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Existence results for a superlinear p-Laplacian equation with indefinite weights ☆

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Abstract

In this paper, using Mountain Pass Lemma and Linking Argument, we prove the existence of nontrivial weak solutions for the Dirichlet problem of the superlinear p-Laplacian equation with indefinite weights in the case where the eigenvalue parameter $\lambda \in (0, \lambda_2)$, λ_2 is the second positive eigenvalue of the p-Laplacian with indefinite weights.

MSC: 35J60

Keywords: p-Laplacian; Indefinite weight; Mountain pass Lemma; Linking argument

1. Introduction

In this paper, we shall investigate the existence of weak solutions for the following Dirichlet problem of the p-Laplacian with indefinite weights:

$$\begin{cases} -\Delta_p u =: -\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda V(x)|u|^{p-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $p > 1$, $\Omega \subset R^N$ is a bounded domain, $V(x)$ is a given function which may change sign and λ is the eigenvalue parameter. Assume that

$$V^+ \not\equiv 0 \text{ and } V \in L^s(\Omega), \tag{1.2}$$

for some $s > N/p$ if $1 < p < N$ and $s = 1$ if $p > N$.

For $p = 2$, $V \equiv 1$, many results on the existence of linking-type critical points of problem (1.1) have been obtained (eg. [2,4,7]); For $p \neq 2$, $V \equiv 1$, by two linking results, Fan and Li [10] obtained the existence results of problem (1.1) when $0 < \lambda < \lambda_2$; while Alves et al. [1] consider the multiple solutions for the resonance involving p-Laplacian, under certain conditions on $f(x,u)$, the authors obtained at least three solutions of problem (1.1) when $p \neq 2$, $\lambda = \lambda_1$, $V = h(x) \in L^\infty(\Omega)$ is a essentially bounded function. Using Morse theory, Liu [13] consider the existence of solutions to problem:

$$\begin{cases} -\Delta_p u =: -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

under condition that $\int_0^s f(x,s)\,\mathrm{d}s$ lies between the first two eigenvalues of p-Laplacian, which includes problem (1.1) with $V \equiv 1$ as special case.

It is interesting here that function V is just belonging to $L^s(\Omega)$ and may change sign. Our results will mainly rely on the results for the eigenvalue problem correspondent to problem (1.1) in [9]. Let us first recall the main results of Cuesta [9]. Consider the nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u =: -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda V(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where V satisfies condition (1.2). Define C^1 functionals Φ and $J : W_0^{1,p}(\Omega) \to R$ by

$$\Phi(u) \equiv \int_\Omega |\nabla u|^p \,\mathrm{d}x, \text{ and } J(u) \equiv V(x)|u(x)|^p \,\mathrm{d}x$$

and set $\mathscr{M}$ by

$$\mathscr{M} = \{u \in W_0^{1,p}(\Omega) : J(u) = 1\}.$$

Assumption (1.2) ensures that $\mathscr{M} \neq \emptyset$. If $\gamma(A)$ denotes the Krasnoselski genus on $W_0^{1,p}(\Omega)$ and for any $k \in \mathscr{N}$, set $\Gamma_k \equiv \{A \subset \mathscr{M}$: A is compact, symmetric and $\gamma(A) \geqslant k\}$. Then by the standard arguments of Ljusternik–Schnirelman critical point theory, value

$$\lambda_k = \inf_{A \in \Gamma_k} \max_{u \in A} \Phi(u) \tag{1.5}$$

is an eigenvalue of problem (1.4). Moreover, $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \to +\infty$, as $k \to +\infty$.

