Int. J. Nonlinear Anal. Appl. 6 (2015) No.2, 133-136
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.262

Modified homotopy method to solve non-linear integral equations

Mohsen Rabbani

Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran

(Communicated by M. Eshaghi Gordji)

Abstract

In this article we decide to define a modified homotopy perturbation for solving non-linear integral equations. Almost, all of the papers that was presented to solve non-linear problems by the homotopy method, they used from two non-linear and linear operators. But we convert a non-linear problem to two suitable non-linear operators also we use from appropriate bases functions such as Legendre polynomials, expansion functions, trigonometric functions and etc. In the proposed method we obtain all of the solutions of the non-linear integral equations. For showing ability and validity proposed method we compare our results with some works.

Keywords: Homotopy perturbation; Integral Equations; Non-linear; Basis Functions; Legendre Polynomials.
2010 MSC: Primary 46B99;, 47H10; Secondary 47J25.

1. Introduction and preliminaries

Non-linear integral equations are the class of important problems in applied sciences and engineering. So, almost all of the scientists in different branch of the sciences are interesting to mathematics specially non-linear problems. In this section we remind a brief of homotopy perturbation method.

Homotopy is an important part of topology and it can convert any non-linear problem in to finite linear problems. A kind of homotopy perturbation method introduced in[5] that is depend on small parameter and this dependence to small parameter can be lead to wrong solution. But in [3, 4] introduced an other homotopy method that it is not depend on small parameter. Also in

Email address: mrabbani@iausari.ac.ir (Mohsen Rabbani)

Received: January 2015 *Revised:* April 2015

[6] a modified was down for homotopy method. In the follow of the above mentions we consider to non-linear problem with boundary condition:

$$\begin{cases} A(u) = f(s), & s \in \Omega \\ B\left(u,\ \frac{\partial u}{\partial n}\right) = 0, & n \in \Gamma \end{cases} \tag{1.1}$$

where A is a general differential operator, B is a boundary operator, $f(s)$ is a known analytic function and Γ is the boundary of the domain Ω, (see[2, 3]). In the above references the operator A, is divided in to the linear and non-linear operators in the names $L$and N respectively. So, Eq.(1.1) was converted in the following form:

$$L(u) + N(u) - f(s) = 0. \tag{1.2}$$

Also in [2, 3] homotopy perturbation was defined to this form,

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[N(v) - f(s)\right] = 0, \;\; p \in [0,1] \tag{1.3}$$

where v is a general approximation and u_0 is an initial approximation for u.

2. Modified Homotopy Perturbation Method

We divide A operator and f function to some sections as follows;

$$A_1(u) + A_2(u) = f_1(s) + f_2(s), \tag{2.1}$$

where A_1, A_2 are non-linear operators, also f_1 and f_2 are functions, such that $f_1(s)$ can be approximated by polynomials with less than k degree, Legendre polynomials, trigomometric functions, expansion functions and etc. it's dependent to the kind of Eq. (1.1) Also we can write, $f_2(s) = f(s) - f_1(s)$.

According to the above subjects we have,

$$N_1(u) + N_2(u) = 0,$$

where $N_1(u) = A_1(u) - f_1(s)$ and $N_2(u) = A_2(u) - f_2(s)$.

Now, we define a homotopy, to this form;

$$H(v,p) = N_1(v) + pN_2(v) = 0, p \in [0,1] \tag{2.2}$$

By considering to Eq. (2.2) , with variations $p = 0$ to $p = 1$ we obtain $A_1(u) = f_1(s)$ to $A(v) = 0$, in other words we find solution of the Eq. (1.1) for $p = 1$. In the follow we use from some approximations such as,

$$v = \sum_{i=0}^{\infty} p^i v_i, \tag{2.3}$$

$$f_1(s) = \sum_{i=0}^{n} \alpha_i \Phi_i(s), \tag{2.4}$$

In the Eq. (2.2) Now, we consider to non-linear integral equations,

$$u(s) = f(s) + \int_a^b k(s,t) g\left(t, u(t)\right) dt, \quad a \le s \le b \tag{2.5}$$

by using Eqs. (2.2-2.4) and Adomain polynomials in the above process for solving Eq. (2.5), we introduce an algorithm as follows:

$$\begin{cases} v_0(s) = f_1(s), \\ v_1(s) = f(s) - f_1(s) + \int_a^b k(s,t)A_0(t)dt, \\ v_{n+1}(s) = \int_a^b k(s,t)A_n(t)dt, \quad n \geq 1 \end{cases} \tag{2.6}$$

where, Adomain polynomials (see [8]) is given by,

$$A_n(t) = \frac{1}{n!}\left(\frac{d^n}{dp^n} g\left(t, \sum_{i=0}^{\infty} p^i v_i(t)\right)\right)_{p=0}. \tag{2.7}$$

For finding solution of Eq. (2.5) by the above algorithm (2.6), we set $v_1(s) = 0$ so, suitable choose of the $v_0(s)$ is important.

