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A Procedure for Investigating the Liapunov Stability of Nonautonomous Linear Second-Order Systems

Utilizing Liapunov's direct method a procedure is presented which generates sufficient conditions for the stability of the null solution of the nonautonomous differential equation $\ddot{x} + b(t)\dot{x} + a(t)x = 0$. This procedure systematically leads to the construction of a Liapunov function for a given differential equation and thus eliminates the normally ad hoc nature of the direct method. Four examples illustrating the procedure are discussed.

Introduction

Normally, when Liapunov's direct method is utilized to establish the stability of a solution to a differential equation governing a given dynamical system, one is faced with the unstructured task of searching for a Liapunov function which will provide sufficient conditions for stability. When nonautonomous differential equations are involved, the search becomes extremely difficult, and one is indeed fortunate to find one at all. The state of affairs in dealing with the Liapunov stability of even linear nonautonomous differential equations by the direct method is characterized in Hahn [1][1] and Brauer and Nohel [2].

In this present paper a systematic procedure is developed for establishing sufficient conditions for the Liapunov stability of the null solution of linear second-order systems with time-dependent coefficients. This is accomplished by applying the direct method but at the same time avoiding the unstructured search for Liapunov function candidates.

In the following sections, the conditions for stability are set forth in a theorem and a corollary, and the procedure is illustrated by application of the corollary to four examples of dynamical systems governed by the linear differential equation $\ddot{x} + b\dot{x} + ax = 0$.

Results

Prior to the presentation of the basic stability theorem, an important lemma regarding two-dimensional quadratic forms is needed and proved. This lemma offers a generalization of Sylvester's theorem to quadratic forms with time-dependent coefficients.

Lemma. Let the quadratic form

$$V = V(t, x, \dot{x}) = p\dot{x}^2 + 2qx\dot{x} + rx^2$$

have continuously differentiable time-dependent coefficients p, q, r which satisfy

$$(i): \quad p > m_1 > 0 \quad \text{for all} \quad t \geq 0$$

$$(ii): \quad \frac{pr - q^2}{p} > m_2 > 0 \quad \text{for all} \quad t \geq 0$$

where m_1, m_2 are positive constants. Then, V is positive-definite for the entire x, $\dot{x}$ plane.

Proof. According to Cesari [3, p. 108], V is positive-definite if V has continuous partial derivatives, $V(t, 0, 0) = 0$, and there exists a positive-definite function $W = W(x, \dot{x})$ such that $V \geq W$ for all t. Obviously the first two conditions are met. To show that the third condition is satisfied, consider the function W

$$W = \begin{cases} m_1\dot{x}^2 & \text{for} \quad x = 0 \\ m_2x^2 & \text{for} \quad x \neq 0 \end{cases}$$

As W is clearly positive-definite and because V may be expressed as

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[1] Numbers in brackets designate References at end of paper.

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$$V = \frac{(p\dot{x} + gx)^2}{p} + \left(\frac{pr - q^2}{p}\right)x^2 \geq \left(\frac{pr - q^2}{p}\right)x^2 > m_2x^2$$

it is easily seen that $V \geq W$. Hence, V is positive-definite.

By use of this lemma, the proof of the basic stability theorem is now possible.

Theorem. The null solution of

$$\ddot{x} + b\dot{x} + ax = 0 \tag{1}$$

where a and b are continuous functions of t, is Liapunov stable if there exists a continuously differentiable real solution p, q, r, u of

$$\dot{p} - 2bp + 2q = -u^2 \tag{2}$$

$$\dot{q} + r - bq - ap = -\left(\frac{q}{p}\right)u^2 \tag{3}$$

$$\dot{r} - 2aq = -\left(\frac{q}{p}\right)^2 u^2 \tag{4}$$

Such that the following conditions are satisfied:

(*i*): $\int_0^t \left(\frac{q}{p}\right)dt$ is bounded below.

(*ii*): $p_0r_0 - q_0^2 = p(0)r(0) - q(0)^2 = C > 0.$

(*iii*): $p > m_1 > 0$ for all $t \geq 0$.