For $p = 2$, $V(x) \equiv 1$, it is well known that the values obtained by (1.5) are all the eigenvalues of problem (1.4) and some other results, such as the first positive eigenvalue λ_1 is simple, isolated and it is the unique eigenvalue with positive eigenfunction. But for the general case $p \neq 2$, the situation becomes far more complicated (cf. [5,11,15] and the references therein). Fortunately, Cuesta [9] proved that the above properties of the first eigenvalue are also true in our case. Moreover, let

$$\underline{\lambda}_2 \equiv \min\{\lambda \in R^+ :\ \lambda \text{ is an eigenvalue and } \lambda > \lambda_1\}. \tag{1.6}$$

Then $\underline{\lambda}_2 = \lambda_2$.

2. Linking results

Let $e_k \in \mathscr{M}$ be the eigenfunction associated to the eigenvalue λ_k, then $\|e_k\|^p = \lambda_k$. Denote $G = \{u \in \mathscr{M} :\ \Phi(u) < \lambda_2\}$. Obviously, G is an open set containing e_1 and e_2. Moreover $-G = G$. First we shall prove the following Lemma.

Lemma 2.1. *e_1 and $-e_1$ do not belong to the same connected component of G.*

Proof. Otherwise, there exists a continuous curve σ connecting e_1 and $-e_1$ in G. Let $A = \sigma \cup \{-\sigma\}$, then from the definition of $\mathscr{M}$, $0 \notin A$, hence $\gamma(A) > 1$. By connectedness of A, $A \in \Gamma_2$. Hence, as A is a compact set in G, and from the definition of G, we will have $\max\{\Phi(u);\ u \in A\} < \lambda_2$, and this contradicts the definition of λ_2. □

Let G_1 be the connected component of G containing e_1, then $-G_1$ is the connected component of G containing $-e_1$. Let

$$K_1 = \{tu :\ u \in G_1, t > 0\},\ K = K_1 \cup \{-K_1\}.$$

Then, we have

$$\int |\nabla u|^p < \lambda_2 \int V(x)|u|^p \quad \forall u \in K \tag{2.1}$$

and

$$\int |\nabla u|^p = \lambda_2 \int V(x)|u|^p \quad \forall u \in \partial K, \tag{2.2}$$

where ∂K is the boundary of K in $X = W_0^{1,p}(\Omega)$. Let $(\partial K)_\rho = \{u \in \partial K :\ \|u\| = \rho\}$.

Set

$$\mathscr{E}_1 = \text{span}\{e_1\},\ \mathscr{E}_2 = \text{span}\{e_1,\ e_2\},$$

$$\mathscr{Z} = \{u \in X :\ \int_\Omega |\nabla u|^p = \lambda_2 \int_\Omega V(x)|u|^p\}.$$

(2.2) implies $\partial K \subset \mathscr{Z}$.

Similar to Proposition 2.1 and Proposition 2.2 in [10], we obtain the following two linking results concerning the p-Laplacian with indefinite weights.

Theorem 2.2. *Assume that* $v \in \mathscr{E}_1$, $v \neq 0$, $Q = [-v, v]$ *is the line segment connecting* $-v$ *and* v, $\partial Q = \{-v, v\}$. *Then* $\partial Q \subset Q$ *and* $\mathscr{Z}$ *link in* X, *that is,*

(i) $\partial Q \cap \mathscr{Z} = \emptyset$ *and*
(ii) *For any continuous map* $\psi : Q \to X$ *with* $\psi|_{\partial Q} = id$, *there holds* $\psi(Q) \cap \mathscr{Z} \neq \emptyset$.

Proof. It is obvious that $\partial Q \cap \mathscr{Z} = \emptyset$. Now let $\psi : Q = [-v, v] \to X$ be continuous and $\psi|_{\partial Q} = \text{id}$. From the definition of K and Lemma 2.1, K has two connected components K_1 and $-K_1$. Assume $v \in K_1$, $-v \in -K_1$, then $\psi(Q)$ is a continuous curve connecting v and $-v$, therefore there holds $\psi(Q) \cap \partial K \neq \emptyset$ and thus $\psi(Q) \cap \mathscr{Z} \neq \emptyset$. □

