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A THEORETICAL APPROACH FOR FREE VIBRATION ANALYSIS OF THE NANO-PLATES CONSIDERING THE SMALL SCALE EFFECT

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ABSTRACT

The free vibration analysis of a nano-plate is investigated based on the first order shear deformation theory considering the small scale effect. The governing equations of motion are obtained using Hamilton's principle by considering the nonlocal constitutive equations of Eringen. These coupled partial differential equations are reformulated into two new equations called the edge-zone and interior equations. Analytical solutions are obtained for a nano-plate with Levy boundary conditions. In order to find the natural frequencies of the nano-plate, the various boundary conditions at one direction of the plate should be imposed. Applying these conditions and setting the determinant of the six order coefficient matrix equal to zero, the natural frequencies of the nano-plate are evaluated. Non-dimensional frequency parameters are presented for over a wide range of nonlocal parameters and different boundary conditions. In addition, the effects of nonlocal parameter on the natural frequency of a nano-plate are discussed in details.

INTRODUCTION

Due to the vast computational expenses of nano-structures analyses when using atomic lattice dynamics and molecular dynamic simulations, there is a great interest in applying continuum mechanics for analysis of nano-structures. Eringen [1] showed that it is possible to represent the integral constitutive relations of nano-structures in an equivalent differential form. Eringen presented a nonlocal elasticity theory to account the small scale effect by specifying the stress at a reference point is a functional of the strain field at every point in the body. Since then, many studies have been carried out for bending, buckling and vibration analyses of nano-structures.

The nonlocal theory of elasticity has been extensively used to study buckling and vibration analyses of carbon nano-tubes with the help of beam and shell theories.

Scale effect on static deformation of micro- and nano-rods or tubes is revealed through nonlocal Euler–Bernoulli and Timoshenko beam theories by Wang and Liew [2]. The constitutive relations of nonlocal elasticity theory for application in the analysis of carbon nanotubes when modelled as Euler–Bernoulli beams, Timoshenko beams or as cylindrical shells were presented by Q.Wang and C.M.Wang [3]. Aydogdu [4] developed vibration analysis of nano-rods considering the small scale effect. Li and Wang [5] investigated a theoretical treatment of Timoshenko beams, in which the influences of shear deformation, rotary inertia, and scale coefficient are taken into account.

Nonlocal elasticity and Timoshenko beam theory are implemented by Murmu and Pradhan [6] to investigate the stability response of single walled carbon nanotube embedded in an elastic medium. Murmu and Pradhan [7] studied vibration response of nano cantilever considering non-uniformity in the cross sections using nonlocal elasticity theory.

Although graphite sheet has many superior properties, such as low electrical and thermal conductivities normal to the sheet but high electrical and thermal conductivities in the plane of the sheet, relatively little research have been reported in the literature for mechanical analyses of graphene sheets.

Kitipornchai et al. [8] used the continuum plate model for mechanical analysis of graphene sheets. He et al. [9] investigated vibration analysis of multi-layered graphene sheets in which the van der Waals interaction between layers is described by an explicit formula.

Behfar and Naghdabadi [10] studied nano scale vibrational analysis of a multi-layered graphene sheet embedded in an elastic medium based on the classical plate theory.

Lu et al. [11] derived the basic equations of nonlocal Kirchhoff and Mindlin plate theories for simply supported nano-plates. Axisymmetric bending of micro/nanoscale circular plates is studied using a nonlocal plate theory by Duan and Wang [12]. Pradhan and Phadikar [13] presented classical and first-order shear deformation plate theories for vibration of nano-plate. Their approach is based on the Navier solution and for a nano-plate with all edges simply supported. Pradhan and Phadikar [14] carried out vibration analysis of multilayered graphene sheets embedded in polymer matrix employing nonlocal continuum mechanics.

In-plane vibration of nano-plates was investigated by Murmu and Pradhan [15] employing nonlocal continuum mechanics and considering small scale effect.

Aghababaei and Reddy [16] developed a higher order plate theory for buckling and vibration analyses of a simply supported plate accounting the small scale effect. Pradhan and Murmu [17] studied the small scale effect on the buckling analysis of biaxially compressed single-layered graphene sheets.