Proof. The theorem can be proved if it can be shown that

$$V = p\dot{x}^2 + 2qx\dot{x} + rx^2 \tag{5}$$

is a Liapunov function for (1). To this end, consider

$$\dot{V} \equiv \frac{\partial V}{\partial t} + \dot{x}\,\frac{\partial V}{\partial x} - (b\dot{x} + ax)\,\frac{\partial V}{\partial \dot{x}} \tag{6}$$

By direct substitution from (5) into (6), it may be seen that $\dot{V}$ can be expressed as

$$\dot{V} = (\dot{p} + 2q - 2bp)\dot{x}^2 + 2(\dot{q} + r - pa - bq)x\dot{x} + (\dot{r} - 2aq)x^2 \tag{7}$$

However, upon substitution from equations (2)–(4), equation (7) becomes

$$\dot{V} = -u^2\left(\dot{x}^2 + 2\left(\frac{q}{p}\right)x\dot{x} + \left(\frac{q}{p}\right)^2x^2\right)$$

$$= -u^2\left(\dot{x} + \frac{q}{p}x\right)^2 \leq 0 \tag{8}$$

Hence, it has been established that (6) is nonpositive. If it can be shown that V is positive-definite, the stability is assured by Liapunov's direct method (eg., see Cesari [3, p. 109]). In this connection, multiply (2) by r, (3) by $-2q$, (4) by p and add the results. One obtains

$$r\dot{p} + p\dot{r} - 2bpr - 2q\dot{q} + 2bq^2$$

$$= -u^2r - p\left(\frac{q}{p}\right)^2u^2 + 2q\left(\frac{q}{p}\right)u^2$$

$$= -\left(\frac{pr - q^2}{p}\right)u^2 \tag{9}$$

Equation (9) may be written in the form

$$\frac{d}{dt}\,(pr - q^2) + \left(\frac{u^2}{p} - 2b\right)(pr - q^2) = 0 \tag{10}$$

However, by eliminating u^2 between (2) and (10), one obtains

$$\frac{d}{dt}\,(pr - q^2) - \left(\frac{\dot{p}}{p} + 2\,\frac{q}{p}\right)(pr - q^2) = 0 \tag{11}$$

or

$$\frac{d}{dt}\left(\frac{pr - q^2}{p}\right) - 2\left(\frac{q}{p}\right)\left(\frac{pr - q^2}{p}\right) = 0 \tag{12}$$

In turn, (12) implies that

$$\frac{pr - q^2}{p} = \frac{C}{p_0}\,e^{2\int_0^t \frac{q}{p}\,dt} \tag{13}$$

With $C/p_0 > 0$ and $\int_0^t \frac{q}{p}\,dt$ bounded below, there exists a positive number m_2 such that

$$\frac{pr - q^2}{p} > m_2 > 0 \tag{14}$$

Now, as $p > m_1 > 0$, the hypotheses of the lemma are met and (5) is positive-definite. Equation (8), together with the fact that V is positive-definite, assures that (5) is a Liapunov function for (1) and that the null solution of (1) is stable.

The system of equations (2)–(4) is underdetermined and solutions are readily available. Any choice of real continuously differentiable, p, q, r, u satisfying equations (2)–(4) will generate a V for which $\dot{V} \leq 0$. However not all such V's are positive-definite. The hypotheses of the theorem provide one with simple criteria which guarantee that V is positive-definite and thus a Liapunov function.

The following corollary removes the explicit requirement of constructing solutions to equations (2)–(4).

Corollary. If the coefficients a and b of the linear second-order differential equation (1) are differentiable and there exists a function f such that

(*i*): $\int_0^t f dt$ is bounded below.

(*ii*): $2b - 4f + \dfrac{\frac{d}{dt}\,(f^2 - bf + a - \dot{f})}{f^2 - bf + a - \dot{f}} \geq 0$ for all t.

(*iii*): $\dfrac{e^{2\int_0^t f dt}}{f^2 - bf + a - \dot{f}} > m_1 > 0$ for all t.

where m_1 is a positive constant, then the null solution of (1) is Liapunov stable.

