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The Influence of Slip Condition on the Thin Film Flow of a Third Order Fluid

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Abstract: This paper deals with the influence of slip condition on a thin film flow of a third order fluid. We investigate the thin film flow of non-Newtonian fluid (i) when moves down an inclined plane and (ii) when moves on a moving belt with slip condition using the traditional perturbation technique and HPM. The results obtained using both techniques are compared. The expressions for volume flux and average film velocity are also expressed.

Keywords: Perturbation method; inclined plane; homotopy perturbation method; moving belt

1 Introduction

In recent years, especially with the emergence of polymer industry, petroleum industries and other types of pulp industries, the non Newtonian fluids have became very much important. Due to complexity of non Newtonian fluids, it becomes difficult to suggest a single model which exhibits all properties of non Newtonian fluids, therefore various empirical and semi empirical models have been proposed. Non Newtonian fluids can mainly be classified into two classes such as differential type fluids and rate type fluids. Among these two classes, the differential type fluids have received great attention from scientists and engineers. A second order fluid is one of the most acceptable fluid in this subclass of non Newtonian fluids. This is because of its mathematical simplicity in comparison to third order and fourth order fluids. However, there are studies available in literature in which the authors have successfully treated the challenging nonlinear equations governing the flow of a third order fluid [1-5].

A survey of literature indicates that much attention is given to slip effect, especially from polymer industry(polymer melts), which exhibits a macroscopic wall slip. It ranges from technological application to medical application, especially in polishing artificial heart valves. Being inspired from such practical applications, several authors discussed the slip effect on fluid flow. T. Hayat [6] and Asghar [7] discuss the effects of slip condition on third order fluid. Ellahi [8] discuss the slip condition of an Oldroyd 8- constant fluid and M. Sajid [9] investigate the effect of slip condition on thin film flow. Being inspired from applications of slip conditions and authors, we have determined the effects of slip condition on viscous flow of third order fluid:

(i) When fluid moves down and inclined plane

(ii) When fluid moves on a belt

This problem was first studied by M. Sajid [10], in which they studied the thin film flow of third order fluid [11].

While dealing with the non Newtonian fluids are of the great challenge in the solution of governing nonlinear differential equations. Number of the numerical and analytical techniques have been proposed by various researchers. However, an efficient analytic solution still finds great appreciations. Keeping this fact in mind, we have solved the governing nonlinear equations of present problem using the two powerful analytic techniques namely, the traditional perturbation method [12] and homotopy perturbation method [13,14]. It is important to mention here that the two solutions are in a complete agreement and the previous results of M. Sajid [10] can easily be recovered by substituting the slip parameter equal to zero. In this study, it is also observe that the homotopy perturbation method is a powerful analytical technique that is simple and straightforward and does not require the existence of any small or large parameter as does traditional perturbation method. Homotopy perturbation method has successfully been applied to a number of nonlinear problems arising in the science and engineering by various researchers [15-17]. This proves the validity and acceptability of HPM as a useful solution technique.

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The distribution of this paper is in five sections. In Section 2, we formulate the problem of thin film flow down an inclined plane with slip conditions, and also includes solution of problem by perturbation method and HPM. In Section 3, we formulate the problem of thin film flow on moving belt with slip conditions, , and also includes solution of problem by perturbation method and HPM. Results and discussion are presented in Section 4 and Some concluding remarks are given in Section 5.

2 Influence of slip condition on a thin film flow down an inclined plane

2.1 Formulation of problem

The governing equations of third order unidirectional thin film flow down an inclined plane of inclination $\alpha \neq 0$ consist of incompressibility condition are

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{B} + div\mathbf{T}, \tag{2}$$

where ρ is fluid density, **V** is velocity vector, p is pressure, **B** is body force, **T** is Cauchy stress tensor and $\frac{D}{Dt}$ denoting the material time derivative. The Cauchy stress tensor in a third order fluid is given by

$$\mathbf{T} = -pI + \mu A_1 + a_1 A_2 + a_2 A_1^2 + S,$$

where I is the identity tensor , μ is coefficient of viscosity and $\alpha_i (i = 1, 2)$ are material coefficients. The kinematical tensors $A_k (k = 1, 2, 3)$ are Rivlin-Ericksen tensors and S is extra stress tensor. The Rivlin-Ericksen tensors $A_k (k = 1, 2, 3)$ and extra stress tensor S for third order fluid is given by

$$S = \beta_1 A_3 + \beta_2 (A_2 A_1 + A_1 A_2) + \beta_3 \left(trA_1^2\right) A_1,$$

where

$$\begin{aligned} A_0 &= I, \;\; A_1 = L + L^T, \\ A_n &= \frac{dA_{n-1}}{dt} + A_{n-1}L + L^T A_{n-1}, \; n = 2, 3, \cdots, \\ L &= \nabla \mathbf{V}, \end{aligned}$$

where ∇ is the gradient operator and $\frac{d}{dt}$ is the material time derivative defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}.\nabla).$$

