

Performance Analysis of Fault Tolerant Control Systems with I.I.D. Upsets

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Abstract—The performance of a class of distributed control systems is analyzed when the interconnected fault tolerant components switch their mode of operation according to independent, identically distributed processes. New expressions for performance metrics are presented for this case, including their sensitivity analysis. These metrics require non-trivial derivations that are not just simplified expressions of the known metrics for the homogeneous Markov case. For a class of distributed control systems, the performance metrics are also computed when the actuators are assumed to have or not have memory. Finally, the results are illustrated with a distributed flight control example.

I. INTRODUCTION

An interconnection of L devices with $L \geq 2$ that are working together to accomplish a certain function represents a high level view of a fault tolerant network architecture. Operation of the network in a harsh environment can result in one or more devices randomly switching their modes of operation. Since these fault tolerant networks are the enabling technology in safety critical control system applications, it is important to analyze the effect of the random jumps of functionality on the controlled dynamical system. Let $(\Omega, \mathcal{F}, \Pr)$ be the ambient probability space. It is assumed that the mode of operation of each device forming a network is characterized with a state of an independent, identically distributed (i.i.d.) process. In particular, suppose that a harsh environment randomly switches each device's mode of operation from among V possibilities such that the mode of operation of the l^{th} device during each sample period is represented by a state of the i.i.d. process $z_l(k)$, where $l \in \{1, \dots, L\}$ and $k \in \mathbb{Z}^+ \triangleq \{0, 1, \dots\}$. When $z_l(k) = 0$, the l^{th} device is operating as intended and, in general, $z_l(k) = v$ denotes the

v^{th} mode of operation during the k^{th} sample period, where $v \in \mathcal{I}_V \triangleq \{0, 1, \dots, V-1\}$. From the point of view of the dynamical control system, it is important to characterize the modes of operation of the fault tolerant network, since they determine the closed-loop system's modes. The network's mode is characterized by a random process, $\rho(k)$, that is a transformation of the joint process $z(k) = (z_1(k), \dots, z_L(k))$. In this paper, a class of networks that result in $\rho(k)$ being an i.i.d. process is characterized. Then the performance of a distributed control system that can be represented as a jump linear system (JLS) switched by this i.i.d. process is analyzed. Most of the JLS literature has addressed the case where the switching process is a homogeneous first order Markov process (see, e.g., [1]–[4]). Some of these papers and others have presented results for i.i.d. switching processes (see, e.g., [1], [2], [5]–[7] and their references). Of course, an i.i.d. process also satisfies the first order Markov property and all the known results would apply in this case. However, simpler formulas can be derived that do not trivially follow from the known Markov results. This has been commented in, e.g., [2], [8] regarding stability results of an i.i.d. JLS. In fact, one motivation for this paper was to reduce the dimensions of the matrices used in the performance analysis of an i.i.d. JLS. If performance metrics derived for a homogeneous first order Markov process are used in an i.i.d. JLS, higher dimensional matrices than necessary are utilized. To avoid possible numerical issues, new closed-form expressions are derived here. Thus, the goals of this paper are to characterize a class of fault tolerant systems that induce an i.i.d. process driving a JLS, derive analytic expressions for two i.i.d. JLS performance metrics, and characterize the sensitivity of these metrics. The presentation of the paper follows this order. Section II characterizes the modes of operation of a class of fault tolerant systems. New analytic expressions for two performance metrics for an i.i.d. JLS are derived in Section III. The sensitivity of one of the performance measures is

This research was supported by the NASA Langley Research Center under grant NNX07AD52A.

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analyzed in Section IV. In Section V these results are applied to a particular distributed control system and an illustrative example is presented. A more thorough description of this distributed control system with a Boeing 747 example is reported in [9]. Finally, the conclusions are given in Section VI.

II. FAULT TOLERANT NETWORKS WITH I.I.D. UPSETS

Consider a fault tolerant network consisting of L devices such that the V modes of operation of each device are represented by the states of the i.i.d. process $z_l(k)$, where $l \in \{1, \dots, L\}$ and $k \in \mathbb{Z}^+$. When each device is independently affected by the harsh environment, a useful preliminary result is given in Lemma 1.

Lemma 1: Let the i.i.d. processes $z_1(k), \dots, z_L(k)$ with common state space \mathcal{I}_V be independent, and let the state probability vector for each process be $\pi_{z_l} \triangleq [\Pr\{z_l(k) = 0\}, \dots, \Pr\{z_l(k) = V - 1\}]$, $l \in \{1, \dots, L\}$. Then the joint process $z(k) = (z_1(k), \dots, z_L(k))$ is i.i.d. with state space $\mathcal{I}_V^L \triangleq \underbrace{\mathcal{I}_V \times \dots \times \mathcal{I}_V}_{L \text{ times}}$ and its state probability vector is $\pi_z \triangleq$

$\pi_{z_1} \otimes \dots \otimes \pi_{z_L}$, where \otimes denotes the Kronecker product.

Proof: This is a special case of [3, Lemma 1]. ■

In this paper, the fault tolerant networks of interest are those where the mode of operation of the network is a structure function of $z(k)$. A definition based on [10] is given next.