Proof. It can be demonstrated that

$$p = \frac{e^{2\int_0^t f dt}}{f^2 - bf + a - \dot{f}} \tag{15}$$

$$q = fp \tag{16}$$

$$r = (2f^2 - bf + a - \dot{f})p \tag{17}$$

$$u^2 = \left[2b - 4f + \frac{\frac{d}{dt}\,(f^2 - bf + a - \dot{f})}{f^2 - bf + a - \dot{f}}\right]p \tag{18}$$

is a real solution of equations (2)–(4). Equations (15)–(18) may be obtained from equations (2)–(4) under the transformation $q/p = f$ and the subsequent integration of the resulting separable differential equations. As $q/p = f$, $\int_0^t q/pdt$ is bounded below. By hypothesis, $p > m_1 > 0$. Finally, because

$$p_0r_0 - q_0^2 = (f_0^2 - b_0f_0 + a_0 - \dot{f}_0)p_0^2$$

where $b_0 = b(0)$, $a_0 = a(0)$, $f_0 = f(0)$, $\dot{f}_0 = \dot{f}(0)$, it follows from (*iii*) that $p_0r_0 - q_0^2 > 0$. Thus the hypotheses of the theorem are met and the Liapunov stability of the null solution of (1) is assured by the Liapunov function

$$V = \frac{e^{2\int_0^t f dt}}{f^2 - bf + a - \dot{f}} [\dot{x}^2 + 2f x\dot{x} + (2f^2 - bf + a - \dot{f})x^2] \tag{19}$$

Observe that the problem is now reduced to simply finding f rather than p, q, r, and u. The conditions which f must satisfy are relatively simple. However, given a pair of differentiable coefficients a and b, it may be difficult to find f for which the conditions (i), (ii), (iii) are simultaneously satisfied. But often, simple choices of f lead to criteria which apply to nontrivial equations. The following examples illustrate both the procedure and some of the resulting stability determinations that can now be readily made with the approach.

Applications

In this section, some examples are given to demonstrate the application of the corollary.

Example 1. Take $f = \lambda$, a constant. Then, the conditions of the corollary become

(i): λt bounded below.

$$(ii): \quad 2b - 4\lambda + \frac{\dot{a} - \lambda\dot{b}}{\lambda^2 - \lambda b + a} \geq 0.$$

$$(iii): \quad \frac{e^{2\lambda t}}{\lambda^2 - \lambda b + a} > m_1 > 0.$$

It is clear from (i) that λ must be nonnegative for stability. Furthermore, it is clear that these conditions will apply to a number of classical equations. For example, if a and b are constants, then, by taking $\lambda = 0$, (ii) and (iii) yield $b \geq 0$ and $\frac{1}{a} > m_1 > 0$, which are the well-known conditions for stability in this case.

Example 2. For the foregoing case, the choice of $\lambda = b/2 \geq 0$ allows the first two conditions to be immediately satisfied, while $a - b^2/4 > 0$ is required for the third. This corresponds to the underdamped motion of the damped, spring-mass dynamical system. The Liapunov function constructed by this analysis is

$$V = \frac{e^{bt}}{a - \frac{b^2}{4}} [\dot{x}^2 + bx\dot{x} + ax^2] \tag{20}$$

Observe that, although the equation being studied is autonomous, the Liapunov function in this case has explicit dependence on t.

Example 3. Leitmann [4] has studied the simple pendulum of variable length. The equation of motion for small oscillations is

$$\ddot{\theta} + 2\frac{\dot{l}}{l}\dot{\theta} + \frac{g}{l}\theta = 0 \tag{21}$$

where θ is the variable measuring the angular displacement, l is the length of the pendulum, and g is a positive constant. With $\lambda = 0$, sufficient conditions for stability as determined from the corollary are as follows:

$$l > m_1' > 0 \tag{22}$$

and

$$\frac{\dot{l}}{l} \geq 0 \tag{23}$$

These conditions are in full agreement with Lietmann's which were obtained by a nonlinear analysis of the companion nonlinear equation

$$\ddot{\theta} + 2\frac{\dot{l}}{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0$$

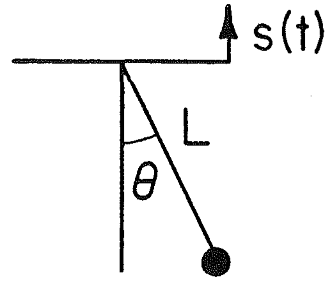

Fig. 1 Oscillating pendulum

Example 4. Consider the stability of small oscillations of a pendulum of length L attached to a support whose motion is a prescribed function of time $s(t)$; see Fig. 1.