In this paper, the vibration analysis of a nano-plate is presented by considering the small scale effect. The three coupled governing equations of motion are obtained based on the nonlocal continuum theory and are decoupled into two new equations. Solving these two decoupled partial differential equations, the natural frequencies of the nano-plate with arbitrary boundary conditions are determined. Finally, a detailed study is carried out to understand the effects of boundary conditions, nonlocal parameter, thickness to length and aspect ratios on the vibration characteristics of nano-plates.

CONSTITUTIVE RELATIONS

According to nonlocal elasticity theory, the stress at a reference point X is considered to be a function of the strain field at every point X' in the body. The nonlocal stress tensor σ^{nl} at point X can be expressed as [1]

$$\sigma^{nl} = \int K(|X'-X|,\tau)\sigma^{l}(X')\,dX' \tag{1}$$

where σ^{l} is the classical stress tensor and $K(|X'-X|)$ is the Kernel function represents the nonlocal modulus. While the constitutive equations of classical elasticity is an algebraic relation between stress and strain tensors, that of nonlocal elasticity involves spatial integrals which represent weighted averages of contributions of the strain of all points in the body to the stress at the given point. Eringen [1] showed that it is possible to represent the integral constitutive relation in an equivalent differential form as

$$(1-\mu\nabla^2)\sigma^{nl} = \sigma^{l} \tag{2}$$

where $\mu = (e_0 a)^2$ is nonlocal parameter, a an internal characteristic length and e_0 a constant. Also, ∇^2 is the Laplacian operator.

GOVERNING EQUATIONS OF MOTION

The first order shear deformation plate theory assumes that the plane sections originally perpendicular to the longitudinal plane of the plate remain plane, but not necessarily perpendicular to the longitudinal plane. This theory accounts for the shear strains in the thickness direction of the plate and is based on the displacement field

$$\begin{aligned} u &= u_0(x,y) + z\psi_x(x,y,t) \\ v &= v_0(x,y) + z\psi_y(x,y,t) \\ w &= w(x,y) \end{aligned} \tag{3}$$

where u_0 and v_0 are displacement components of the midplane, w is transverse displacement, t is time, and ψ_x and ψ_y are the rotation functions of the midplane normal in the x and y directions, respectively. Using the Hamilton's principle, the nonlocal bending governing equations of motion for a single layered nano-plate are obtained as follows [13]

$$D(\psi_{x,xx}+\psi_{y,xy}) + \frac{D(1-\nu)}{2}(\psi_{x,yy}-\psi_{y,xy}) - \kappa^2 Gh(\psi_x + w_{,x}) = I_2(\ddot{\psi}_x - \mu\nabla^2\ddot{\psi}_x) \tag{4a}$$

$$D(\psi_{y,yy}+\psi_{x,xy}) + \frac{D(1-\nu)}{2}(\psi_{y,xx}-\psi_{x,xy}) - \kappa^2 Gh(\psi_y + w_{,y}) = I_2(\ddot{\psi}_y - \mu\nabla^2\ddot{\psi}_y) \tag{4b}$$

$$\kappa^2 Gh(\psi_{x,x}+\psi_{y,y}+w_{,xx}+w_{,yy}) + q(x,y,t) = I_1(\ddot{w} - \mu\nabla^2\ddot{w}) \tag{4c}$$

In the above equations, dot above each parameter denotes derivative with respect to time, G is the shear modulus, $D = Eh^3/12(1-\nu^2)$ denotes the bending rigidity of the plate, E and ν elastic modulus and Poisson's ratio, respectively and κ^2 the shear correction factor. Also, q is the transverse loading and the mass moments of inertia, I_1 and I_2, are defined as

$$(I_1, I_2) = \int_{-h/2}^{h/2} \rho(1,z)\,dz \tag{5}$$

in which ρ is the density of the plate. It can be seen that the governing equations (4) are generally a system of six-order coupled partial differential equations in terms of x and y variables.