Definition 1: Let $z(k) = (z_1(k), \dots, z_L(k))$ be i.i.d. with state space \mathcal{I}_V^L , and let $\mathcal{I}_\ell = \{0, \dots, \ell - 1\}$ be a finite set with $1 < \ell < V^L$. The onto, memoryless function $\phi : \mathcal{I}_V^L \rightarrow \mathcal{I}_\ell$ mapping $z(k)$ into $\phi(z(k))$ is called a *structure function*.

The set \mathcal{I}_ℓ denotes the modes of operation of the network. Then during each sampling period, the network's mode of operation is given by the random variable $\rho(k) = \phi(z(k))$ for $k \in \mathbb{Z}^+$, which is assumed to have ℓ states satisfying $1 < \ell < V^L$. Since ϕ reduces the number of states of $z(k)$ from V^L to ℓ , it is a *lumping transformation*, and $\rho(k)$ is called a *lumped process*. It is well-known that lumping transformations of a homogeneous Markov chain (HMC) do not, in general, preserve the Markov property [11]–[13]. In Theorem 1 below it is shown that a lumping transformation does preserve the zeroth order Markov property when applied to an i.i.d. process. This theorem also characterizes the distribution of $\rho(k)$. First, observe that

the lumping transformation ϕ partitions the state space of $z(k)$ as follows: $\mathcal{I}_V^L = \bigcup_{j=0}^{\ell-1} I_j$, where for each $j \in \mathcal{I}_\ell$, $I_j = \phi^{-1}(j) = \{\xi \in \mathcal{I}_V^L : \phi(\xi) = j\}$.

Theorem 1: Let $z_1(k), \dots, z_L(k)$ be independent i.i.d. processes with common state space \mathcal{I}_V and defined over the same probability space. Let ϕ be a lumping transformation of the joint process $z(k) = (z_1(k), \dots, z_L(k))$ mapping \mathcal{I}_V^L into \mathcal{I}_ℓ such that $1 < \ell < V^L$. Then $\rho(k) = \phi(z(k))$ is an i.i.d. process, and its probability distribution function is given by

$$\Pr\{\rho(k) = j\} = \sum_{\xi \in I_j} \prod_{l=1}^L \pi_{z_l} \begin{bmatrix} 1_{\{\xi_l=0\}} \\ \vdots \\ 1_{\{\xi_l=V-1\}} \end{bmatrix}, \quad (1)$$

where $1_{\{\cdot\}}$ is the indicator function of the event $\{\cdot\}$, and ξ_l is the l^{th} component of ξ .

Proof: Since the joint process $z(k)$ is i.i.d., the sigma algebras $\sigma(\{z(k)\})$, $k \in \mathbb{Z}^+$ are independent. Thus, the claim follows immediately from the fact that the lumping transformation ϕ is a memoryless measurable function implying that $\sigma(\{\rho(k)\}) = \sigma(\{z(k)\})$. Finally, (1) follows from [14, Lemma 1] for the i.i.d. case. ■

III. PERFORMANCE ANALYSIS

In this section the performance of a JLS driven by an i.i.d. sequence $\rho(k)$ that takes values in the set \mathcal{I}_ℓ with distribution characterized by $p_i \triangleq \Pr\{\rho(k) = i\}$ for $i \in \mathcal{I}_\ell$ is analyzed. The i.i.d. JLS that models the dynamical system effect of the randomly switched modes of operation of an interconnection of components is

$$\mathbf{x}(k+1) = A_{\rho(k)} \mathbf{x}(k) + B_{\rho(k)} \mathbf{w}(k), \quad (2a)$$

$$\mathbf{x}(0) = \mathbf{x}_0,$$

$$\mathbf{y}(k) = C_{\rho(k)} \mathbf{x}(k), \quad (2b)$$

where $\mathbf{x}(k) \in \mathbb{R}^n$, $\mathbf{y}(k) \in \mathbb{R}^p$, \mathbf{x}_0 is a random vector with finite second moment, and $\mathbf{w}(k) \in \mathbb{R}^q$ is a zero mean white noise process with identity covariance matrix I_q and independent of $\rho(k)$ and \mathbf{x}_0 . Since $\rho(k)$ is i.i.d., the initial distribution $\rho(0)$ is the same as for $\rho(k)$, $k \geq 1$. A standard mean square stability (MSS) definition applied to the i.i.d. case is given next.

Definition 2: The i.i.d. JLS (2) is MSS if there exists a non-negative constant α such that for any initial condition $\mathbf{x}(0) = \mathbf{x}_0$ with finite second moment, it follows that $\lim_{k \rightarrow \infty} E\{\|\mathbf{x}(k)\|^2\} = \alpha$. If $\mathbf{w}(k) = 0$ for $k \in \mathbb{Z}^+$ then $\alpha = 0$.

Two useful properties of the i.i.d. JLS are introduced in Lemma 2. Denote by $\mathcal{F}_k \triangleq \sigma(\{\rho(k)\})$, the sigma-algebra generated by $\rho(k)$, $k \in \mathbb{Z}^+$.

Lemma 2: Let $\rho(k)$ be the process driving the i.i.d. JLS (2). Then $\mathbf{x}(k)$ and $\mathbf{1}_{\{\rho(k)=i\}}$ are independent for $i \in \mathcal{I}_\ell$ and $k \geq 1$. In addition, for each $k \in \mathbb{Z}^+$ the random variables $\mathbf{x}(k)$ and $\mathbf{w}(k)$ are independent.

