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ANALYTICAL AND EXPERIMENTAL NATURAL FREQUENCIES OF TRANSVERSE VIBRATION OF SANDWICH BEAMS INTERCONNECTED BY WINKLER ELASTIC FOUNDATION

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ABSTRACT

The free transverse vibration of an elastically connected axially loaded double beam system for different materials and geometry were measured experimentally and analyzed theoretically. The theory predicts that natural frequencies of the system are composed of two infinite sets, describing in-phase and out-of-phase vibrations. It is observed, for the case of identical beams, that the in-phase frequencies are independent of the elastic foundation stiffness and its frequencies are identical to a single beam with the same boundary conditions. To compare and verify the accuracy and reliability of theoretical models, experimental measurements of natural frequencies of free vibration of axially tensioned, double beams interconnected by a silicone rubber foundation with fixed-fixed supported conditions are conducted. The first four synchronous natural frequencies were measured, and they were found to increase with increasing tension. The experiments showed that the synchronous natural frequencies of axially tensioned double beam system with fixed-fixed end conditions are in excellent agreement with those for a tensioned single beam with the same end conditions. The asynchronous mode frequencies are not observed, and believed to be due to the existence of damping properties in the elastic foundation, which suppressed the out-of-phase (asynchronous) mode frequencies.

1. INTRODUCTION

Layered (sandwich) structures are used in many applications such as airplanes, bridges, ships, and space vehicles. Sandwich structures are composed of several beam/plate layers separated by flexible core(s), all bonded together, to support loads as a whole [1]. Such composite beam structures have been developed in many industries to meet important structural performance criteria such as weight reduction, strength improvement and vibration absorption. In general, the material, the cross sectional areas, the mass

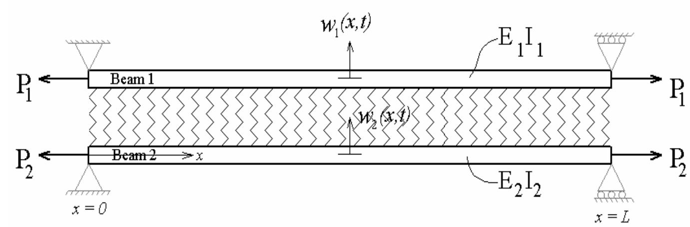

Figure 1 Two beams connected by an elastic foundation.

densities, as well as the support conditions of the individual beams that make up the sandwich structure could be different.

The vibration of beams connected by elastic foundation is very common in engineering, and a variety of problems adopt it as a model. The basic mathematical model uses the Winkler foundation, in which a beam rests continuously on a system of closely spaced unconnected linear springs. A considerable theoretical and experimental work on the transverse vibrations of such systems has been done. Most of the studies considered different aspects of the vibration of double beam systems as sandwich structures [2-10], or as vibration absorbers [11-19]. While in some studies the constrained layer connecting the two beams are treated as an elastic and/or a viscoelastic foundation [3, 7-8, 12-13, 16, 18 20-22], in others it is treated as discrete springs [14, 20, 23-27]. Although most of the studies on transverse vibrations of double beam systems consider numerical approximations, many researchers analyze the vibrations using analytical methods [2, 6-7, 11-15, 20-21,23-32], and some experimental measurements were reported to verify the predicted results [2, 33].

Jensen *et al*. measured the synchronous mode frequencies of sandwich beams with fixed-fixed end-support conditions [34]. Sokolinsky *et al*. measured the natural frequencies of sandwich beams with fixed-free boundaries [2]. They used two thin aluminum sheets, flexible foam core, and two accelerometers to measure the in-phase and out-of-phase mode frequencies.

In this paper, the experimental measurements of natural frequencies of free vibration of two axially tensioned, beams interconnected by a silicone rubber foundation with fixed-fixed supported conditions are described. Experimental results are compared to theoretical predictions.