SOLUTION

In order to solve the governing equations of motion (4) for various boundary cinditions, it is reasonable to find a method to decouple these equations. Let us introduce two new functions ς and φ such that

$$\varsigma = \psi_{x,x} + \psi_{y,y} \tag{6a}$$

$$\varphi = \psi_{x,y} - \psi_{y,x} \tag{6b}$$

Using Eqs. (6), the governing equations (4) can be rewritten as

$$D\varsigma_{,x} + \frac{D(1-\nu)}{2}\varphi_{,y} - \kappa^2 Gh(\psi_x + w_{,x}) = I_2(\ddot{\psi}_x - \mu\nabla^2\ddot{\psi}_x) \quad (7a)$$

$$D\varsigma_{,y} - \frac{D(1-\nu)}{2}\varphi_{,x} - \kappa^2 Gh(\psi_y + w_{,y}) = I_2(\ddot{\psi}_y - \mu\nabla^2\ddot{\psi}_y) \quad (7b)$$

$$\kappa^2 Gh(\varsigma + \nabla^2 w) + q = I_1(\ddot{w} - \mu\nabla^2\ddot{w}) \quad (7c)$$

Doing some algebraic operations on Eqs. (7), the three coupled partial differential equations in (4) can be replaced by the following two uncoupled equations

$$C\nabla^2\varphi - \kappa^2 Gh\varphi = I_2(1-\mu\nabla^2)\ddot{\varphi} \quad (8a)$$

$$D\nabla^2\nabla^2 w = (1-\mu\nabla^2)\{q - \frac{D}{\kappa^2 Gh}\nabla^2 q - I_1\ddot{w} + (\frac{I_1 D}{\kappa^2 Gh} + I_2)\nabla^2\ddot{w} + I_2\} + \frac{(1-\mu\nabla^2)^2}{\kappa^2 Gh}\{I_2\ddot{q} - I_1 I_2 \ddddot{w}\} \quad (8b)$$

where C denotes $D(1-\nu)/2$. It can be seen that the above equations becomes as the classical equations of the Mindlin plate when $\mu = 0$. Like the classical elasticity [18], equations (8a) and (8b) are called edge-zone (boundary layer) and interior equations, respectively. Also, the rotation functions ψ_x and ψ_y can be defined in terms of w and φ as

$$\kappa^2 Gh\psi_x + I_2\ddot{\psi}_x = \frac{\partial}{\partial x}[-\frac{D(1-\mu\nabla^2)}{\kappa^2 Gh}q + \frac{I_1 D(1-\mu\nabla^2)}{\kappa^2 Gh}\ddot{w} - D\nabla^2 w - \kappa^2 Ghw + I_2\mu(-\frac{(1-\mu\nabla^2)}{\kappa^2 Gh}\ddot{q} - \nabla^2\ddot{w} + \frac{I_1(1-\mu\nabla^2)}{\kappa^2 Gh}\ddddot{w})] + \frac{\partial}{\partial y}[C\varphi + I_2\mu\ddot{\varphi}] \quad (9a)$$

$$\kappa^2 Gh\psi_y + I_2\ddot{\psi}_y = \frac{\partial}{\partial y}[-\frac{D(1-\mu\nabla^2)}{\kappa^2 Gh}q + \frac{I_1 D(1-\mu\nabla^2)}{\kappa^2 Gh}\ddot{w} - D\nabla^2 w - \kappa^2 Ghw + I_2\mu(-\frac{(1-\mu\nabla^2)}{\kappa^2 Gh}\ddot{q} - \nabla^2\ddot{w} + \frac{I_1(1-\mu\nabla^2)}{\kappa^2 Gh}\ddddot{w})] - \frac{\partial}{\partial x}[C\varphi + I_2\mu\ddot{\varphi}] \quad (9b)$$

With deriving w, ψ_x and ψ_y, the stress components of the nano-plate can be computed by using the nonlocal constitutive relations in the following forms

$$\begin{aligned}
\sigma_{xx}^{nl} - \mu\nabla^2\sigma_{xx}^{nl} &= \frac{E}{1-\nu^2}(\psi_{x,x} + \nu\psi_{y,y})z \\
\sigma_{xy}^{nl} - \mu\nabla^2\sigma_{xy}^{nl} &= \frac{E}{2(1+\nu)}(\psi_{x,y} + \psi_{y,x})z \\
\sigma_{yy}^{nl} - \mu\nabla^2\sigma_{yy}^{nl} &= \frac{E}{1-\nu^2}(\psi_{y,y} + \nu\psi_{x,x})z \\
\sigma_{xz}^{nl} - \mu\nabla^2\sigma_{xz}^{nl} &= G(\psi_x + w_{,x}) \\
\sigma_{yz}^{nl} - \mu\nabla^2\sigma_{yz}^{nl} &= G(\psi_y + w_{,y})
\end{aligned} \quad (10)$$