Proof: From (2a) it follows that $\mathbf{x}(k)$ is \mathcal{F}_{k-1} -measurable for $k \geq 1$. Since $\mathbf{1}_{\{\rho(k)=i\}}$ is \mathcal{F}_k -measurable for each $i \in \mathcal{I}_\ell$, the claim follows because $\rho(k)$ is an i.i.d. process implying that the sigma algebras \mathcal{F}_{k-1} and \mathcal{F}_k are independent. The independence between $\mathbf{x}(k)$ and $\mathbf{w}(k)$ follows from the assumption that $\mathbf{w}(k)$ is independent of $\rho(k)$. ■

For each $k \in \mathbb{Z}^+$ define $Q(k) \triangleq E\{\mathbf{x}(k)\mathbf{x}^T(k)\}$. The following lemma gives another characterization of MSS.

Lemma 3: The i.i.d. JLS (2) is MSS if and only if there exists a positive semi-definite matrix $Q \in \mathbb{R}^{n \times n}$ not depending on $\mathbf{x}(0)$ such that $\lim_{k \rightarrow \infty} Q(k) = Q$. Moreover, when (2) is MSS, Q satisfies the Lyapunov equation

$$Q = \sum_{i=0}^{\ell-1} A_i Q A_i^T p_i + \sum_{i=0}^{\ell-1} B_i B_i^T p_i, \quad (3)$$

and

$$Q = \text{vec}^{-1} \left((I_{n^2} - \mathcal{A})^{-1} \text{vec}(\mathcal{B}) \right), \quad (4)$$

where

$$\mathcal{A} \triangleq \sum_{i=0}^{\ell-1} (A_i \otimes A_i) p_i, \quad (5)$$

$$\mathcal{B} = \sum_{i=0}^{\ell-1} B_i B_i^T p_i, \quad (6)$$

and vec denotes the column stacking operator.

Proof: The claim follows from [1, Theorem 3.33] and [15, Theorem 2.3.3]. Equations (3) and (4) follow from the definition for $Q(k)$. ■

When \mathcal{B} is positive definite, the Lyapunov equation is a particular case of [1, Corollary 3.26]. A test for mean square stability is given next.

Lemma 4: The i.i.d. JLS (2) is MSS if and only if the spectral radius of \mathcal{A} is less than 1, where \mathcal{A} is defined in (5).

Proof: This stability test has appeared in, e.g., [2], [5]–[7], and it is proven in [5]. ■

The matrix \mathcal{A} in (5) has dimension $n^2 \times n^2$. If $\rho(k)$ had been an HMC taking values in \mathcal{I}_ℓ then the corresponding MSS test would compute the spectral radius of an $\ell n^2 \times \ell n^2$ matrix. The lower dimension of \mathcal{A} in (5) is a benefit of working with an i.i.d. JLS instead of a Markov JLS. An additional benefit is that an equivalent MSS test for a Markov JLS requires solving a set of coupled algebraic generalized Lyapunov equations [1, Theorem 3.9]. For an i.i.d. JLS, only one algebraic generalized Lyapunov equation needs to be solved, (3).

To characterize the performance of the i.i.d. JLS (2) with a white noise input, analytic expressions are derived for the average energy and power of the output signals based on the definitions given in [1], [4]. Analogous expressions appeared in [4] when $\rho(k)$ is a first order HMC. Since an i.i.d. process is an HMC of order zero, Theorem 9 in [4] can be also used when $\rho(k)$ is i.i.d., however, simpler and lower dimensional formulas are derived here. The output performance metrics for (2) are defined as follows:

$$J = \begin{cases} J_w \triangleq \lim_{k \rightarrow \infty} E\{\|\mathbf{y}(k)\|^2\} & , w(k) \neq 0 \\ J_0 \triangleq \sum_{k=0}^{\infty} E\{\|\mathbf{y}(k)\|^2\} & , w(k) = 0 \end{cases}$$

where J_w is called the steady-state mean output power, and J_0 is the mean output energy. The order of the sum and the expectation in J_0 have been changed with respect to [4] to match the order given in [1]. Analytic expressions for J_w and J_0 are given in Theorems 2 and 3, respectively.

Theorem 2: If the i.i.d. JLS (2) is MSS then it follows that $J_w < \infty$ and

$$J_w = \sum_{i=0}^{\ell-1} \text{tr}(C_i Q C_i^T) p_i \quad (7)$$

with Q defined in (4).

Proof: From (2b) and Lemma 2 it follows that

$$E\{\|\mathbf{y}(k)\|^2\} = \text{tr} \left\{ \sum_{i=0}^{\ell-1} C_i E\{\mathbf{x}(k)\mathbf{x}^T(k)\} C_i^T \right\} p_i.$$

When (2) is MSS, taking limits as $k \rightarrow \infty$ on both sides of this equation gives (7) by Lemma 3. ■

For each $k \in \mathbb{Z}^+$ define $M(k) = \sum_{i=0}^k Q(i)$. When $w(k) = 0$, the following lemma gives another equivalent characterization of MSS.