2. THEORY

The system shown in Fig. 1 consists of two parallel, slender, prismatic and homogeneous beams, joined by a Winkler foundation of stiffness k. Both beams have the same length, and axially tensioned, to p_1 and p_2 as shown. The coupled governing equations of the free transverse vibrations of the system are derived using Euler- Bernoulli beam theory and can be written as [24]:

$$m_1\frac{\partial^2 W_1}{\partial t^2}+E_1I_1\frac{\partial^4 W_1}{\partial x^4}-p_1\frac{\partial^2 W_1}{\partial x^2}+k(W_1-W_2)+ c(\frac{\partial W_1}{\partial t}-\frac{\partial W_2}{\partial t})=0 \qquad (1)$$

$$m_2\frac{\partial^2 W_2}{\partial t^2}+E_2I_2\frac{\partial^4 W_2}{\partial x^4}-p_2\frac{\partial^2 W_2}{\partial x^2}+k(W_2-W_1)+ c(\frac{\partial W_2}{\partial t}-\frac{\partial W_1}{\partial t})=0 \qquad (2)$$

where $W_i = W_i(x,t)$ are the transverse deflections of the two beams (i=1, 2), x is the spatial coordinate, t is the time, m_i the mass per unit length, E_i is the Young's modulus, I_i is the second moment of area of the beam cross section, k is the stiffness of the Winkler foundation, c is the damping coefficient of the foundation which is taken to be zero in the theoretical analysis in this paper, L is the distance between the simple supports; and p_i is the applied axial force on the beam cross section. The beams are supported by built-in boundaries. The built-in boundary conditions are:

$$W_1=\frac{\partial W_1}{\partial x}=0 \text{ at } x=0 \text{ and } x=L \qquad (3)$$

$$W_2=\frac{\partial W_2}{\partial X}=0 \text{ at } x=0 \text{ and } x=L \qquad (4)$$

2.1 NATURAL FREQUENCY ANALYSIS

The solution of Eqs. (1) and (2) can be written in the form:

$$W_1=\mathrm{Re}\left(\hat{\phi}_1(x)e^{\bar{\lambda}T}\right) \text{ and } W_2=\mathrm{Re}\left(\hat{\phi}_2(x)e^{\bar{\lambda}T}\right) \qquad (5)$$

where the eigenvalues $\bar{\lambda}$ (= $i\omega$) are purely imaginary, with i = √-1, and the eigenfunctions $\hat{\phi}_j(x)$ are complex. The orthonormality of the eigenfunctions can be shown by casting the equations in state space representation, and by investigating the symmetry and skew symmetry of the operator matrices [28-32]. This is demonstrated for this problem in reference [35][35].

In order to obtain the natural frequencies and mode shapes of the system, the homogenous form of equations (1) and (2) are combined into a single eighth-order PDE:

$$\frac{\partial^4 W_1}{\partial t^4}+A_1\frac{\partial^4 W_1}{\partial t^3\partial x}+A_2\frac{\partial^4 W_1}{\partial x^2\partial t^2}+A_3\frac{\partial^4 W_1}{\partial x^3\partial t}+ A_4\frac{\partial^2 W_1}{\partial t^2}+A_5\frac{\partial^2 W_1}{\partial x\partial t}+A_6\frac{\partial^2 W_1}{\partial x^2}+A_7\frac{\partial^4 W_1}{\partial x^4}=0 \qquad (6)$$

The constant coefficients A_n (n = 1…7) depend on the system parameters [35]. Using Eqn. (5) eigenfunctions become:

$$\hat{\phi}_1(x)=\sum_{k=1}^{8}c_k e^{i\gamma_k x} \qquad (7)$$

where c_k are constant coefficients, and γ_k are the roots of the characteristic equation of Eq. (6), which can be obtained by substituting $e^{i\gamma_k x}e^{\bar{\lambda}T}$ into Eq. (6). The eigenfunction $\hat{\phi}_2$ is, using Eqs. (1), (2) and (7);

$$\hat{\phi}_2(x)=\sum_{k=1}^{8}B_k c_k e^{i\gamma_k x} \qquad (8)$$

with,

$$B_k=\frac{1}{K}\left[(1-\nu^2)\gamma_k^2+2\nu\ i\gamma_k\bar{\lambda}+\bar{\lambda}^2+K\right]. \qquad (9)$$