Here, a rectangular plate $(a \times b)$ with two opposite simply supported edges at $x = 0$ and $x = a$ and arbitrary boundary conditions at two other edges is considered. For free harmonic vibration of the plate, the transverse loading q is put equal to zero and the transverse deflection w and boundary layer function φ are assumed as

$$w = \sum_{n=1}^{\infty} w_n(y)\sin(\beta_n x)e^{i\omega_n t} \quad (11a)$$

$$\varphi = \sum_{n=1}^{\infty} \varphi_n(y)\cos(\beta_n x)e^{i\omega_n t} \quad (11b)$$

which exactly satisfy the simply supported conditions at $x = 0$ and $x = a$. In these relations, ω_n is the natural frequency of the nano-plate and β_n denotes $n\pi/a$. Substituting the proposed series solutions (11) into Eqs. (8) yields

$$\lambda_1\frac{\partial^4 w_n(y)}{\partial y^4} + \lambda_2\frac{\partial^2 w_n(y)}{\partial y^2} + \lambda_3 w_n(y) = 0 \quad (12a)$$

$$\lambda_4\frac{\partial^2\varphi_n(y)}{\partial y^2} + \lambda_5\varphi_n(y) = 0 \quad (12b)$$

where the constant coefficients $\lambda_i\,(i = 1,..,5)$ are given in Appendix. The above equations are two ordinary differential equations with total order of six. The solutions of Eqs. (12) can be expressed as

$$w_n(y) = C_1\sin(\eta_1 y) + C_2\cos(\eta_1 y) + C_3\sinh(\eta_2 y) + C_4\cosh(\eta_2 y) \quad (13a)$$

$$\varphi_n(y) = C_5\sinh(\eta_3 y) + C_6\cosh(\eta_3 y) \quad (13b)$$

where $C_i\,(i = 1,..,6)$ are constants of integration and parameters η_1, η_2 and η_3 are defined as

$$\eta_1 = \sqrt{\frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{2\lambda_1}} \quad (14a)$$

$$\eta_2 = \sqrt{\frac{-\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{2\lambda_1}} \quad (14b)$$

$$\eta_3 = \frac{\sqrt{\lambda_4\lambda_5}}{\lambda_4} \quad (14c)$$

Six independent linear equations must be written among the integration constants to solve the free vibration problem. Applying arbitrary boundary conditions along the edges of the plate at $y=0$ and $y=b$, leads to six algebraic equations. Here, three types of boundary conditions along the edges of the nano-plate in y direction are considered

Simply supported (S) $w = M_{yy} = \psi_x = 0$ (14a)

Clamped (C) $w = \psi_x = \psi_y = 0$ (14b)

Free (F) $M_{yy} = M_{xy} = Q_y = 0$ (14c)

where the resultant moments M_{yy} and M_{xy} and resultant force Q_y are expressed as

$$M_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}^{nl} z\, dz\,,\; M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}^{nl} z\, dz\,,\; Q_y = \int_{-h/2}^{h/2} \sigma_{yz}^{nl}\, dz \qquad (15)$$

In order to find the natural frequencies of the nano-plate, the various boundary conditions at $y=0$ and $y=b$ should be imposed. Applying these conditions and setting the determinant of the six order coefficient matrix equal to zero, the natural frequencies of the nano-plate are evaluated.

NUMERICAL RESULTS AND DISCUSSION

For numerical results, the following material properties are used throughout the investigation [13]

$$E = 1.2TPa\,,\; \nu = 0.3\,,\; \kappa^2 = 5/6 \qquad (16)$$

In order to verify the accuracy of the present formulations, a comparison has been carried out with the results given by Pradhan and Phadikar [13] for an all edges simply supported nano-plate. To this end, a four edges simply supported nano-plate is considered. The non-dimensional natural frequency parameter $\Omega = \omega a^2 \pi^4 \sqrt{I_1/D}$ is listed in Table 1 for some nonlocal parameters. From this table, it can be found that the present results are in good agreement with those of Ref. [13] when the rotary inertia terms have been neglected. It can be also seen that the rotary inertia terms have considerable effects especially in second mode of vibration and cause the natural frequency decreases. Hereafter, the rotary inertia terms are considered in numerical results.

To study the effects of boundary condition, the nonlocal parameter (μ) and thickness to length ratio (h/a) on the vibrational behavior of the nano-plate, the first two non-dimensional frequencies are listed for a single layered nano-plate. The results are tabulated for the nano-plates with two boundary conditions at $y=0$ and $y=b$ as clamped-clamped (**C-C**) and clamped-simply (**C-S**) in Tables 2-3. Nano-plates are identified by their boundary conditions at $y=0$ and $y=b$.