Lemma 5: The i.i.d. JLS (2) with $w(k) = 0$ is MSS if and only if there exists a positive semi-definite matrix

M not depending on $\mathbf{x}(0)$ such that $M = \sum_{k=0}^{\infty} Q(k)$. Moreover, when (2) is MSS, M satisfies the Lyapunov equation

$$M = \sum_{i=0}^{\ell-1} A_i M A_i^T p_i,$$

and

$$M = \text{vec}^{-1}(I_{n^2} - \mathcal{A})^{-1} \quad (8)$$

with \mathcal{A} defined in (5).

Proof: By [16, Theorem 2] the following inequalities hold for each $k \in \mathbb{Z}^+$:

$$\frac{1}{n} E \{ \|\mathbf{x}(k)\|^2 \} \leq \|Q(k)\| \leq E \{ \|\mathbf{x}(k)\|^2 \}. \quad (9)$$

Suppose that (2) is MSS. Since MSS is equivalent to stochastic stability [17], $\sum_{i=0}^{\infty} E \{ \|\mathbf{x}(i)\|^2 \} < \infty$. Thus, the sequence of partial sums, $\sum_{i=0}^k E \{ \|\mathbf{x}(i)\|^2 \}$, is Cauchy. This implies that $M(k)$ is Cauchy since

$$\begin{aligned} \|M(m) - M(n)\| &\leq \sum_{k=n+1}^m \|Q(k)\| \\ &\leq \sum_{k=n+1}^m E \{ \|\mathbf{x}(k)\|^2 \}, \end{aligned}$$

where the last inequality follows from the second inequality in (9). Thus, the series $\sum_{k=0}^{\infty} Q(k)$ is convergent. Assume now that the series $\sum_{k=0}^{\infty} Q(k)$ is convergent. Then $\lim_{k \rightarrow \infty} Q(k) = 0$ and, therefore, the first inequality in (9) implies that $\lim_{k \rightarrow \infty} E(\|\mathbf{x}(k)\|^2) = 0$, i.e., (2) is MSS. The expressions for M follow from its definition. This completes the proof. ■

Theorem 3: If the i.i.d. JLS (2) with $w(k) = 0$ is MSS then $J_0 < \infty$, and

$$J_0 = \sum_{i=0}^{\ell-1} \text{tr}(C_i M C_i^T) p_i,$$

with M defined in (8).

Proof: When (2) is MSS, equation (2b) and Lemmas 2 and 5 directly give the formula for J_0 . ■

IV. SENSITIVITY PERFORMANCE ANALYSIS

When the i.i.d. JLS (2) is MSS, the output performance metrics J_w and J_o can be seen as the real-valued functions $J_w(p)$ and $J_o(p)$, mapping a mean-square stabilizing subset of $[0, 1]^\ell \triangleq \underbrace{[0, 1] \times \dots \times [0, 1]}_{\ell \text{ times}}$ into \mathbb{R} , where $p \triangleq (p_0, \dots, p_{\ell-1})$ and $p_j \triangleq \Pr\{\boldsymbol{\rho}(k) = j\}$,

$j \in \mathcal{I}_\ell$. In fact, from Theorems 2 and 3 it follows that the performance metrics are rational functions of these mean-square stabilizing probabilities. Moreover, the following lemma makes possible the evaluation of partial derivatives.

Lemma 6: Let $p^* \in [0, 1]^\ell$ be such that the i.i.d. JLS (2) is MSS. Then there exist a neighborhood of p^* such that for each p in this neighborhood the i.i.d. JLS (2) remains MSS.

Proof: The result follows because the spectral radius of \mathcal{A} is a continuous function of p . ■

In Theorem 4 the sensitivity with respect to p_j for $j \in \mathcal{I}_\ell$ is evaluated at the mean-square stabilizing probability $p^* \triangleq (p_0^*, \dots, p_{\ell-1}^*) \in [0, 1]^\ell$. A less local result is given in Theorem 5, where the intervals over which the performance metric is monotonic are characterized for a special case. Similar arguments can be followed for J_0 .

Theorem 4: Let $p^* \in [0, 1]^\ell$ be such that the i.i.d. JLS (2) is MSS and let $Q^* \triangleq Q(p^*)$ be the value of Q at this point. Then for each $j \in \mathcal{I}_\ell$

$$\begin{aligned} \left. \frac{\partial J_w(p)}{\partial p_j} \right|_{p=p^*} &= \left(\sum_{i=0}^{\ell-1} \text{tr} \left(C_i \left. \frac{\partial Q(p)}{\partial p_j} \right|_{p=p^*} C_i^T \right) p_i^* \right) \\ &\quad + \text{tr}(C_j Q^* C_j^T), \end{aligned}$$

where

$$\begin{aligned} \left. \frac{\partial Q(p)}{\partial p_j} \right|_{p=p^*} &= \\ \text{vec}^{-1} \left((I - \mathcal{A})^{-1} ((A_j \otimes A_j) (I - \mathcal{A})^{-1} \right. \\ &\quad \left. \text{vec}(\mathcal{B}) + \text{vec}(B_j B_j^T)) \right) \end{aligned}$$

with \mathcal{A} and \mathcal{B} defined in (5) and (6), respectively.