For simplicity, some assumptions are made
(i) The ambient air is stationary
(ii) Surface tension is negligible
(iii) Thin film is of uniform thickness δ
(iv) Thermal effects are negligible
(v) Pressure gradient is absent.
We have a velocity field of the form

$$\mathbf{V} = (u(y), 0, 0).$$

By using the above assumptions and substituting the values of **V** and **T** in eqs.
(1) and (2), we get the following non-linear second order ordinary differential eqution

$$\mu \frac{\partial^2 u}{\partial y^2} + 6(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y}\right)^2 \left(\frac{\partial^2 u}{\partial y^2}\right) + \rho g \sin \alpha = 0 \tag{3}$$

The boundary conditions on u are

$$u - \gamma \left[\mu \left(\frac{\partial u}{\partial y}\right) + 2(\beta_2 + \beta_3) \left(\frac{\partial u}{\partial y}\right)^3\right] = 0 \;\; at \;\; y = 0, \tag{4}$$

$$\frac{du}{dy} = 0 \;\; at \;\; y = \delta. \tag{5}$$

Eq. (4) is slip condition, where γ is coefficient of slip and eq. (5) comes from $\tau_{yx} = 0$ at $y = \delta$. In order to carry out the non-dimensional analysis, we define the following variables

$$\begin{cases} u = \frac{u^*\nu}{\delta}, & y = \delta y^*, \\ \beta^* = \frac{(\beta_2+\beta_3)\nu^2}{\delta^4\mu}, & m^* = \frac{\delta^3 g sin\alpha}{\nu^2}. \end{cases} \tag{6}$$

Using Eq. (6) in Eqs. (3)-(5), we get

$$\begin{aligned} \frac{d^2u^*}{dy^{*2}} + 6\beta^*\left(\frac{du^*}{dy^*}\right)^2\frac{d^2u^*}{dy^{*2}} + m^* &= 0, \\ u^* - \gamma\left[\frac{\mu}{\delta}\left(\frac{\partial u^*}{\partial y^*}\right) + 2\left(\beta^*\mu\right)\left(\frac{\partial u^*}{\partial y^*}\right)^3\right] &= 0 \ \ at \ \ y^* = 0, \\ \frac{du^*}{dy^*} &= 0 \ \ at \ \ y^* = 1. \end{aligned}$$

For simplicity, we drop the asterisks

$$\frac{d^2u}{dy^2} + 6\beta\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} + m = 0, \tag{7}$$

$$u - \gamma\left[\frac{\mu}{\delta}\left(\frac{\partial u}{\partial y}\right) + 2\beta\mu\left(\frac{\partial u}{\partial y}\right)^3\right] = 0 \ \ at \ \ y = 0, \tag{8}$$

$$\frac{du}{dy} = 0 \ \ at \ \ y = 1. \tag{9}$$

We take $\epsilon = \beta$ and solve the system of equations (7)-(9) by traditional perturbation method and also by homotopy perturbation method.

2.2 Solution of the problem by perturbation method

We assume ϵ to be a small parameter and expand $u(y, \epsilon)$ in the Poincare-type series of the form

$$u\left(y, \epsilon\right) = u_0(y) + \epsilon u_1(y) + \epsilon^2 u_2(y) + \cdots. \tag{10}$$

Substituting (10) into (7)-(9) and equating coefficients of like powers of ϵ, we get the following problems of different orders.

Zeroth order problem and its solution
The differential equation of zeroth order problem is

$$O\left(\epsilon^0\right): \qquad \frac{d^2u_0}{dy^2} = -m, \tag{11}$$

with boundary conditions

$$u_0 - \frac{\gamma\mu}{\delta}\left(\frac{\partial u_0}{\partial y}\right) = 0 \ \ at \ \ y = 0, \tag{12}$$

$$\left(\frac{\partial u_0}{\partial y}\right) = 0 \ \ at \ \ y = 1. \tag{13}$$

The solution of system of equations (11)-(13) is given by

$$u_0 = \left(\frac{-my^2}{2} + my\right) + \frac{\gamma\mu m}{\delta}. \tag{14}$$

First order problem and its solution
The differential equation of first order problem is

$$O\left(\epsilon^1\right): \qquad \frac{d^2u_1}{dy^2} + 6\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_0}{dy^2}\right) = 0, \tag{15}$$

with boundary conditions

$$u_1 - \frac{\gamma\mu}{\delta}\left(\frac{\partial u_1}{\partial y}\right) - 2\gamma\mu\left(\frac{\partial u_0}{\partial y}\right)^3 = 0 \ \ at \ \ y=0, \tag{16}$$

$$\left(\frac{\partial u_1}{\partial y}\right) = 0 \ \ at \ \ y=1. \tag{17}$$

The solution of system of equations (15)-(17) is given by

$$u_1 = 6m^3\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{12} - \frac{y}{3} + \left(\frac{\gamma\mu}{3}\right)\left(1 - \frac{m^3}{\delta}\right)\right] \tag{18}$$