Based on the results in these tables, it can be concluded that the frequency parameter decreases for all modes as the nonlocal parameter μ increases. The reason is that with increasing the nonlocal parameter, the stiffness of the nano-plate decreases. i.e. the small scale effect makes the nano-plate more flexible as the nonlocal model may be viewed as atoms linked by elastic springs while the local continuum model assumes the spring constant to take on an infinite value. In sum, the nonlocal plate theory should be used if one needs accurate predictions of natural frequencies of nano-plates.

The influence of thickness-length ratio on the frequency parameter can also be examined by keeping the nonlocal parameter constant while varying the thickness to length ratio. It can be easily observed that as h/a increases, the frequency parameter decreases. The decrease in the frequency parameter is due to effects of the shear deformation, rotary inertia and use of term a^2/h in the definition of the non-dimensional frequency. These effects are more considerable in the second mode than in the first modes.

In Fig. 1, the relation between natural frequency and nonlocal parameter of a square **C-C** nano-plate is depicted for different thickness to length ratios. It can be seen that nonlocal theories predict smaller values of natural frequencies than the local theories especially for higher thickness to length ratios. Thus the local theories, in which the small length scale effect between the individual carbon atoms is neglected, overestimate the natural frequencies. Also, for lower values of nonlocal parameters, variation of the natural frequency is significant. In addition, it is seen that changes of natural frequencies are less significant for lower values of thickness to length ratios. Therefore, the small scale effect is more significant for thicker nano-plates. Conversely, for a thin nano-plate, the frequency for the nonlocal plate theory is close to that furnished by the local plate model, indicating the negligible effect of small scale in such plates. The effect of boundary conditions on the natural frequency of a nano-plate is shown in Fig. 2. It can be concluded that the boundary condition has significant effect on the vibrational characteristic of the nano-plates. It can be seen that the lowest and highest values of frequency parameters correspond to F-F and C-C edges, respectively.

TABLE 1. COMPARISON OF NON-DIMENSIONAL FREQUENCY PARAMETER $\Omega = \omega a^2 \pi^4 \sqrt{I_1 / D}$ OF A NANO-PLATE WITH ALL EDGES SIMPLY SUPPORTED

[a] Neglecting the rotary inertia terms

μ	h/b		Mode 1	Mode 2
1nm	0.1	Present	0.1322 0.1332[a]	0.1994 0.2026[a]
		Ref. [13]	0.1332	0.2026
	0.2	Present	0.1210 0.1236[a]	0.1673 0.1730[a]
		Ref. [13]	0.1236	0.1730
2nm	0.1	Present	0.0935 0.0942[a]	0.1410 0.1432[a]
		Ref. [13]	0.0942	0.1432
	0.2	Present	0.0855 0.0874[a]	0.1183 0.1224[a]
		Ref. [13]	0.0874	0.1224
3nm	0.1	Present	0.0763 0.0769[a]	0.1151 0.1170[a]
		Ref. [13]	0.0769	0.1170
	0.2	Present	0.0698 0.0714[a]	0.0966 0.0999[a]
		Ref. [13]	0.0714	0.0999
4nm	0.1	Present	0.0661 0.0666[a]	0.0997 0.1013[a]
		Ref. [13]	0.0666	0.1013
	0.2	Present	0.0605 0.0618[a]	0.0836 0.0865[a]
		Ref. [13]	0.0618	0.0865

TABLE 2. FIRST TWO NON-DIMENSIONAL FREQUENCY PARAMETERS $\Omega = \omega a^2 \pi^4 \sqrt{I_1 / D}$ OF A C-C NANO-PLATE

μ	h/b	Mode 1	Mode 2
1nm	0.1	0.1757	0.2124
	0.2	0.1494	0.1735
2nm	0.1	0.1242	0.1502
	0.2	0.1057	0.1227
3nm	0.1	0.1014	0.1226
	0.2	0.0863	0.1002
4nm	0.1	0.0878	0.1062
	0.2	0.0747	0.0868

TABLE 3. FIRST TWO NON-DIMENSIONAL FREQUENCY PARAMETERS $\Omega = \omega a^2 \pi^4 \sqrt{I_1 / D}$ OF A C-S NANO-PLATE