Proof: Since J_w is a rational function, it is infinitely differentiable at any point where it is well-defined. The partial derivatives of J_w and Q follow by direct application of $\frac{\partial}{\partial p_j}$ and noting that the trace, vec, and vec^{-1} are linear transformations. Thus, these transformations commute with the partial derivative. ■

To present a less local result let $\ell = 2$. Then the i.i.d. JLS (2) has two modes of operation that are selected by $\boldsymbol{\rho}(k)$. The mode denoted by ‘0’ represents a nominal mode in which the closed-loop system is working correctly. The mode denoted by ‘1’ represents the upset state in which the system is not performing its intended function due to the harsh environment. Then the probability $p_1 = \Pr\{\boldsymbol{\rho}(k) = 1\}$ can be interpreted

as the probability that the closed-loop system is in the upset state, and the performance J_w can be seen as a function of this probability. Let \mathcal{U} denote the union of all the disjoint subintervals of $[0, 1]$ containing the values of p_1 that result in (2) being MSS. When \mathcal{U} is nonempty, the endpoints of each open subinterval are consecutive points taken from the sequence $0 \leq \hat{p}_0 < \hat{p}_1 < \dots < \hat{p}_{r-1} \leq 1$, where \hat{p}_i , $i = 0, \dots, r-1$, satisfy one or more of the following conditions: $\hat{p}_0 = 0$ ($\hat{p}_{r-1} = 1$) when A_0 (A_1) is Hurwitz; \hat{p}_i are the values of p_1 that result in a unit spectral radius for \mathcal{A} ; and \hat{p}_i can also be the distinct real roots of $\frac{dJ_w(p_1)}{dp_1}$. If $\hat{p}_0 = 0$ ($\hat{p}_{r-1} = 1$) then its subinterval is closed on the left (right).

Theorem 5: When the i.i.d. JLS (2) is MSS, the sign of $\frac{dJ_w(p_1)}{dp_1}$ is constant over each subinterval in \mathcal{U} , that is, $J_w(p_1)$ is monotonic on these subintervals.

Proof: Since J_w and $\frac{dJ_w(p_1)}{dp_1}$ are rational functions of p_1 , the only possible endpoints for the subintervals are those in \mathcal{U} . ■

V. DISTRIBUTED CONTROL SYSTEM REGULATION PERFORMANCE

An application of the results of this paper to a distributed control system is presented in this section. Consider the following discretized state space realization of a plant:

$$\begin{aligned} \mathbf{x}_p(k+1) &= A_p \mathbf{x}_p(k) + B_p \mathbf{u}(k) \\ \mathbf{y}_p(k) &= C_p \mathbf{x}_p(k), \end{aligned} \quad (10)$$

where $\mathbf{x}_p(k) \in \mathbb{R}^{n_p}$ is the plant's state vector, $\mathbf{y}_p(k) \in \mathbb{R}^m$ is the plant's output, and $\mathbf{u}(k) \in \mathbb{R}^m$ is the plant's input. The nominal control law used to close the loop to attain a desired level of regulation performance is $\mathbf{u}(k) = \mathbf{w}(k) - \mathbf{y}_c(k)$, where $\mathbf{w}(k) \in \mathbb{R}^m$ is a zero mean white noise process with identity covariance matrix I_m and independent of $\mathbf{x}_p(0)$, and $\mathbf{y}_c(k) \in \mathbb{R}^m$ is the controller's output. The designed observer-based controller's state-space representation is

$$\begin{aligned} \mathbf{x}_c(k+1) &= A_p \mathbf{x}_c(k) + B_p \mathbf{u}(k) \\ &\quad + L_p (\mathbf{y}_p(k) - C_p \mathbf{x}_c(k)) \\ \mathbf{y}_c(k) &= K \mathbf{x}_c(k), \end{aligned} \quad (11)$$

where $\mathbf{x}_c(k) \in \mathbb{R}^{n_p}$ is the controller's state vector, and K and L_p are the pole placement and observer matrices, respectively. The nominal closed-loop system is obtained when the nominal control law is applied. It results in a nominal regulation level of closed-loop

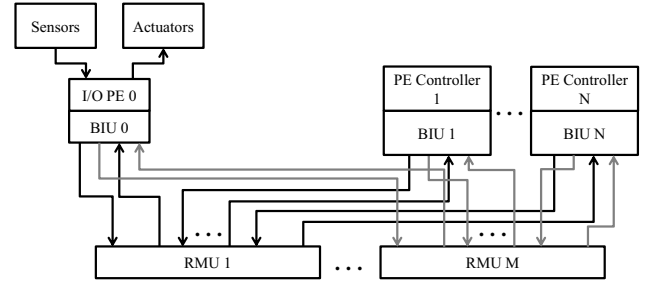


Fig. 1. Schematic of a distributed closed-loop system implemented with a ROBUS-2 fault tolerant communication system.

performance given by $J_w = \lim_{k \rightarrow \infty} E\{\|\mathbf{y}_p(k)\|^2\}$. The results in this paper make it possible to determine the performance degradation when an update to the control law is not received by the actuators at each control cycle due to random events caused by a harsh environment acting on a distributed control system as shown in Fig. 1. It consists of redundant and equivalent implementations of the controller dynamics in N Processing Elements (PEs). Each of the PEs connects to a fault tolerant communication network with a Bus Interface Unit (BIU) and each BIU is connected to M Redundancy Management Units (RMUs). For simplicity, all the sensors and actuators are connected using a single I/O PE and BIU. This PE-BIU node is assumed not to fail. This network is based on NASA's SPIDER (Scalable Processor-Independent Design for Enhanced Reliability) architecture, which uses the ROBUS-2 communication system [18]–[20]. A more detailed description of the distributed control system and a Boeing 747 control application is given in [9]. The network shown in Fig. 1 is referred to as an N PE \times M RMU distributed control system, where the N PE-BIU nodes and M RMUs will be assumed to be the only components that can randomly fail silently, i.e., the devices produce no output during a control cycle but can recover and restart operation at the next control cycle.