Second order problem and its solution
The second order problem together with boundary conditions is

$$O\left(\epsilon^2\right): \qquad \frac{d^2u_2}{dy^2} + 6\left(\frac{du_0}{dy}\right)^2\left(\frac{d^2u_0}{dy^2}\right) + 12\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)\left(\frac{d^2u_0}{dy^2}\right) = 0, \tag{19}$$

$$u_2 - \frac{\gamma\mu}{\delta}\left(\frac{\partial u_2}{\partial y}\right) - 6\gamma\mu\left(\frac{\partial u_0}{\partial y}\right)^2\left(\frac{\partial u_1}{\partial y}\right) = 0 \, at \, y=0, \tag{20}$$

$$\left(\frac{\partial u_2}{\partial y}\right) = 0 \, at \, y=1. \tag{21}$$

The solution of system of equations (19)-(21) is given by

$$u_2 = 36m^5\left[-\frac{y^6}{18} + \frac{y^5}{3} - 5\frac{y^4}{6} + 10\frac{y^3}{9} - 5\frac{y^2}{6} + \frac{y}{3} - \gamma\mu\left(\frac{1}{\delta} - \frac{1}{3}\right)\right], \tag{22}$$

By using eqs.(14), (18) and (22) in eq.(10), we obtain

$$\begin{aligned} u(y) = {} & \left(\frac{-my^2}{2} + my\right) + \frac{\gamma\mu m}{\delta} + 6\epsilon m^3\left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{12} - \frac{y}{3} + \left(\frac{\gamma\mu}{3}\right)\left(1 - \frac{m^3}{\delta}\right)\right] \\ & + 36\epsilon^2 m^5\left[-\frac{y^6}{18} + \frac{y^5}{3} - 5\frac{y^4}{6} + 10\frac{y^3}{9} - 5\frac{y^2}{6} + \frac{y}{3} - \gamma\mu\left(\frac{1}{\delta} - \frac{1}{3}\right)\right] \end{aligned}$$

2.3 Solution of the problem by homotopy perturbation method

The problem under consideration i.e. eqs. (7)-(9) can be written as

$$L(V) - L\left(u_0\right) + qL\left(u_0\right) + q\left[6\beta\left(\frac{dv}{dy}\right)^2\left(\frac{d^2v}{dy^2} + m\right)\right] = 0, \tag{23}$$

where $L = \frac{d^2}{dy^2}$ and $u_0 = \left(\frac{-my^2}{2} + my\right)$ is initial guess approximation. Substitute
$V(y) = v_0 + qv_1 + q^2v_2 + \cdots$. in eq. (23), we have

Zeroth order problem and its solution
The zeroth order problem is given by

$$L\left(v_0\right) - L\left(u_0\right) = 0, \tag{24}$$

subject to the boundary conditions

$$v_0 - \frac{\gamma\mu}{\delta}\left(\frac{\partial v_0}{\partial y}\right) = 0 \ \ at \ \ y=0, \tag{25}$$

$$\left(\frac{\partial v_0}{\partial y}\right) = 0 \ \ at \ \ y=1. \tag{26}$$

The solution of system of equations (24)-(26) is given by

$$v_0 = \left(\frac{-my^2}{2} + my \right) + \frac{\gamma \mu m}{\delta}. \tag{27}$$

First order problem and its solution
The differential equation of first order problem is given by

$$L(v_1) - L(v_0) + 6\beta \left(\frac{du_0}{dy} \right)^2 \left(\frac{d^2 u_0}{dy^2} \right) + m = 0 \tag{28}$$

with boundary conditions

$$v_1 - \frac{\gamma\mu}{\delta} \left(\frac{\partial v_1}{\partial y} \right) - 2\gamma\mu \left(\frac{\partial v_1}{\partial y} \right)^3 = 0 \;\; at \;\; y = 0, \tag{29}$$

$$\left(\frac{\partial v_1}{\partial y} \right) = 0 \;\; at \;\; y = 1. \tag{30}$$

The solution of system of equations (28)-(30) is given by

$$v_1 = 6m^3 \beta \left[\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2} - \frac{y}{3} + \left(\frac{\gamma\mu}{3} \right) \left(1 - \frac{m^3}{\delta} \right) \right] \tag{31}$$

Second order problem and its solution
The second order problem is given by

$$L(v_2) + 6 \left(\frac{du_0}{dy} \right)^2 \left(\frac{d^2 u_0}{dy^2} \right) + 12 \left(\frac{du_0}{dy} \right) \left(\frac{du_1}{dy} \right) \left(\frac{d^2 u_0}{dy^2} \right) = 0, \tag{32}$$

with the boundary conditions

$$v_2 - \frac{\gamma\mu}{\delta} \left(\frac{\partial v_2}{\partial y} \right) - 6\gamma\mu \left(\frac{\partial u_0}{\partial y} \right)^2 \left(\frac{\partial v_1}{\partial y} \right) = 0 \;\; at \;\; y = 0, \tag{33}$$

$$\left(\frac{\partial v_2}{\partial y} \right) = 0 \;\; at \;\; y = 1. \tag{34}$$

