \infty$, *then* $\sqrt[z^*]{\sqrt[z^*]{N}} = \sqrt[z^*]{N}$.
(3) $\sqrt[z^*]{N \cap N'} = \sqrt[z^*]{N} \cap \sqrt[z^*]{N'}$.

Proof. (1) Since M is multiplication, we have $\nu^*(N) = \{Q \in \mathcal{X} \mid \sqrt{(N : M)} \subseteq \sqrt{(Q : M)}\}$ for a submodule N of M. Hence the assertion follows from the fact that $(Q : M)$ is a primary ideal for $Q \in \mathcal{X}$.
(2) $\nu^*(\sqrt[z^*]{N}) = \nu^*(N)$, by Lemma 3.8. Therefore $\xi(\nu^*(\sqrt[z^*]{N})) = \xi(\nu^*(N))$. Thus $\sqrt[z^*]{\sqrt[z^*]{N}} = \sqrt[z^*]{N}$.
(3) $\sqrt[z^*]{N \cap N'} = \xi(\nu^*(N \cap N')) = \xi(\nu^*(N) \cup \nu^*(N')) = \xi(\nu^*(N)) \cap \xi(\nu^*(N')) = \sqrt[z^*]{N} \cap \sqrt[z^*]{N'}$, by (1). □

Lemma 3.10. *Let M be an R-module such that $|\ \mathcal{X}\ | < \infty$ and for every submodule K of M, $K \subseteq \sqrt[z^*]{K}$. If N, N' are submodules of M, then $\gamma(\nu^*(N)) \cap \gamma(\nu^*(N')) = \gamma(\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}))$.*

Proof. Suppose $\nu^*(N'') \in \gamma(\nu^*(N)) \cap \gamma(\nu^*(N'))$. So $\nu^*(N'') \in \gamma(\nu^*(N))$ and $\nu^*(N'') \in \gamma(\nu^*(N'))$. Hence $\nu^*(N) \subseteq \nu^*(N'')$ and $\nu^*(N') \subseteq \nu^*(N'')$. By Lemma 3.1, $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N}$ and $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N'}$. Therefore $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N} \cap \sqrt[z^*]{N'}$. So $\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}) \subseteq \nu^*(\sqrt[z^*]{N''})$. Thus $\nu^*(N'') \in \gamma(\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}))$. For the reverse inclusion, let $\nu^*(N'') \in \gamma(\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}))$. Then $\nu^*(\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}) \subseteq \nu^*(N'')$. Hence by Lemma 3.1 and Lemma 3.9 $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{\sqrt[z^*]{N} \cap \sqrt[z^*]{N'}} = \sqrt[z^*]{\sqrt[z^*]{N}} \cap \sqrt[z^*]{\sqrt[z^*]{N'}} = \sqrt[z^*]{N} \cap \sqrt[z^*]{N'}$. Thus $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N}$ and $\sqrt[z^*]{N''} \subseteq \sqrt[z^*]{N'}$. By Lemma 3.1, $\nu^*(N) \subseteq \nu^*(N'')$ and $\nu^*(N') \subseteq \nu^*(N'')$. Thus $\nu^*(N'') \in \gamma(\nu^*(N)) \cap \gamma(\nu^*(N'))$ □

Lemma 3.11. *Let M be an R-module such that $|\ \mathcal{X}\ | < \infty$ and for every submodule N of M, $N \subseteq \sqrt[z^*]{N}$. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$, then Δ is a subtractive linearly independent set of $\eta^*(M)$ if and only if $\nu^*(0) \notin \Delta$ and $\sqrt[z^*]{N_i} \cap \sqrt[z^*]{\sum_{j \neq i} N_j} = \sqrt[z^*]{0}$, for each i, $(1 \leq i \leq n)$.*

Proof. Suppose $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$. Therefore $\gamma(\sum_{i=1}^n \eta^*(R) * \nu^*(N_i)) = \gamma(\nu^*(\sum_{i=1}^n N_i))$ by Theorem 2.17. Thus Δ is a subtractive linearly independent set of $\eta^*(M)$ if and only if $\nu^*(0) \notin \Delta$ and $\gamma(\nu^*(N_i)) \cap \gamma(\nu^*(\sum_{j\neq i} N_j)) = \{\nu^*(0)\}$ for each i, $(1 \leq i \leq n)$. Therefore $\gamma(\nu^*(\sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j})) = \{\nu^*(0)\}$ for each i, $(1 \leq i \leq n)$ by Lemma 3.10, so $\nu^*(\sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j}) = \nu^*(0)$ for each i, $(1 \leq i \leq n)$. Thus by Lemma 3.1 and Lemma 3.9 we have $\sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j} = \sqrt[Z^*]{0}$ for each i, $(1 \leq i \leq n)$. □