To analyze this distributed control system suppose that for each control cycle $k \in \mathbb{Z}^+$ the modes of operation of the i^{th} PE and the j^{th} RMU are denoted by the indicator random variables $z_i(k)$ and $\tilde{z}_j(k)$, respectively. The convention for all the indicator random variables is that a value of '0' denotes that the device is available and that a value of '1' denotes that the device has failed silently. Assume that a valid controller output is delivered to the actuators if at least one PE and

one RMU are available; otherwise, no controller output is delivered to the actuators. This event is denoted with the indicator random variable $z_v(k)$ that uses the same convention assumed for the components. An application of the results in this paper leads to the following statistical characterization of $z_v(k)$.

Lemma 7: Consider an N PE \times M RMU distributed control system as in Fig. 1. Assume that all the availability processes $\{z_i(k), i = 1, \dots, N\}$ and $\{\tilde{z}_j(k), j = 1, \dots, M\}$ are i.i.d. and mutually independent. Let $p_{\theta_i} \triangleq \Pr\{z_i(k) = 1\}$ and $p_{\nu_j} \triangleq \Pr\{\tilde{z}_j(k) = 1\}$ then $z_v(k)$ is an i.i.d. process with distribution characterized by

$$p_1 \triangleq \Pr\{z_v(k) = 1\} \\ = 1 - \left(1 - \prod_{i=1}^N p_{\theta_i}\right) \left(1 - \prod_{j=1}^M p_{\nu_j}\right).$$

Proof: The proof follows by repeated application of Theorem 1, since

$$z_v(k) = \phi_{2|2} \left(\phi_{1|N} (z_1(k), \dots, z_N(k)), \right. \\ \left. \phi_{1|M} (\tilde{z}_1(k), \dots, \tilde{z}_M(k)) \right),$$

where the lumping transformations $\phi_{1|N}$ (1-out-of- N) and $\phi_{1|M}$ (1-out-of- M) are parallel structure functions, and $\phi_{2|2}$ (2-out-of-2) is a series structure function [14]. ■

The effect of the random upsets affecting the N PEs and M RMUs on the closed-loop system can be characterized as follows. When $z_v(k) = 1$ no control input is delivered to the plant's actuators and the communication system restarts the N PEs resulting in the controllers' state vectors getting reset to zero. When $z_v(k) = 0$ the closed-loop system behaves as the nominal one. Thus, the random upsets result in a switched control system indexed by $z_v(k)$. In particular, the control law is also switched, i.e., $\mathbf{u}(k) \triangleq \mathbf{u}_{z_v(k)}(k)$. The value of $\mathbf{u}_{z_v(k)}(k)$ depends on the type of actuators, which can be memoryless or have memory. Memoryless actuators assume a zero command when no data is received. The effective control input is then

$$\mathbf{u}_{z_v(k)}(k) = \mathbf{w}(k) - (1 - z_v(k)) \mathbf{y}_c(k), \quad (12)$$

where the white noise now needs to be also assumed to be independent of $z_v(k)$. Actuators with memory belong to a class of smart actuators. When no data

is received, these actuators use the previous control command. The effective control input is

$$\mathbf{u}_{z_v(k)}(k) = \mathbf{w}(k) - (1 - z_v(k)) \mathbf{y}_c(k) \\ - z_v(k) \mathbf{y}_c(k-1). \quad (13)$$

A realization of the switched closed-loop system follows from (10), (11) and either (12) or (13) to be

$$\mathbf{x}_{\text{cl}}(k+1) = \bar{A}_{z_v(k)} \mathbf{x}_{\text{cl}}(k) + \bar{B}_{z_v(k)} \mathbf{w}(k) \\ \mathbf{y}_{\text{cl}}(k) = \bar{C}_{z_v(k)} \mathbf{x}_{\text{cl}}(k), \quad (14)$$

where $\mathbf{y}_{\text{cl}}(k) = \mathbf{y}_p(k)$. For memoryless actuators the state vector is $\mathbf{x}_{\text{cl}}(k) = [\mathbf{x}_p^T(k) \ \mathbf{x}_c^T(k)]^T \in \mathbb{R}^{2n_p}$. The state equation realization pairs $(\bar{A}_{z_v(k)}, \bar{B}_{z_v(k)})$ for $z_v(k) \in \{0, 1\}$ are