The coefficients $\boldsymbol{c}=\{c_1\ c_2\ \dots\ c_8\}^T$ can be obtained by using Eqs. (7)-(9) in Eqs. (1)-(2). This results in eight homogeneous algebraic equations, represented as:

$$\boldsymbol{D}.\boldsymbol{c}=0 \qquad (10)$$

where $\boldsymbol{D}$ is the matrix of coefficients. In order to have a nontrivial solution, the determinant of matrix $|\boldsymbol{D}|$ must be zero, where the variable is the frequency $\bar{\lambda}$. This gives the characteristic equation of the system. The mode shapes are found from Eqs. (7) and (8).

The natural frequencies of the system are composed of two infinite sets, ω_{1n} and ω_{2n} [35-36] with n = 1,2,3... The free vibration is composed of *in-phase* vibrations with ω_{1n}, and *out-of-phase* vibrations with ω_{2n}. In case the two beams are identical, these modes become *synchronous* and *asynchronous*, respectively. The synchronous natural frequencies are independent of the elastic foundation stiffness and identical to those of a single tensioned beam. The asynchronous natural frequencies are dependent on the elastic foundation stiffness and identical to those for a single tensioned beam on an elastic foundation of stiffness 2k [35]

The synchronous mode shapes stem from the fact that the two continuous media are vibrating with the same amplitude and direction anti-symmetrically, with respect to an axis passing through the mid-thickness of the system. On the other hand, the asynchronous mode shapes are vibrating with the same amplitude but in opposite directions symmetrically with respect to the mid-thickness. Therefore, synchronous modes do not cause stretching of the elastic foundation, while asynchronous modes do. This synchronization and asynchronization of the mode shapes deteriorate if the masses, flexural rigidities, or axial tensions are not identical. A general tendency is seen to increase the natural frequencies in the case of increasing axial tension, and the elastic stiffness of the foundation [34-35].

3. TEST PROCEDURE

In the present study, the double beam system, installed in an Instron (Norwood, MA) tensile testing machine, was subjected to axial tension. The vibrations, excited by a small hammer, were measured using a laser-Doppler vibrometer (Polytec, Hopkinton, MA). The test specimens consisted of two beams, either both aluminum, or brass, or mixed, bonded by a silicone rubber layer (00684 Dow Corning). The geometrical dimensions of all beams used in this study were 3.18 × 25.4 mm^2 (0.125 × 1 in^2) in cross section and 419.1 mm (16.5 in) in length. The silicone rubber layer had the same width and length of the solid beams, but the thickness was different for each experiment. The mechanical properties of the materials are listed in Table 1.

The specimen is installed in the jaws of the Instron tensile testing machine and clamped. The Instron machine was controlled through its interface software with deformation rate of 1.27 mm/min (0.05 in/min) until the desired axial load was reached. The axial tension was applied on both beams at ends, then, the laser head of the vibrometer was placed perpendicular to the mid-span of one of beams with the recommended stand-off distance of 20.5 cm (8.07 in). The laser head was aligned so that the light was reflected back with signal strength of at least 50 – 60%. Laser-Doppler vibrometry is a non-contact, optical technique for determining the velocity of a point on a vibrating surface. The technique involves directing a laser beam onto the test object and collecting the reflected light. Vibration of the object surface introduces a Doppler frequency shift into the reflected beam, which is used to measure the component of velocity lying along the laser beam axis. This device has practical advantages compared to the accelerometers and displacement sensors, as it provides vibration measurements without adding a mass, and as it is easier to relocate for different measurement positions. The vibrometer software was used with the following setting: the signal is delivered via the laser head to the connected computer, using *Polytec Vibrometer* and *Software Version 3.3*, and the Fourier Transform (FT), of the data was taken. The bandwidth was (0-1.3 kHz). The method involves setting the sample time to 2 – 3 seconds so that several excitations can be made for one measurement.