Theorem 2.3. *Assume* $0 < \rho < r < \infty$, *let* $\tilde{e}_1 = e_1/\lambda_1^{1/p}, \tilde{e}_2 = e_2/\lambda_2^{1/p}$, *and*

$$Q = Q_r = \{u = t_1\tilde{e}_1 + t_2\tilde{e}_2 : \ \|u\| \leqslant r, t_2 \geqslant 0\},$$

$$\partial Q = \partial Q_r = \{u = t_1\tilde{e}_1 : \ |t_1| \leqslant r\} \cup \{u \in Q_r : \ \|u\| = r\},$$

$$Z_\rho = \{u \in \mathscr{Z} : \ \|u\| = \rho\}.$$

Then $\partial Q_r \subset Q_r$ *and* Z_ρ *link in* X.

Proof. $\partial Q_r \cap Z_\rho = \emptyset$ is obvious. Let $\psi : Q_r \to X$ be continuous and $\psi|_{\partial Q_r} = \text{id}$. Denote $d_1 = \text{dist}(\tilde{e}_1, \partial K)$ and define mapping $P : X \to \mathscr{E}_2$ as follows:

$$P(u) = (\min\{\text{dist}(u, \partial K), rd_1\})\tilde{e}_1 + (\|u\| - \rho)\tilde{e}_2, \text{ if } u \notin -K_1;$$

$$-(\min\{\text{dist}(u, \partial K), rd_1\})\tilde{e}_1 + (\|u\| - \rho)\tilde{e}_2, \text{ if } u \in -K_1.$$

It is easy to see that P is continuous, and P maps $v = r\tilde{e}_1$ to $v_1 = Pv = rd_1\tilde{e}_1 + (r - \rho)\tilde{e}_2$, the origin 0 to $0_1 = P0 = -\rho\tilde{e}_2$, the line segment $[v, 0]$ onto the line segment $[v_1, 0_1]$ homeomorphically; $-v = -r\tilde{e}_1$ to $v_2 = P(-v) = -rd_1\tilde{e}_1 + (r - \rho)\tilde{e}_2$, the line segment $[0, -v]$ onto a line segment $[0_1, v_2]$ homeomorphically; and the half circle $\{u \in \partial Q : \|u\| = r\}$ which is from v to $-v$ in ∂Q onto the line segment $[v_1, v_2]$, where $P(r\tilde{e}_2) = (r - \rho)\tilde{e}_2$.

Let $f = P \circ \psi : Q \to \mathscr{E}_2$. From the discussion above, there holds $0 \notin f(\partial Q)$, and when u turns a circuit along ∂Q anticlockingly, $f(u)$ also moves a circuit around the original 0 in $\mathscr{E}_2$ anticlockingly. Hence, by a degree argument, there holds $\deg(f, Q, 0) = 1$. So there exists some $u \in Q$ such that $f(u) = 0$, i.e., $P(\psi(u)) = 0$, which implies that $\psi(u) \in \partial K$; and $\|\psi(u)\| = \rho$. Thus $\psi(u) \in (\partial K)_\rho$ and $\psi(Q) \cap (\partial K)_\rho \neq \emptyset$. Since $(\partial K)_\rho \subset Z_\rho$, hence $\psi(Q) \cap \mathscr{Z} \neq \emptyset$. □

3. Existence results for problem (1.1)

In this section, we will give some conditions on $f(x, u)$ to guarantee the functional associated to problem (1.1) satisfies the Palais–Smale condition ((PS) condition) and the geometric assumptions of Mountain Pass Lemma (cf. Theorem 6.1 in Chapter 2 of [14]) in the case of $0 < \lambda < \lambda_1$; the $(C)_c$ condition due to Cerami and the assumptions of the linking theorem (cf. Theorem 8.4 in Chapter 2 of [14]) in the case of $\lambda_1 \leqslant \lambda < \lambda_2$.