$$\bar{A}_0 = \begin{bmatrix} A_p & -B_p K \\ L_p C_p & A_c \end{bmatrix}, \quad \bar{B}_0 = \begin{bmatrix} B_p \\ B_p \end{bmatrix}, \\ \bar{A}_1 = \begin{bmatrix} A_p & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

where $A_c = A_p - B_p K - L_p C_p$. The output equation is not switched. It is characterized by $C_{\text{cl}} = \bar{C}_0 = \bar{C}_1 = [C_p \ 0]$. When the actuators have memory, the closed-loop system is augmented with an additional state vector that remembers the previous value of the controller's state vector. So the state vector in (14) is $\mathbf{x}_{\text{cl}}(k) = [\mathbf{x}_p^T(k) \ \mathbf{x}_c^T(k) \ \mathbf{x}_a^T(k)]^T \in \mathbb{R}^{3n_p}$, $\mathbf{x}_a(k) = \mathbf{x}_c(k-1)$. The state equation realization pairs in this case are

$$\bar{A}_0 = \begin{bmatrix} A_p & -B_p K & 0 \\ L_p C_p & A_c & 0 \\ 0 & I & 0 \end{bmatrix}, \quad \bar{B}_0 = \begin{bmatrix} B_p \\ B_p \\ 0 \end{bmatrix}, \\ \bar{A}_1 = \begin{bmatrix} A_p & 0 & -B_p K \\ 0 & 0 & I \\ 0 & 0 & I \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The output equation is characterized by $C_{\text{cl}} = \bar{C}_0 = \bar{C}_1 = [C_p \ 0 \ 0]$.

The reset and one sample period rollback recovery models presented here are special cases of more general error recovery techniques. Comparison of the effect of these techniques on MSS is given in [21].

The degradation in regulation performance can now be characterized. The case of memoryless actuators and actuators with memory are considered in parallel. First, the nominal closed-loop realization for $z_v(k) = 0$ for $k \in \mathbb{Z}^+$ follows from (14) to be

$$\mathbf{x}_n(k+1) = \bar{A}_0(k) \mathbf{x}_n(k) + \bar{B}_0(k) \mathbf{w}(k) \\ \mathbf{y}_n(k) = C_{\text{cl}(k)} \mathbf{x}_n(k), \quad (15)$$

where $\mathbf{x}_n(k) = \mathbf{x}_{\text{cl}}(k)$ for $k > 0$ is the nominal closed-loop state vector. The regulation error caused by the random upsets is $\mathbf{y}_e(k) \triangleq \mathbf{y}_{\text{cl}}(k) - \mathbf{y}_n(k)$ when (14) and (15) have the same disturbance input $\mathbf{w}(k)$. A realization of this error system is

$$\begin{bmatrix} \mathbf{x}_{\text{cl}}(k+1) \\ \mathbf{x}_n(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{z_v(k)} & 0 \\ 0 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{\text{cl}}(k) \\ \mathbf{x}_n(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_{z_v} \\ \bar{B}_0 \end{bmatrix} \mathbf{w}(k), \quad (16a)$$

$$\begin{bmatrix} \mathbf{x}_{\text{cl}}(0) \\ \mathbf{x}_n(0) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_{n,0} \end{bmatrix}, \quad \mathbf{y}_e(k) = [C_{\text{cl}} \quad -C_{\text{cl}}] \begin{bmatrix} \mathbf{x}_{\text{cl}}(k) \\ \mathbf{x}_n(k) \end{bmatrix}. \quad (16b)$$

The error system in (16) is an i.i.d. JLS switched by $z_v(k)$. Let its realization be denoted by $(\tilde{A}_{z_v(k)}, \tilde{B}_{z_v(k)}, \tilde{C})$ and the state vector be $\tilde{\mathbf{x}}(k) \triangleq [\mathbf{x}_{\text{cl}}^T(k), \mathbf{x}_n^T(k)]^T$. The performance metrics for the i.i.d. JLS (16) have been derived in Section III. In particular, the steady-state mean error power is $J_{w,e} \triangleq \lim_{k \rightarrow \infty} E\{\|\mathbf{y}_e(k)\|^2\}$. When the white noise, $\mathbf{w}(k)$, is applied to (16), and if it is MSS, then Theorem 2 gives the closed form expression for $J_{w,e}$. The sensitivity of this metric with respect to $p_1 = \Pr\{z_v(k) = 1\}$ follows from Theorem 4. For the distributed closed-loop system in Fig. 1 the sensitivities with respect to the upset probabilities of the PEs and RMUs can also be derived. A special case is considered next.

Lemma 8: Consider an N PE \times N RMU distributed control system as in Fig. 1. Assume that all the availability processes $\{z_i(k), i = 1, \dots, N\}$ and $\{\tilde{z}_j(k), j = 1, \dots, N\}$ are i.i.d. and mutually independent. Let $p_\theta \triangleq \Pr\{z_i(k) = 1\} = p_\nu = \Pr\{\tilde{z}_j(k) = 1\}$. Let p_θ^* be such that (16) is MSS and $Q^* = Q(p_\theta^*)$. Then $p_1 = 1 - (1 - (p_\theta^*)^N)^2$ and

$$\left. \frac{dJ_{w,e}(p_\theta)}{dp_\theta} \right|_{p_\theta^*} = \left[\frac{\partial J_w(p_0, p_1)}{\partial p_1} - \frac{\partial J_w(p_0, p_1)}{\partial p_0} \right] \Big|_{(p_0^*, p_1^*)} \cdot (2N(1 - p_\theta^*)(p_\theta^*)^{N-1}).$$

Proof: Apply Theorem 4 and Lemma 7. ■

A numerical example is presented next.