The tests were conducted under several tensions, and the first several modes were observed. The specimen was excited by hand, using a small hammer four times during the sampling time. The natural frequencies were read from the frequency spectrum determined by the FT of the vibrometer software. Several such measurements were taken at each tension and observed to have consistent frequency peaks.

Table 1 The material properties of the beams

Material	E (GPa)	ρ (Kg/m^2)
Aluminum	69	2790
Brass	105	8410
Silicone foundation	0.62	1330

4. RESULTS AND DISCUSSION

Figures 2 shows the frequency spectrum for the first synchronous natural frequency peaks for double aluminum beam system tensioned with $p_1 = p_2 = 5000$ N. The figure shows clearly the frequency peaks in a bandwidth (BW) of (0-1.3 kHz). In these experiments, one laser spot was used. The experimental frequencies were measured in four different ways: pointing the laser head on the front side of the beam and excite it from both sides, then, pointing the laser head on the other side, and excite it from either sides. The measured peaks were consistent, which indicates that when the laser pointed from the two sides and the excitation of the specimen from either side showed the same peaks.

Three experiments were conducted in this study;

- one test involved single beams using aluminum; and,
- two tests involved double beam systems, involving aluminum and brass, bonded by a silicone rubber layer.

The purpose of the conducting the first two tests was to calibrate the experimental measurements with respect to the well established theoretical predictions for single fixed-fixed beams (e.g., [35]). The first test was conducted with a single aluminum beam, clamped at both ends on the Intsron machine. Figure 3 shows the first four experimental and analytical natural frequencies of this system as a function of tension. The experimental frequencies are compared to the analytically predicted ones. It is observed that all the measured natural frequencies are lower than the analytical frequencies within 10%.

Next experiment was conducted using a pair of aluminum beams, and a pair of brass beams each bonded with 0.46 and 0.91 mm (0.018 and 0.036 in) thick silicone rubber layers, respectively. Figures 4 and 5 show the first four natural frequencies obtained experimentally, and the first four synchronous natural frequencies obtained analytically. In these experiments the asynchronous frequencies were not observed.

Same observation was made in References [21, 34]; Jensen *et al.* reported on the difficulty to measure the asynchronous modes [34], and Vu *et al.* have demonstrated analytically that damping in the elastic layer suppresses the asynchronous modes [21]. Nevertheless, in summary, the experimental results are all within 10% error from the predicted results which show the success of the setup, as it was able to measure the frequencies of a single beam within the same error range.

The double beam theory states that the natural frequencies of two identical beams connected by an elastic foundation are separated into synchronous and asynchronous modes [14,23-24,15,21,37]. The theoretical synchronous frequencies of the double beam system, supported by fixed boundaries are identical to those of a single beam with fixed at ends [15, 21]. This is demonstrated in Figs. 6 and 7 where the first four experimentally measured frequencies of single and double beam systems are compared. These figures demonstrate excellent agreement to the theoretical predictions stated above.

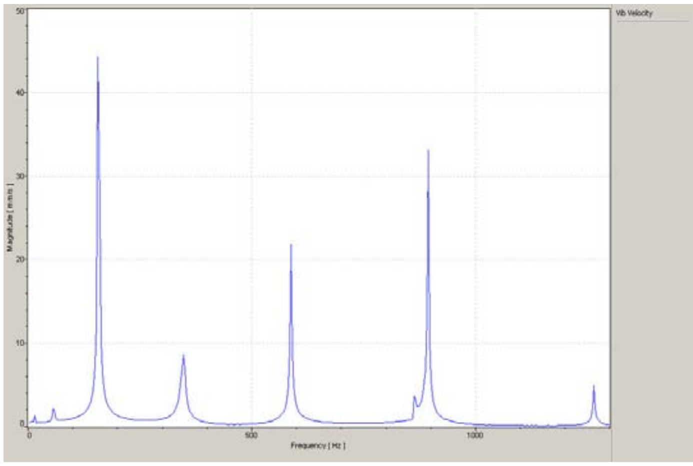

Figure 2 The frequency spectrum for double aluminum beam system (tension = 5000 N).