Assume $f : \Omega \times R \to R$ satisfies:

(f_1) (*Subcritical growth*). $|f(x,s)| \leqslant c_1|s|^{r-1} + c_2\ \forall s \in R$, a.e. $x \in \Omega$, where $1 < r < p^* = \frac{NP}{N-P}$, if $1 < p < N$; $1 < r < +\infty$ if $p \geqslant N$;
(f_2) $f \in C(\bar{\Omega} \times R, R)$, $f(x,0) = 0$, $uf(x,u) \geqslant 0$, $u \in R$ and a.e. $x \in \Omega$;
(f_3) (*Asymptotic property at infinity*). $\exists \theta > p$ and $M > 0$ such that $0 < \theta F(x,u) \leqslant uf(x,u)$ for $|u| \geqslant M$ and a.e. $x \in \Omega$;
(f_4) (*Asymptotic property at $u = 0$*). $\lim_{s\to 0} f(x,s)/|s|^{p-1} = 0$ uniformly a.e. $x \in \Omega$.

Assumptions of (1.2) and (f_1) imply that functional $I : W_0^{1,p}(\Omega) \to R$:

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega V(x)|u|^p\,dx - \int_\Omega F(x,u)\,dx$$

is well-defined and $I \in C^1(W_0^{1,p}(\Omega); R)$, where $F(x,s) = \int_0^s f(x,t)\,dt$, and the weak solutions of problem (1.1) is equivalent to the critical points of I. (f_2) implies that 0 is a trivial solution to problem (1.1).

Lemma 3.1. *If f satisfies assumptions (f_1)–(f_3), then I satisfies the (PS) condition for $\lambda \in (0, \lambda_1)$.*

Proof. (1) The boundedness of (PS) sequence of I.

Suppose $\{u_m\}$ is a (PS) sequence of I, that is, there exists $C > 0$ such that $|I(u_m)| \leqslant C$ and $I'(u_m) \to 0$ in X', the dual space of X, as $m \to \infty$. The properties of the first eigenvalue λ_1 imply that for any $u \in X$, there holds

$$\lambda_1 \int_\Omega V(x)|u|^p\,dx \leqslant \int_\Omega |\nabla u|^p\,dx.$$

Let $d := \sup_m I(u_m)$. Then by the above inequality and (f_3), as $m \to \infty$, there holds

$$\begin{aligned} d - \frac{1}{\theta}o(1)\|u_m\| &= \left(\frac{1}{p} - \frac{1}{\theta}\right)\int_\Omega |\nabla u_m|^p - \lambda\left(\frac{1}{p} - \frac{1}{\theta}\right)\int_\Omega V(x)|u_m|^p \\ &\quad + \int_\Omega \left(\frac{1}{\theta} f(x,u_m)u_m - F(x,u_m)\right) \\ &\geqslant \left(\frac{1}{p} - \frac{1}{\theta}\right)\left(1 - \frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla u_m|^p \\ &\quad + \int_{\Omega(u_m \geqslant M)} \left(\frac{1}{\theta} f(x,u_m)u_m - F(x,u_m)\right) \end{aligned}$$

$$+\int_{\Omega(u_m<M)}\left(\frac{1}{\theta}f(x,u_m)u_m - F(x,u_m)\right)$$
$$\geqslant \left(\frac{1}{p}-\frac{1}{\theta}\right)\left(1-\frac{\lambda}{\lambda_1}\right)\|u_m\|^p - C_1,$$

where $C_1 \geqslant 0$ is a constant independent of u_m. The above estimate implies the boundedness of $\{u_m\}$ for $0<\lambda<\lambda_1$.