μ	h/b	Mode 1	Mode 2
1nm	0.1	0.1501	0.2049
	0.2	0.1333	0.1700
2nm	0.1	0.1062	0.1449
	0.2	0.0942	0.1202
3nm	0.1	0.0867	0.1183
	0.2	0.0769	0.0982
4nm	0.1	0.0751	0.1024
	0.2	0.0666	0.0850

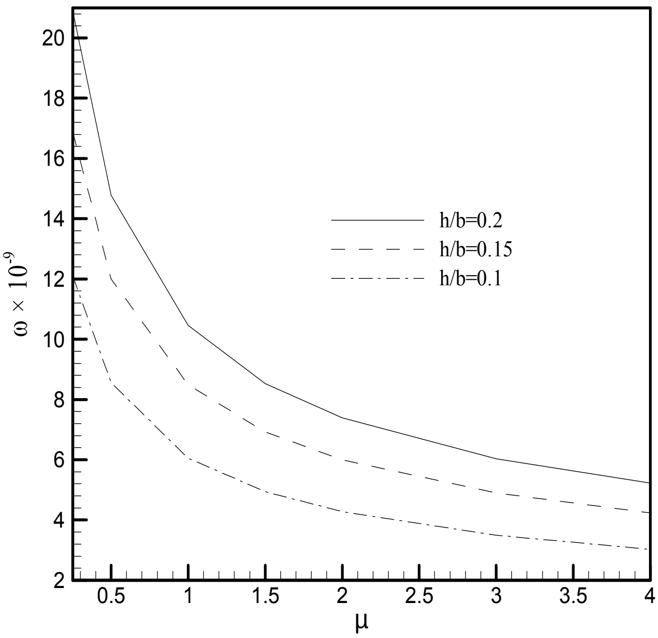

FIG. 1 VARIATION OF NATURAL FREQUENCY WITH NONLOCAL PARAMETER FOR A C-C NANO-PLATE

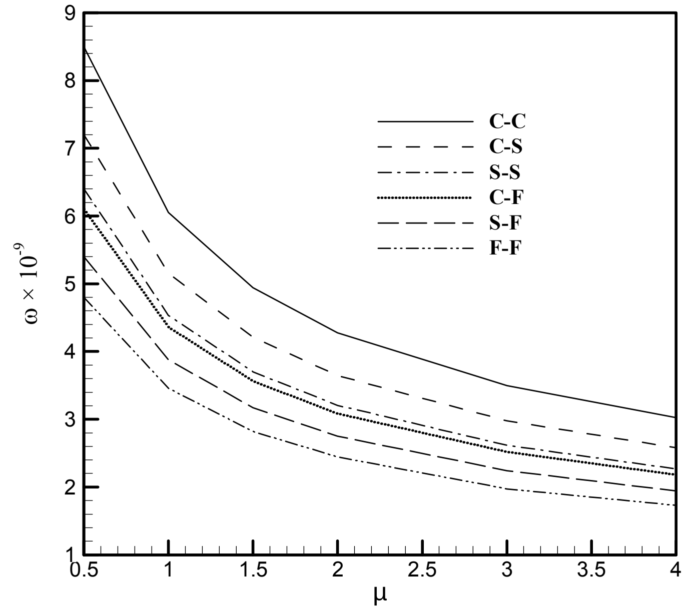

FIG. 2 VARIATION OF NATURAL FREQUENCY WITH NONLOCAL PARAMETER FOR NANO-PLATES WITH DIFFERENT BOUNDARY CONDITIONS AT TWO EDGES

CONCLUSION

Presented herein is a variational derivation of the governing equations and boundary conditions for the free vibration of nano-plates based on Eringen's nonlocal elasticity and first order shear deformation plate theory. This nonlocal plate theory accounts for both the scale effect and the effects of transverse shear deformation and rotary inertia which become significant when dealing with nano-plates. Coupled partial differential equations have been reformulated and the generalized levy type solutions have been presented for free vibration analysis of a nano-plate considering the small scale effect. The accurate natural frequencies of nano-plates have been tabulated for various nonlocal parameters, some thickness to length ratios and different boundary conditions. The effects of boundary conditions, variation of nonlocal parameter, thickness to length and aspect ratios on the frequency values of a nano-plate have been examined and discussed in detail. The effect of different boundary conditions on natural frequencies of nano-plates has been investigated for the first time.

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