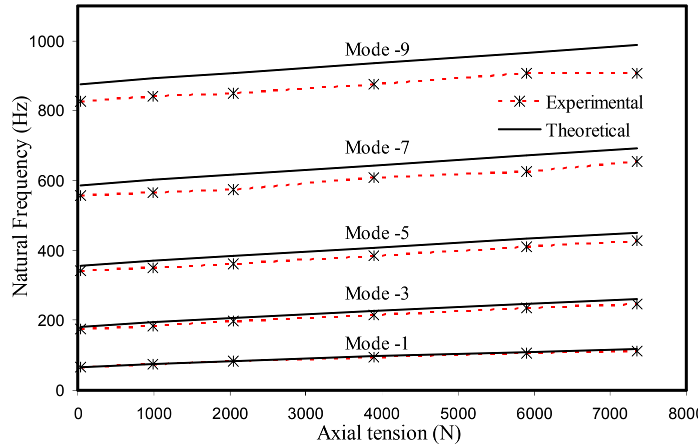

Figure 5 The experimental and theoretical frequencies for double brass beam system with silicone layer thickness 0.91 mm. Brass beams were 3.18 mm thick.

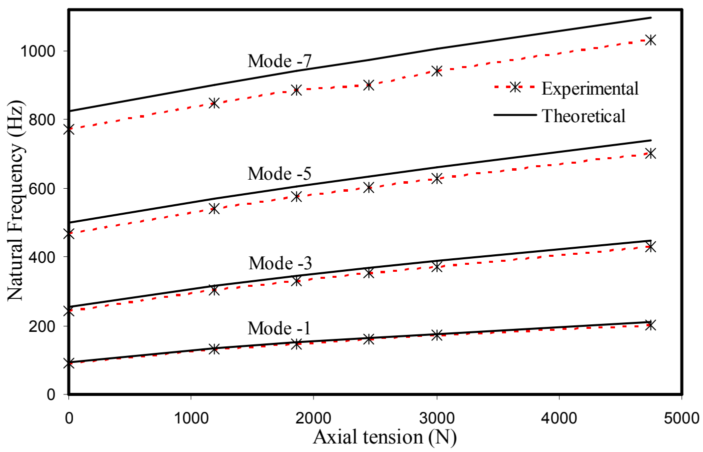

Figure 3 The experimental and theoretical frequencies for single aluminum beam. 3.18 mm thick.

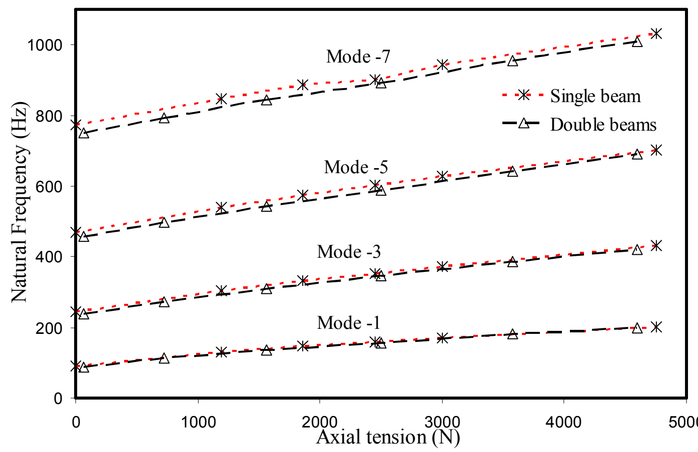

Figure 6 Comparison of the experimental frequencies for single and double aluminum beam systems. Silicone thickness for double beam system is 0.46 mm.