The solution of system of equations (32)-(34) is given by

$$v_2 = 36m^5 \beta^2 \left[-\frac{y^6}{18} + \frac{y^5}{3} - 5\frac{y^4}{6} + 10\frac{y^3}{9} - 5\frac{y^2}{6} + \frac{y}{3} - \gamma\mu \left(\frac{1}{\delta} - \frac{1}{3} \right) \right], \tag{35}$$

The velocity field obtained by the homotopy perturbation method is

$$u(y) = \lim_{p \to 1} V = v_0 + v_1 + v_2 + \cdots.$$

By substituting the values of v_0, v_1 and v_2 from eqs.(27), (31) and (35) in the above expression, we get the solution up to second order as

$$\begin{aligned} u(y) &= \left(\frac{-my^2}{2} + my \right) + \frac{\gamma\mu m}{\delta} + 6m^3\beta \left[\frac{y^4}{12} - \frac{y^3}{3} + \frac{y^2}{2} - \frac{y}{3} + \left(\frac{\gamma\mu}{3} \right) \left(1 - \frac{m^3}{\delta} \right) \right] \\ &+ 36m^5\beta^2 \left[-\frac{y^6}{18} + \frac{y^5}{3} - 5\frac{y^4}{6} + 10\frac{y^3}{9} - 5\frac{y^2}{6} + \frac{y}{3} - \gamma\mu \left(\frac{1}{\delta} - \frac{1}{3} \right) \right]. \end{aligned} \tag{36}$$

2.4 Volume flux and average velocity

The flow rate per unit width is given by

$$\frac{Q}{W} = \int_0^\delta u(y) dy,$$

By substituting the values of u(y) from eq. (36) in above equation, we get

$$\frac{Q}{W} = \left(\frac{-m\delta^3}{6}+\frac{m\delta^2}{2}\right)+\gamma\mu m+6m^3\beta\left[\frac{\delta^5}{60}-\frac{\delta^4}{12}+\frac{\delta^3}{6}-\frac{\delta^2}{6}+\left(\frac{\gamma\mu\delta}{3}\right)\left(1-\frac{m^3}{\delta}\right)\right]$$
$$+36m^5\beta^2\left[-\frac{\delta^7}{126}+\frac{\delta^6}{18}-\frac{\delta^5}{6}+5\frac{\delta^4}{18}-5\frac{\delta^3}{18}+\frac{\delta^2}{6}-\gamma\mu\delta\left(\frac{1}{\delta}-\frac{1}{3}\right)\right].$$

The average velocity $\bar{u}(y)$ over the cross section of the film is given by

$$\bar{u}\left(y\right)=\frac{Q}{W\delta}.$$

$$\bar{u}(y) = \left(\frac{-m\delta^2}{6}+\frac{m\delta}{2}\right)+\frac{\gamma\mu m}{\delta}+6\beta m^3\left[\frac{\delta^4}{60}-\frac{\delta^3}{12}+\frac{\delta^2}{6}-\frac{\delta}{6}+\left(\frac{\gamma\mu}{3}\right)\left(1-\frac{m^3}{\delta}\right)\right]$$
$$+36\beta^2m^5\left[-\frac{\delta^6}{126}+\frac{\delta^5}{18}-\frac{\delta^4}{6}+5\frac{\delta^3}{18}-5\frac{\delta^2}{18}+\frac{\delta}{6}-\gamma\mu\left(\frac{1}{\delta}-\frac{1}{3}\right)\right].$$

3 Influence of slip condition on a thin film flow on a moving belt

3.1 Problem formulation

The governing equations of third order, unidirectional thin film flow on a moving belt, consists of incompressibility condition are

$$\mu\frac{\partial^2 v}{\partial x^2}+6\left(\beta_2+\beta_3\right)\left(\frac{\partial v}{\partial x}\right)^2\left(\frac{\partial^2 v}{\partial x^2}\right)+\rho g = 0, \tag{37}$$

$$v-\gamma\left[\mu\delta\left(\frac{\partial v}{\partial x}\right)+2\left(\beta_2+\beta_3\right)\left(\frac{\partial v}{\partial x}\right)^3\right] = 0 \;\; at \;\; x=0, \tag{38}$$

$$\frac{dv}{dx} = 0 \;\; at \;\; x=\delta. \tag{39}$$

Eq.(38) is slip condition, where γ is coefficient of slip and eq. (39) comes from $\tau_{yx}=0$ at $y=\delta$, where v is fluid velocity, μ is dynamic viscosity, ρ the density, β_2 and β_3 are material moduli of third order fluid, g the acceleration due to gravity and δ is thickness of the thin film flow. In order to carry out the non-dimensional analysis, we define the following variables

$$v^*=\frac{v}{U_0}, \quad x^*=\frac{x}{\delta}. \tag{40}$$

The flow problem consisting of eqs. (37)-(39) becomes

$$\frac{d^2v}{dx^2}+\epsilon\left(\frac{dv}{dx}\right)^2\frac{d^2v}{dx^2}-\lambda = 0, \tag{41}$$

$$v-\gamma\left[\frac{\mu}{\delta}\left(\frac{\partial v}{\partial x}\right)+\left(\frac{\mu}{3*\delta}\right)\left(\frac{\partial v}{\partial x}\right)^3\right] = 1 \;\; at \;\; x=0, \tag{42}$$

$$\frac{dv}{dx} = 0 \;\; at \;\; x=1. \tag{43}$$

where

$$\epsilon=\frac{6\left(\beta_2+\beta_3\right)U_0^2}{\mu\delta^2}, \quad k=\frac{\rho g\delta^2}{\mu U_0^2}.$$

Note: For simplicity, we droped the asterisks.