(2) By (f_1), f satisfies the subcritical growth condition, by a standard argument, one can obtain that there exists a convergent subsequence of $\{u_m\}$ from the boundedness of $\{u_m\}$ in X. □

Theorem 3.2. *If f satisfies assumptions (f_1)–(f_4), then problem (1.1) has a nontrivial weak solution $u\in W_0^{1,p}(\Omega)$ provided that $0<\lambda<\lambda_1$.*

Proof. We will verify the geometric assumptions of the Mountain Pass Lemma (cf. [14] Chapter 2, Theorem 6.1):

(1) $I(0)=0$ is obvious;
(2) $\exists\rho>0,\ \alpha>0:\ \|u\|=\rho \Rightarrow I(u)\geqslant\alpha$;

In fact, $\forall u\in X$, there holds

$$I(u)=\frac{1}{p}\int_\Omega|\nabla u|^p-\frac{\lambda}{p}\int_\Omega V(x)|u|^p-\int_\Omega F(x,u)$$
$$\geqslant\frac{1}{p}\left(1-\frac{\lambda}{\lambda_1}\right)\int_\Omega|\nabla u|^p-\int_\Omega F(x,u). \tag{3.1}$$

From (f_4), $\forall\varepsilon>0$, $\exists\rho_0=\rho_0(\varepsilon)$ such that: if $0<\rho=\|u\|<\rho_0$, then $|f(x,u)|<\varepsilon|u|^{p-1}$, thus

$$\int_\Omega F(x,u)\,\mathrm{d}x\leqslant\int_\Omega\int_0^{u(x)}f(x,t)\,\mathrm{d}t\,\mathrm{d}x\leqslant\frac{\varepsilon}{p}\int_\Omega|u(x)|^p\,\mathrm{d}x\leqslant\frac{c_0\varepsilon}{p}\|u\|^p_{W_0^{1,p}}.$$

Choose $c_0\varepsilon_0=(1-\frac{\lambda}{\lambda_1})/2>0$, $\rho=\frac{\rho_0(\varepsilon_0)}{2}$, from (3.1), one has

$$I(u)\geqslant\frac{1}{p}\left(1-\frac{\lambda}{\lambda_1}-c_0\varepsilon_0\right)\int_\Omega|\nabla u|^p\geqslant\frac{\lambda_1-\lambda}{2\lambda_1 p}\cdot\rho\ =:\alpha>0. \tag{3.2}$$

(3) $\exists u_1\in X:\|u_1\|\geqslant\rho$ and $I(u_1)<0$.

In fact, from (f_2) and (f_3), one can deduce that there exist constants c_3, $c_4>0$ such that

$$F(x,u)\geqslant c_3|u|^\theta-c_4\quad\forall u\in R. \tag{3.3}$$

Since $\theta > p$, a simple calculation shows that as $t \to \infty$, there holds

$$I(te_1) = \frac{t^p}{p}\int_\Omega |\nabla e_1|^p - \frac{\lambda t^p}{p}\int_\Omega V(x)|e_1|^p - \int_\Omega F(x,te_1)$$
$$\leqslant \frac{t^p}{p}\int_\Omega |\nabla e_1|^p - \frac{\lambda t^p}{p}\int_\Omega V(x)|e_1|^p - c_3 t^\theta \int_\Omega |e_1|^\theta + c_4|\Omega|$$
$$\to -\infty. \tag{3.4}$$

Eq. (3.4) implies that $I(te_1) < 0$ for $t > 0$ large enough.

Thus Lemma 3.1 and the Mountain Pass Lemma imply that value

$$\beta = \inf_{p\in P}\sup_{u\in p} E(u) \geqslant \alpha > 0$$

is critical, where $P = \{p \in C^0([0,1];X) : p(0) = 0, p(1) = u_1\}$. That is, there is a $u \in X$, such that

$$E'(u) = 0,\ E(u) = \beta > 0.$$

$E(u) = \beta > 0$ implies $u \not\equiv 0$. □

Lemma 3.3. *Assume that f satisfies assumptions (f_1)–(f_3). Furthermore, $\theta > ps/(s-1)$ in* (f_3). *Then for any $\lambda \in R$, I satisfies the $(C)_c$ condition introduced by Cerami in* [6], *that is, any sequence $\{u_m\} \subset X$ such that $I(u_m) \to c$ and $(1 + \|u_m\|)\|I'(u_m)\|_{X'} \to 0$ possesses a convergent subsequence.*

Proof. The boundedness of $(C)_c$ sequence in X.