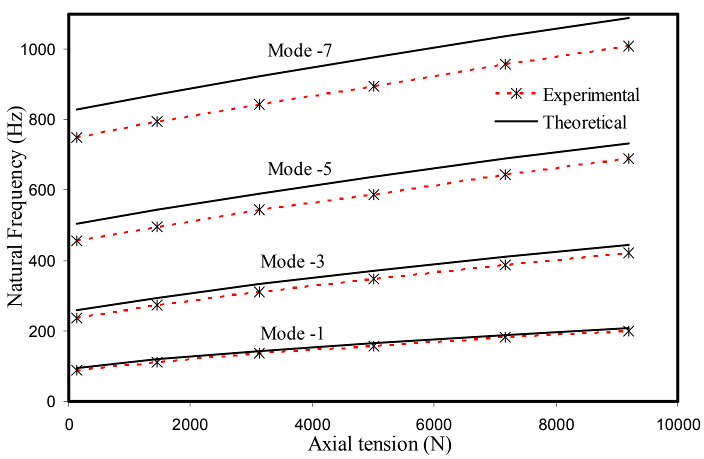

Figure 4 The experimental and theoretical frequencies for double aluminum beam system with silicone layer thickness 0.46 mm. Aluminum beams were 3.18 mm thick.

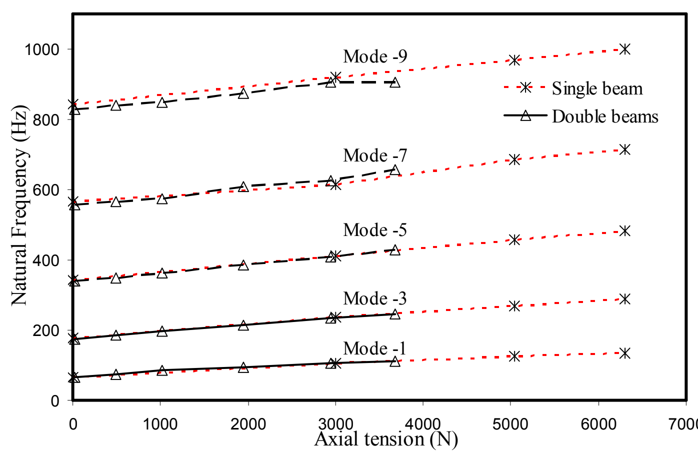

Figure 7 Comparison of the experimental frequencies for single and double brass beam systems. Silicone thickness for double beam system is 0.91 mm.

5. SUMMARY AND CONCLUSIONS

The natural frequencies of the system are composed of two infinite sets, ω_{1n} and ω_{2n} [35-36]. The free vibration of such system is described by two types of motions; in-phase vibrations with ω_{1n}, and out-of-phase vibrations with ω_{2n}. In case the two beams are identical, these modes become synchronous and asynchronous, respectively. The synchronous natural frequencies are independent of the elastic foundation stiffness and identical to those of a single tensioned beam.

The asynchronous natural frequencies depend on the elastic foundation stiffness. Experiments to measure the natural frequencies of free vibration of tensioned double beams, with fixed-fixed end conditions, interconnected by a silicone layer were presented. The measured results are in very good agreement with the analytical results within 10% error. By exciting the specimen from either side, or pointing the laser head from either side, the measured peaks were found to be consistent. The first four synchronous natural frequencies were measured, and they were found to increase with increasing tension. It was observed that the double beams in the experiments, were acting perfectly as a double beam system predicted by the theory [34-35]. The experiments showed that the synchronous natural frequencies of axially tensioned double beam system with fixed-fixed at ends are in excellent agreement with those for a tensioned single beam with the same end conditions as predicted from the physics of the system. The asynchronous mode frequencies were not observed. This is believed to be due to the existence of damping in the elastic foundation, which suppressed the out-of-phase (asynchronous) mode frequencies [28, 38-40].

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