3.2 Solution of the problem by perturbation method

We assume ϵ to be a small parameter and expand $v(x,\epsilon)$ in the Poincare-type series of the form

$$v\left(x,\epsilon\right)=v_0(x)+\epsilon v_1(x)+\epsilon^2 v_2(x)+\cdots. \tag{44}$$

Zeroth order problem and its solution
The differential equation of zeroth order problem is

$$O\left(\epsilon^0\right): \quad \frac{d^2v_0}{dx^2}=k, \tag{45}$$

with boundary conditions

$$v_0-\frac{\gamma\mu}{\delta}\left(\frac{\partial v_0}{\partial x}\right) = 1 \;\; at \;\; x=0, \tag{46}$$

$$\left(\frac{\partial v_0}{\partial x}\right) = 0 \;\; at \;\; x=1. \tag{47}$$

The solution of system of equations (45)-(47) is given by

$$v_0=kx\left(\frac{x}{2}-1\right)+\left(1-\frac{\gamma\mu k}{\delta}\right), \tag{48}$$

First order problem and its solution
The first order problem is given by

$$O\left(\epsilon^1\right): \quad \frac{d^2v_1}{dx^2}+6\left(\frac{dv_0}{dx}\right)^2\left(\frac{d^2v_0}{dx^2}\right)=0, \tag{49}$$

with boundary conditions

$$v_1-\frac{\gamma\mu}{\delta}\left(\frac{\partial v_1}{\partial x}\right)-(\frac{\gamma\mu}{3*\delta})\left(\frac{\partial v_0}{\partial x}\right)^3 = 0 \;\; at \;\; x=0, \tag{50}$$

$$\left(\frac{\partial v_1}{\partial x}\right) = 0 \;\; at \;\; x=1. \tag{51}$$

The solution of system of equations (49)-(51) is given by

$$v_1=-k^3\left(\frac{x^4}{12}-\frac{x^3}{3}+\frac{x^2}{2}-\frac{x}{3}\right). \tag{52}$$

Second order problem and its solution
The differential equation of second order problem is

$$O\left(\epsilon^2\right): \frac{d^2v_2}{dx^2}+6\left(\frac{dv_0}{dx}\right)^2\left(\frac{d^2v_0}{dx^2}\right)+12\left(\frac{dv_0}{dx}\right)\left(\frac{dv_1}{dx}\right)\left(\frac{d^2v_0}{dx^2}\right)=0, \tag{53}$$

with boundary conditions

$$v_2-\frac{\gamma\mu}{\delta}\left(\frac{\partial v_2}{\partial x}\right)-\frac{\gamma\mu}{\delta}\left(\frac{\partial v_0}{\partial x}\right)^2\left(\frac{\partial v_1}{\partial x}\right) = 0 \;\; at \;\; x=0, \tag{54}$$

$$\left(\frac{\partial v_2}{\partial x}\right) = 0 \;\; at \;\; x=1. \tag{55}$$

The solution of system of equations (53)-(55) is given by

$$v_2=k^5\left[\frac{x^6}{18}-\frac{x^5}{3}+5\frac{x^4}{6}-10\frac{x^3}{9}+5\frac{x^2}{6}-\frac{x}{3}\right]. \tag{56}$$

Substituting the values of v_0, v_1 and v_2 from eqs.(48), (52) and (56) in eq.(44), we get the homotopy perturbation solution up to second order as

$$\begin{aligned} v(x,\epsilon) &= kx\left(\frac{x}{2}-1\right)+\left(1-\frac{\gamma\mu k}{\delta}\right)-\epsilon k^3\left(\frac{x^4}{12}-\frac{x^3}{3}+\frac{x^2}{2}-\frac{x}{3}\right) \\ &+\epsilon^2 k^5\left[\frac{x^6}{18}-\frac{x^5}{3}+5\frac{x^4}{6}-10\frac{x^3}{9}+5\frac{x^2}{6}-\frac{x}{3}\right]. \end{aligned} \tag{57}$$

3.3 Solution of the problem by homotopy perturbation method

The problem under consideration i.e. eqs. (41)-(43) can be written as

$$L(w)-L(v_0)+qL(v_0)+q\left[\beta\left(\frac{dw}{dx}\right)^2\left(\frac{d^2w}{d^2x}\right)-k\right]=0, \tag{58}$$

where $L=\frac{d^2}{dx^2}$ and $v_0=\frac{k}{2}\left(x^2-2x\right)+1$ is initial guess approximation. Substitute $w=w_0+qw_1+q^2w_2+\cdots$ in eq. (58), we have