Example 1: Consider the simplified longitudinal dynamics of the AFTI-F16 aircraft given in [22], where the aircraft model has four states (change in speed, angle of attack, pitch rate, and pitch angle) and the output of interest is the pitch rate. The

sampled-data closed-loop system has sampling period $T = 0.004$ sec., the pole placement controller places the nominal continuous-time closed-loop poles at $\{-0.2 \pm j0.9798, -0.01 \pm j0.0995\}$, and the observer's discrete-time poles were chosen to be five times faster than the plant's closed-loop poles. The distributed control system consists of 2 PEs and 2 RMUs. When these four devices are allowed to randomly fail independently then \mathcal{U} consists of one interval and (14) is MSS for $p_\theta^* \in [0, 0.0174)$ when memoryless actuators are used and $p_\theta^* \in [0, 0.2461)$ when actuators with memory are used. Figure 2 shows the analytic steady-state mean error power for both actuator cases. Assuming zero initial conditions for the closed-loop and nominal state vectors in (16), $J_{w,e}$ starts at zero and is finite only for each value p_θ^* that results in MSS. By Theorem 4 this error metric is known to be monotonically increasing since the nominal closed-loop system (15) is asymptotically stable. Finally, the sensitivity of the error metric with respect to p_θ is shown in Figure 3. Therefore, by using actuators with memory, the closed-loop is MSS over a larger interval, the error metric is smaller and has less sensitivity.

VI. CONCLUSIONS

Closed-form expressions for two performance metrics, the steady-state mean output power and mean output energy, for an i.i.d. JLS were derived. It was shown that the i.i.d. JLS can be used to analyze the performance of a class of distributed control systems such as those based on NASA's SPIDER distributed fault-tolerant architecture when the interconnected components fail independently according to finite state i.i.d. processes. The sensitivity of the steady-state mean output power with respect to perturbations of the probability distribution of the i.i.d. process driving the JLS was also characterized. A numerical example was presented that compared the steady-state mean output power and its sensitivity for two closed-loop systems: one where the actuators have memory and one where they do not. This example showed the advantage of embedding memory in actuators for fault-tolerant applications.

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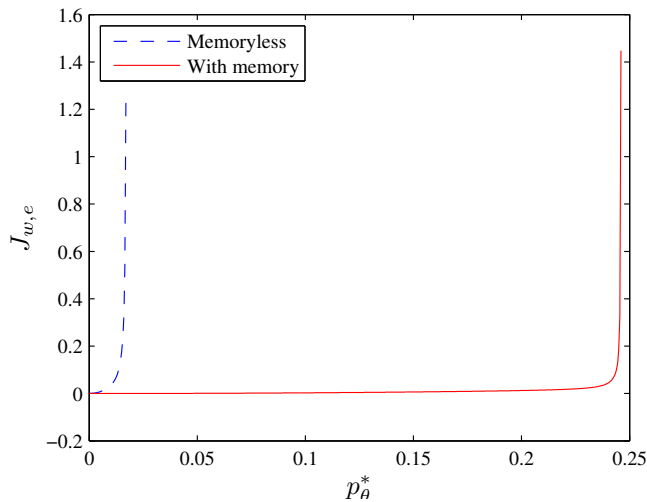


Fig. 2. $J_{w,e}$ for the pitch rate output vs. p_{θ}^* when $p_{\theta} = p_{\nu}$ for a $2 \text{ PE} \times 2 \text{ RMU}$ distributed control system.

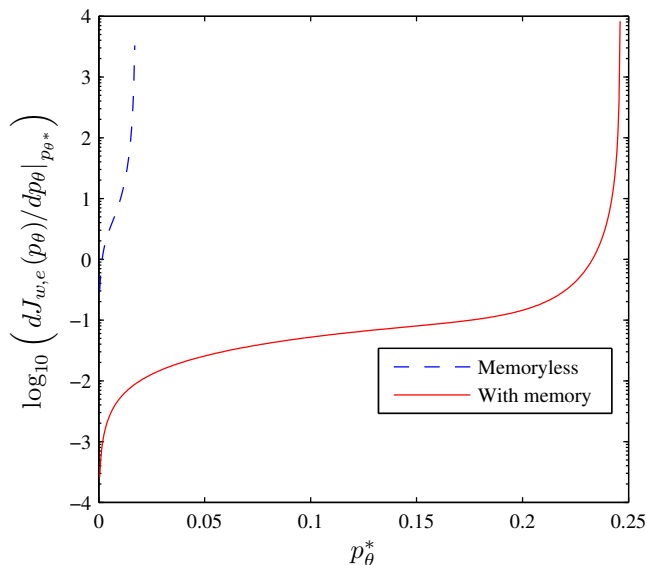


Fig. 3. The sensitivity with respect to p_{θ} is shown in a log scale when $p_{\theta} = p_{\nu}$ for a $2 \text{ PE} \times 2 \text{ RMU}$ distributed control system.

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