Let $\{u_m\} \subset X$ be such that $I(u_m) \to c$ and $(1 + \|u_m\|)\|I'(u_m)\|_{X'} \to 0$. Then from ($f_2$), ($f_3$) and (3.3), as $m \to \infty$, there holds

$$pc + o(1) = pI(u_m) - \langle I'(u_m), u_m\rangle = \int_\Omega (u_m f(x,u_m) - pF(x,u_m))\,dx$$
$$= \int_\Omega (u_m f(x,u_m) - \theta F(x,u_m))\,dx + (\theta - p)\int_\Omega \theta F(x,u_m)\,dx$$
$$\geqslant -C_1 + (\theta - p)c_3|u_m|_\theta^\theta - c_4|\Omega|. \tag{3.5}$$

Thus $\theta > p$ implies the boundedness of $\{u_m\}$ in $L^\theta(\Omega)$. Since $\theta > ps/(s-1)$, the Hölder inequality and the boundedness of Ω show that

$$\left|\int_\Omega V(x)|u_m|^p\,dx\right| \leqslant \|V(x)\|_{L^s}\|u_m\|^p_{L^{ps/(s-1)}}$$
$$\leqslant C\|V(x)\|_{L^s}\|u_m\|^p_{L^\theta}, \tag{3.6}$$

which, together with the boundedness of $\{u_m\}$ in $L^\theta(\Omega)$, means that $\{|\int_\Omega V(x)|u_m|^p\,dx|\}$ is bounded. Then from (f_3), a simple calculation shows that

$$\begin{aligned}
\theta c + o(1) &= \theta I(u_m) - \langle I'(u_m), u_m\rangle \\
&= \left(\frac{\theta}{p}-1\right)\|\nabla u_m\|_{L^p}^p - \lambda\left(\frac{\theta}{p}-1\right)\int_\Omega V(x)|u_m|^p\,dx \\
&\quad + \int_\Omega (u_m f(x,u_m) - \theta F(x,u_m))\,dx \\
&\geqslant \left(\frac{\theta}{p}-1\right)\int_\Omega |\nabla u_m|^p\,dx - C \\
&\quad + \int_{\Omega(u_m<M)} (u_m f(x,u_m) - \theta F(x,u_m))\,dx \\
&\quad + \int_{\Omega(u_m\geqslant M)} (u_m f(x,u_m) - \theta F(x,u_m))\,dx \\
&\geqslant \left(\frac{\theta}{p}-1\right)\|\nabla u_m\|_{L^p}^p - C, \qquad (3.7)
\end{aligned}$$

where $C>0$ is a universal constant independent of u_m, which may be different from line to line. Thus $\theta > p$ and (3.7) imply the boundedness of $\{u_m\}$ in X.

(2) By (f_1), f satisfies the subcritical growth condition, by a standard argument, one can obtain that there exists a convergent subsequence of $\{u_m\}$ from the boundedness of $\{u_m\}$ in X. □

Theorem 3.4. *Suppose f satisfies assumptions (f_1)–(f_4), and furthermore, $\theta > ps/(s-1)$ in (f_3). Then problem (1.1) has a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ provided that $\lambda_1 \leqslant \lambda < \lambda_2$.*

Proof. It was shown in [3] that $(C)_c$ condition actually suffices to get a deformation theorem (Theorem 1.3 in [3], see also Proposition 2.1 in [12]), which is crucial for minimax type critical point theory, and it also remarked in [3,8,12] that the proofs of the standard Mountain Pass Lemma and saddle-point theorem go through without change once the deformation theorem (Theorem 1.3 in [3]) is obtained with $(C)_c$ condition. Here we verify the assumptions of standard Linking Argument Theorem (cf. [15] Chapter 2, Theorem 8.4) hold with $(C)_c$ condition replacing $(PS)_c$ condition.