Zeroth order problem and its solution
The zeroth order problem is given by

$$L(w_0)-L(v_0)=0, \tag{59}$$

subject to the boundary conditions

$$w_0-\frac{\gamma\mu}{\delta}\left(\frac{\partial w_0}{\partial x}\right) = 1 \ \ at \ \ x=0, \tag{60}$$

$$\left(\frac{\partial w_0}{\partial x}\right) = 0 \ \ at \ \ x=1. \tag{61}$$

The solution of system of equations (59)-(61) is given by

$$w_0=kx\left(\frac{x}{2}-1\right)+\left(1-\frac{\gamma\mu k}{\delta}\right). \tag{62}$$

First order problem and its solution
The first order problem is given by

$$L(w_1)+L(v_0)+\beta\left[\left(\frac{dw_0}{dx}\right)^2\left(\frac{d^2w_0}{dx^2}\right)-k\right]=0, \tag{63}$$

with boundary conditions

$$w_1-\frac{\gamma\mu}{\delta}\left(\frac{\partial w_1}{\partial x}\right)-2\gamma\mu\left(\frac{\partial w_1}{\partial x}\right)^3 = 0 \ \ at \ \ x=0, \tag{64}$$

$$\left(\frac{\partial w_1}{\partial x}\right) = 0 \ \ at \ \ x=1. \tag{65}$$

The solution of system of equations (63)-(65) is given by

$$w_1=-k^3\beta\left(\frac{x^4}{12}-\frac{x^3}{3}+\frac{x^2}{2}-\frac{x}{3}\right). \tag{66}$$

Second order problem and its solution
The second order problem is given by

$$L(w_2)+\beta\left[\left(\frac{dw_0}{dx}\right)^2\left(\frac{d^2w_1}{dx^2}\right)+2\left(\frac{dw_0}{dx}\right)\left(\frac{dw_1}{dx}\right)\left(\frac{d^2w_0}{dx^2}\right)\right]=0, \tag{67}$$

with boundary conditions

$$w_2 - \frac{\gamma\mu}{\delta}\left(\frac{\partial w_2}{\partial x}\right) - 6\gamma\mu\left(\frac{\partial v_0}{\partial x}\right)^2\left(\frac{\partial w_1}{\partial x}\right) = 0 \ at \ x = 0, \quad (68)$$

$$\left(\frac{\partial w_2}{\partial x}\right) = 0 \ at \ x = 1. \quad (69)$$

The solution of system of equations (67)-(69) is given by

$$w_2 = k^5\beta^2\left[\frac{x^6}{18} - \frac{x^5}{3} + 5\frac{x^4}{6} - 10\frac{x^3}{9} + 5\frac{x^2}{6} - \frac{x}{3}\right]. \quad (70)$$

The velocity field obtained by the homotopy perturbation method is

$$v(x) = \lim_{q\to 1} w = \lim_{q\to 1}\left(w_0 + qw_1 + q^2 w_2 + \cdots\right).$$

By substituting the values of w_0, w_1 and w_2 from eqs.(62),(66) and (70) in the above equation, we get the solution up to second order as

$$\begin{aligned} v(x) = {} & kx\left(\frac{x}{2} - 1\right) + \left(1 - \frac{\gamma\mu k}{\delta}\right) - k^3\beta\left(\frac{x^4}{12} - \frac{x^3}{3} + \frac{x^2}{2} - \frac{x}{3}\right) \\ & + k^5\beta^2\left[\frac{x^6}{18} - \frac{x^5}{3} + 5\frac{x^4}{6} - 10\frac{x^3}{9} + 5\frac{x^2}{6} - \frac{x}{3}\right]. \quad (71) \end{aligned}$$

3.4 Volume flux and average velocity

The flow rate per unit width is given by

$$\begin{aligned} \frac{Q}{W} &= \int_0^\delta v(x)dx, \\ \frac{Q}{W} &= k\frac{\delta^2}{2}\left(\frac{\delta}{3} - 1\right) - k^3\beta\left(\frac{\delta^5}{60} - \frac{\delta^4}{12} + \frac{\delta^3}{6} - \frac{\delta^2}{6}\right) \\ &\quad + k^5\beta^2\left[\frac{\delta^7}{126} - \frac{\delta^6}{18} + \frac{\delta^5}{6} - 5\frac{\delta^4}{18} + 5\frac{\delta^3}{18} - \frac{\delta^2}{6}\right]. \end{aligned}$$