Lemma 3.12. *Let M be a Z^*-radical R-module such that $\mid \mathcal{X} \mid < \infty$ and for every submodule N of M, $N \subseteq \sqrt[Z^*]{N}$. If $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ is a subtractive linearly independent set of $\eta^*(M)$, then $\sum_{i=1}^n N_i$ is direct.*

Proof. By Lemma 3.11, $\sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j} = \sqrt[Z^*]{0} = 0$ for each i, $(1 \leq i \leq n)$. By assumption $N_i \cap \sum_{j\neq i} N_j = 0$. Thus $\sum_{i=1}^n N_i$ is direct. □

Theorem 3.13. *Let M be a Noetherian Z^*-radical R-module such that for every submodule N of M and $Q \in \mathcal{X}$, $N \subseteq \sqrt[Z^*]{N}$ and $rad(Q) \cap N = rad(Q \cap N)$. If $\mid \mathcal{X} \mid < \infty$, then $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n) \mid N_i \neq 0\}$ is a subtractive linearly independent set of $\eta^*(M)$ if and only if $M = \bigoplus_{i=1}^n N_i$.*

Proof. Suppose $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ is a subtractive linearly independent set of $\eta^*(M)$. Hence by Lemma 3.5(2), $M = \sum_{i=1}^n N_i$. Thus by Lemma 3.12, $M = \bigoplus_{i=1}^n N_i$. Conversely, assume $M = \bigoplus_{i=1}^n N_i$. Hence by Lemma 3.5(2), Δ is a subtractive generating set of $\eta^*(M)$. Moreover, for every i, $(1 \leq i \leq n)$ we have $\sqrt[Z^*]{0} = 0 = N_i \cap \sum_{j\neq i} N_j = \sqrt[Z^*]{N_i} \cap \sqrt[Z^*]{\sum_{j\neq i} N_j}$ by Lemma 3.4. Since $N_i \neq 0$ for every i, $(1 \leq i \leq n)$ we have $\nu^*(0) \notin \Delta$. Thus Δ is a subtractive linearly independent set of $\eta^*(M)$ by Lemma 3.11. □

Let $\Delta = \{\nu^*(N_1), \cdots, \nu^*(N_n)\}$ be a subtractive linearly independent set of $\eta^*(M)$. Assume that for some j, $(1 \leq j \leq n)$ there exist submodules N'_{j_1} and N'_{j_2} of M such that $\Gamma = \{\nu^*(N_1), \cdots, \nu^*(N_{j-1}), \nu^*(N'_{j_1}), \nu^*(N'_{j_2}), \nu^*(N_{j+1}), \nu^*(N_n)\}$ is likewise a subtractive linearly independent set of $\eta^*(M)$. Then Γ is said to be a simple refinement of Δ. A subtractive linearly independent set Δ of $\eta^*(M)$ is said to be a subtractive basis if there does not exist a simple refinement of Δ.

Corollary 3.14. *Let M be a Noetherian Z^*-radical R-module such that for every submodule N of M and $Q \in \mathcal{X}$, $N \subseteq \sqrt[Z^*]{N}$ and $rad(Q) \cap N = rad(Q \cap N)$. If $|\mathcal{X}| < \infty$, then $\eta^*(M)$ has a subtractive basis.*

Proof. Since M is Noetherian, it has a finite indecomposable direct sum decomposition such as $M = \bigoplus_{i=1}^{n} N_i$. Thus by Theorem 3.13 $\{\nu^*(N_i)\}_{i=1}^{n}$ is a subtractive basis for M. □

Corollary 3.15. *Let M be a Noetherian Z^*-radical R-module such that $|\mathcal{X}| < \infty$ and for every submodule N of M and $Q \in \mathcal{X}$, $N \subseteq \sqrt[Z^*]{N}$ and $rad(Q) \cap N = rad(Q \cap N)$. If N' is a direct summand of M and N'' is a submodule of M such that $\sqrt[Z^*]{N''} = N'$, then $N'' = N'$.*

Proof. By Lemma 3.4, $\sqrt[Z^*]{N'} = N'$. Hence $\sqrt[Z^*]{N''} = N' = \sqrt[Z^*]{N'}$. So by Lemma 3.1, $\nu^*(N') = \nu^*(N'')$. Hence by Theorem 3.13 N'' is a direct summand of M. Then by Lemma 3.4, $\sqrt[Z^*]{N''} = N''$. Thus $N'' = N'$. □

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