Since $\partial Q_r \subset Q_r$ and Z_ρ link in X, it suffices to show that

(1) $\alpha_0 = \sup_{u\in\partial Q_r} I(u) \leqslant 0$, when $r>0$ is large enough;
(2) $\alpha = \inf_{u\in Z_\rho} I(u) > 0$, when $\rho > 0$ is small enough.

In fact, let $u=te_1\in\mathscr{E}_1$, from assumption (f_2), $F(x,s)\geqslant 0$ for all $s\in R$ and almost every $x\in\Omega$, thus there holds

$$I(u)=I(te_1)\leqslant\frac{|t|^p}{p}\int_\Omega|\nabla e_1|^p\,\mathrm{d}x-\frac{|t|^p\lambda}{p}\int_\Omega V(x)|e_1|^p\,\mathrm{d}x$$
$$=\frac{|t|^p}{p}\left(1-\frac{\lambda}{\lambda_1}\right)\|e_1\|\leqslant 0. \tag{3.8}$$

Noticing that

$$\|u\|_\theta=\left(\int_\Omega|u|^\theta\right)^{1/\theta}$$

is a norm on $\mathscr{E}_2$, and the norms of finite dimensional space are equivalent, thus there exists a constant $c_5>0$ such that

$$\int_\Omega|u|^\theta\,\mathrm{d}x\geqslant c_5\|u\|^\theta_{W_0^{1,p}}.$$

From (3.3), there holds

$$I(u)\leqslant\frac{1}{p}\|u\|^p_{W_0^{1,p}}+\frac{S_1^p\lambda}{p}\|V(x)\|_{L^s}\|u\|^p_{W_0^{1,p}}-c_3c_5\|u\|^\theta_{W_0^{1,p}}+c_4|\Omega|, \tag{3.9}$$

where S_1 is the best constant of imbedding $X\hookrightarrow L^{ps/(s-1)}(\Omega)$. Since $\theta>p$, there holds

$$I(u)\to-\infty,\ \text{as}\ \|u\|\to\infty,\ u\in\mathscr{E}_2.$$

This implies (1).

From (f_4) and (f_1), there holds

$$\int_\Omega F(x,u)\,\mathrm{d}x=\mathrm{o}(\|u\|^p)\ \text{as}\ u\to 0\ \text{in}\ X,$$

then for any $u\in Z$, there holds

$$I(u)=\frac{1}{p}\left(1-\frac{\lambda}{\lambda_2}\right)\int_\Omega|\nabla u|^p\,\mathrm{d}x+\mathrm{o}(\|u\|^p). \tag{3.10}$$

Since $\lambda<\lambda_2$, (3.10) implies (2).

Thus the Linking Argument Theorem (cf. [14] Chapter 2, Theorem 8.4)with $(C)_c$ condition replacing $(\mathrm{PS})_c$ condition implies that value

$$\beta=\inf_{h\in\Gamma}\sup_{u\in Q}E(h(u))\geqslant\alpha>0$$

is critical, where $\Gamma=\{h\in C^0(X;X);\ h|_{\partial Q}=\mathrm{id}\}$. That is, there is a $u\in X$, such that

$$E'(u)=0,\ E(u)=\beta>0.$$

$E(u)=\beta>0$ implies $u\not\equiv 0$. $\square$

Remark 3.1.

(1) If $V(x) \in L^\infty$, then $ps/(s-1) = p$. In particular, $V(x) \equiv 1$, then Theorem 3.4 recovers a partial result in [13].
(2) It is well-known that problem (1.1) has a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ for all $\lambda \in R$ if $p = 2$, $V(x) \equiv 1$ (cf. [2] and [7]). We conjucture that problem (1.1) has a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ for all $\lambda \in R$ in our case, that is, $p > 1$, $V(x)$ satisfies condition (1.2).

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