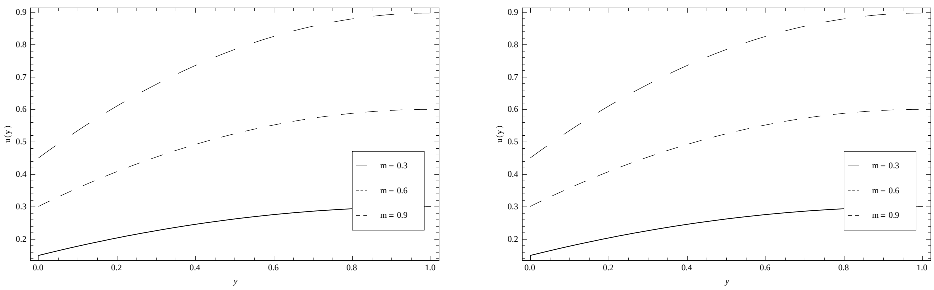
The average velocity $\bar{v}(x)$ over the cross section of the film is given by

$$\begin{aligned} \bar{v}(x) &= \frac{Q}{W\delta}, \\ \bar{v}(x) &= k\frac{\delta}{2}\left(\frac{\delta}{3} - 1\right) - k^3\beta\left(\frac{\delta^4}{60} - \frac{\delta^3}{12} + \frac{\delta^2}{6} - \frac{\delta}{6}\right) \\ &\quad + k^5\beta^2\left[\frac{\delta^6}{126} - \frac{\delta^5}{18} + \frac{\delta^4}{6} - 5\frac{\delta^3}{18} + 5\frac{\delta^2}{18} - \frac{\delta}{6}\right]. \end{aligned}$$

4 Results and discussion

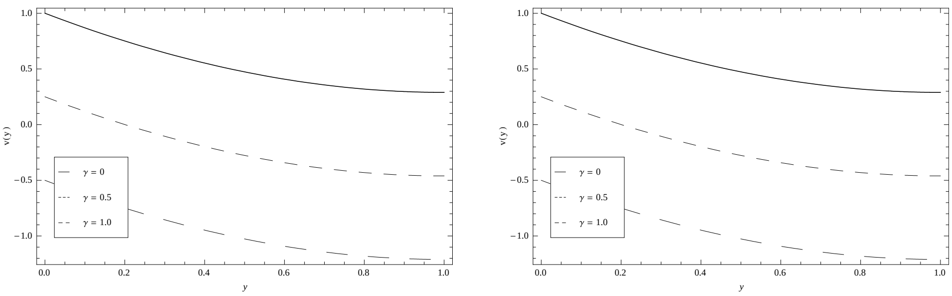
Figure 1 shows that solution for fluid flow down an inclined plane obtained by perturbation and homotopy perturbation method are same for identical values of ϵ and β. Similarly Figure 2 shows that the solution for fluid flow on a moving belt obtained by perturbation and homotopy perturbation method are same for the identical values of ϵ and β. This shows that solutions are identical, hence for further study, we discuss only solution obtained by homotopy perturbation method. To see the effect of different parameters of interest on velocity field, Figures 3 and 4 are displayed in the case of fluid flow down an inclined plane and moving belt.

Figure 3 (a)-(c) exhibits the effect of parameters m, β, γ on the velocity field. In Figure 3 (a), velocity function $u(y)$ is plotted against y for different values of m. Clearly, increasing values of m cause to increase in the velocity. This



(a) Perturbation method (b) Homotopy perturbation method

Figure 1: *Dimensionless velocity profiles for the third order fluid flow down an inclined plane and for different values of* $m = 0.3, 0.6, 0.9$*: (a) (23) for* $\epsilon = 0.01, \mu = 1, \delta = 1, \gamma = 1$ *, (b) (36) for* $\beta = 0.01, \mu = 1, \delta = 1, \gamma = 1$.



(a) Perturbation method (b) Homotopy perturbation method

Figure 2: *Dimensionless velocity profiles for the third order fluid flow on moving belt for different values of* $\gamma = 0, 0.5, 1.0$*: (a) (57) for* $\epsilon = 0.2, \mu = 1, \delta = 1, k = 1.5$*, (b) (71) for* $\beta = 0.2, \mu = 1, \delta = 1, k = 1.5$.

is because the reason that increasing value of m correspond to the increasing angle of inclination, which shows that by increasing the angle of inclination of inclined plane, the velocity increases. Figure 3 (b) is plotted for different values of non-Newtonian parameter β, we see that with increase in β, the velocity increases in this region and as a consequence, the velocity gradient also increases, due to which skin friction also increases. In Figure 3 (c), graph is plotted against different values of slip parameter γ, as the values of γ increases the velocity decreases.
Figure 4 is prepared for the flow of third order fluid on moving belt for different values of parameter k, β, γ. Figure 4 (a) depicts that the velocity decreases as the value of k increases. Figure 2 (b) shows that the velocity of the fluid flow decreases with increase of β in this region. Figure 4 (c) elucidates that the velocity decreases with increase in slip parameter γ.

5 Concluding remarks

A thin film flow of a third order fluid with slip conditions has been discussed in two cases: (i) when moves down an inclined plane and (ii) when moves on a moving belt using analytical techniques. We see that the solution obtained by perturbation and homotopy perturbation method are identical at the same values of β and ϵ, this is witnessed by graphs in Figure 1 & 2. The effect of non-Newtonian parameter, slip parameter and other parameters involved in the problem are discussed and results are displayed in graphs to visualize their effects.