The equation of motion of the pendulum bob is

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{\ddot{s}}{L}\right)\theta = \ddot{\theta} + a\theta = 0 \tag{24}$$

Suppose further that the support motion is prescribed in such a way that

$$a = \frac{g}{L} + \frac{\ddot{s}}{L} = Ae^{-pt}\sin\omega t + a_0 \tag{25}$$

where $A \geq 0$, $\omega \geq 0$, $p > 0$, $a_0 > 0$ are constants. Consider

$$f = -\alpha e^{-\beta t} \tag{26}$$

where $\alpha > 0$, $\beta > 0$ are constants. Obviously,

$$-\frac{\alpha}{\beta} \leq \int_0^t f dt \leq 0 \tag{27}$$

so that condition (i) of the corollary is met. By substitution of f into conditions (ii) and (iii) of the corollary, the conditions for stability become

$$4\alpha^2 e^{-3\beta t} + 2\alpha^2\beta e^{-2\beta t} - \alpha\beta^2 e^{-\beta t} + 4\alpha e^{-\beta t}(Ae^{-pt}\sin\omega t + a_0) + (-Ape^{-pt}\sin\omega t + A\omega e^{-pt}\cos\omega t) \geq 0 \tag{28}$$

and

$$m_2' > \alpha^2 e^{-2\beta t} + \alpha\beta e^{-\beta t} + Ae^{-pt}\sin\omega t + a_0 > m_1' > 0 \tag{29}$$

where m_1' and m_2' are positive constants. Equation (29) will be satisfied for all values of $t \geq 0$ when

$$\alpha^2 e^{-2\beta t} + \alpha\beta e^{-\beta t} - Ae^{-pt} + a_0 > 0 \tag{30}$$

This inequality is satisfied if $p > 3\beta$ and $\alpha^2 + \alpha\beta + a_0 - A > 0$. Equation (28) will be satisfied if

$$4\alpha^2 e^{-3\beta t} + 2\alpha^2\beta e^{-2\beta t} + 4\alpha a_0 e^{-\beta t} \geq \alpha\beta^2 e^{-\beta t} + 4\alpha Ae^{-(\beta+p)t} + A\sqrt{p^2 + \omega^2}\, e^{-pt} \tag{31}$$

Now, as

$$4\alpha^2 e^{-3\beta t} + 2\alpha^2\beta e^{-2\beta t} + 4\alpha a_0 e^{-\beta t} > (4\alpha^2 + 2\alpha^2\beta)e^{-3\beta t} + 4\alpha a_0 e^{-\beta t} \tag{32}$$

and

$$\alpha\beta^2 e^{-\beta t} + (4\alpha A + A\sqrt{p^2 + \omega^2})e^{-3\beta t} > \alpha\beta^2 e^{-\beta t} + 4\alpha Ae^{-(\beta+p)t} + A\sqrt{p^2 + \omega^2}e^{-pt} \tag{33}$$

under the condition that $p > 3\beta$, it follows that $4a_0 > \beta^2$ and $4\alpha^2 + 2\alpha^2\beta > 4\alpha A + A\sqrt{p^2 + \omega^2}$ assure the satisfaction of (31). However, this means that $p > 0$ is enough to guarantee that α and β can always be chosen so that (28) and (29) are satisfied.

Hence, the null solution of (24) is Liapunov stable under the conditions given in (25).

Observe that the coefficient a in this example can change its sign for certain positive values of A, ω, p, and a_0. For such systems, stability criteria are generally difficult to establish, yet this procedure can clearly be applied. Furthermore, the conditions of the example, for this choice of f, cannot be weakened, as $p = 0$ immediately causes (28) to fail for large values of t.

Conclusions

Sufficient conditions for stability are established by means of a construction which leads directly to a Liapunov function when the stability conditions in the stated corollary are met. A function f, whose integral is bounded below, is required. Any choice of f with this property establishes a pair of inequalities involving the time-dependent coefficients a and b of the differential equation. Simultaneous satisfaction of these inequalities is sufficient for stability. As demonstrated through the examples, simple choices of f lead to criteria which apply to a number of well-known equations.

Finally, the application of the corollary set forth here holds the promise of yielding stability criteria for some nonautonomous systems that heretofore were untractable by the Liapunov direct method.

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