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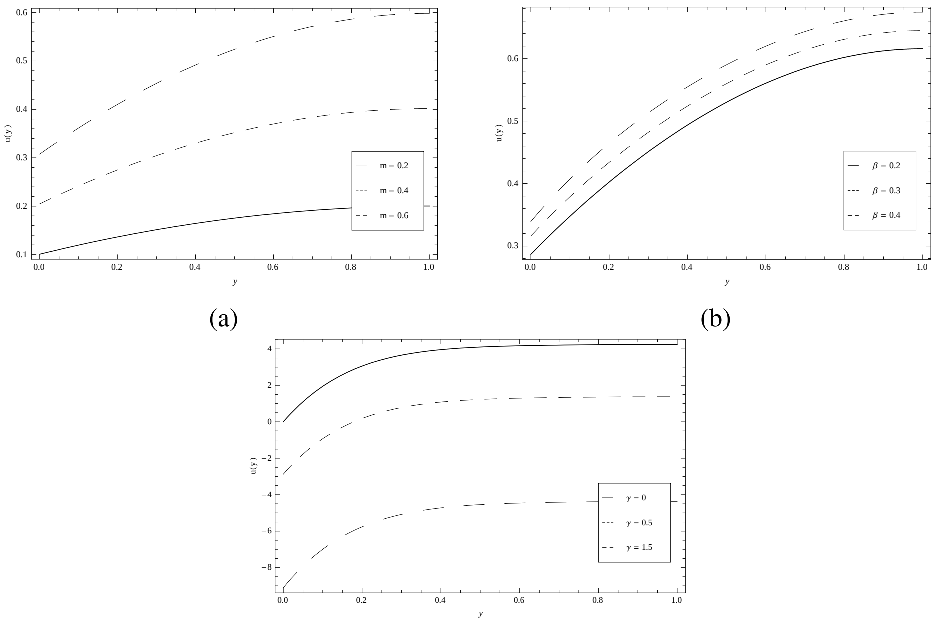

(a) (b)

(c)

Figure 3: *Dimensionless velocity profiles for the third order fluid flow down an inclined plane: (a) Velocity profile for* $\beta = 0.1, \mu = 1, \delta = 1, \gamma = 1$ *and different values of* $m = 0.2, 0.4, 0.6$*, (b) velocity profile for* $m = 0.8, \mu = 1, \delta = 1, \gamma = 1.5$ *and different values of* $\beta = 0.2, 0.3, 0.4$*, (c) velocity profile for* $m = 1, \mu = 0.5, \delta = 1, \beta = 1.5$ *and different values of the parameter* $\gamma = 0, 0.5, 1.5$.

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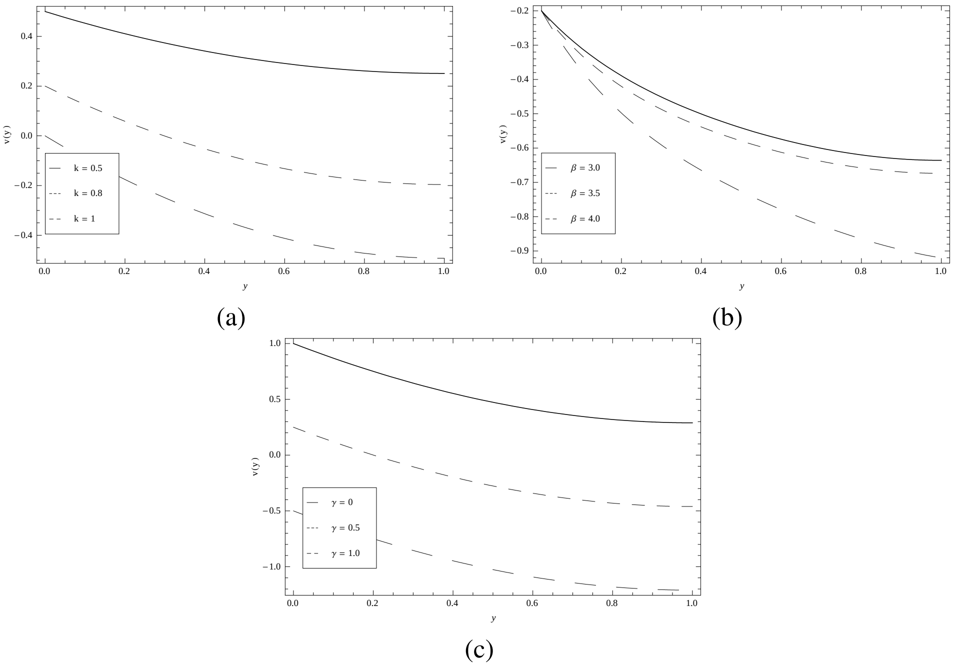

Figure 4: *Dimensionless velocity profiles for the third order fluid flow on moving belt: (a) Velocity profile for* $\beta = 0.1, \mu = 1, \delta = 1, \gamma = 1$ *and different values of* $k = 0.5, 0.8, 1$*, (b) Velocity profile for* $k = 0.8, \mu = 1, \delta = 1, \gamma = 1.5$ *and different values of* $\beta = 3.0, 3.5, 4.0$*, (c) Velocity profile for* $\beta = 0.2, \mu = 1, \delta = 1, k = 1.5$ *and different values of* $\gamma = 0, 0.5, 1.0$.