3. Applications

In this section we solve two examples that they are in [1, 7] respectively.

Example 3.1. Consider the non-linear integral equation,

$$u(s) = \cos\, s^2 + \int_0^{\frac{\pi}{6}} \sin\, s^2 \;\; (u(t))^2\, dt. \tag{3.1}$$

In[1] with Adomain and Simpson role methods in 5th stage is given by

$$u_5(s) = \cos\, s^2 + 0.000164953 \sin\, s^2,$$

also absolute error is 3×10^{-4}, but we use basis functions as $\{\cos is^2,\ \sin is^2\}_{i=0}^{\infty}$ and applying it for solving Eq. (2.5), so by using (2.6) algorithm, we can write,

$$\begin{aligned} v_0(s) &= \alpha_0, \\ v_1(s) &= \cos s^2 - \alpha_0 + \textstyle\int_0^{\frac{\pi}{6}} \sin\, s^2\ \ \alpha_0^2 dt, \\ &= (-\alpha_0)(1) + (1)\cos s^2 + \left(\alpha_0^2 \tfrac{\pi}{6}\right) \sin\, s^2, \end{aligned} \tag{3.2}$$

with condition of $v_1(s) = 0$ and linear independence of the elements of $\{1, \cos s^2,\ \sin\, s^2\}$ we conclude that relations of (3.2) is invalid. So, we increase number of elements of basis to construct $v_0(s)$, for this we choose $v_0(s) = \alpha_1 \cos s^2 + \alpha_2 \sin\, s^2$ then, according to (2.6) we have,

$$v_1(s) = (1 - \alpha_1)\cos s^2 + (a\, \alpha_2\alpha_1^2 + b\, \alpha_2^2 + c\, \alpha_1\alpha_2) \sin\, s^2,$$

where $a = 0.515837,\ b = 0.00776208$ and $c = 0.0936628$.

By considering of the linear independence for elements of $\{1, \cos s^2,\ \sin\, s^2\}$ we can get $\alpha_1 = 1,\ \alpha_2 = 0.571946,\ \alpha_2 = 116.193$. So, the solutions of the relations (3.2) are as,

$$\begin{cases} v(s) \approx v_0(s) = \cos s^2 + 0.571946 \sin\, s^2, \\ v(s) \approx v_0(s) = \cos s^2 + 116.193 \sin\, s^2. \end{cases} \tag{3.3}$$

For the above solutions the absolute errors are 1.1×10^{-16} and 8.3×10^{-13} respectively.

So, the proposed method is shown the high accuracy in the compare of [1].

Example 3.2. In this example, we consider the non-linear integral equation solved by wavelet basis in [7], also with absolute error equal to 10^{-7} for $u(s) = e^s$ as an exact solution,

$$u(s) = e^{s+1} - \int_0^1 e^{s-2t} (u(t))^3 \, dt. \tag{3.4}$$

So, solution of the above integral equation is given by $u(s) = e^s$. But we give all of the solutions for Eq.(3.4). In this way it's enough that we choose $v_o(s) \in \mathit{linear\,span}\ \left\{1, \overset{is}{e}\right\}_{i=0}^{\infty}$, then we have,

$$\begin{aligned} &v_0(s) = \alpha_0 + \alpha_1 e^s, \\ &v_1(s) = f(s) - v_0(s) + \textstyle\int_0^1 k(s,t)\,(v_0(t))^3 \, dt, \end{aligned}$$

by condition $v_1(s) = 0$, solutions of the Eq. (3.4) are in the following form

$$\begin{aligned} &u(s) \approx v_o(s) = e^s, \\ &u(s) \approx v_o(s) = (-0.5 + i(1.15411))\, e^s, \\ &u(s) \approx v_o(s) = (-0.5 - i(1.15411))\, e^s. \end{aligned}$$

It is obvious that, we find exact solution of the non-linear integral equation in one step and also, we could find others solutions of it.

4. Conclusion

In this paper, we try for introducing a modified homotopy perturbation method to solve non-linear integral equations. In the proposed method we found almost all of the solutions of problem in the only one step with high accuracy. These are ability and validity our method.

References

[1] E. Babolian and A. Davari, *Numerical implementation of Adomiam decomposition method*, Appl. Math. Comput. 153 (2004) 301–305.

[2] J.H. He, *Homotopy Perturbation Technique*, Appl. Mech. Engin. 178 (1999) 257–262.

[3] J.H. He, *A coupling method of a hamotopy technique and a perturbation technique non-linear problems*, Int. J. Non-Linear Mech. 35 (2000) 37–43.

[4] S.J. Liao, *An approximate solution technique not depending on small parameters: a special example*, Int. J. Non-Linear Mech. 30 (1995) 371–380.

[5] A.H. Nayfeh, *Problems in Perturbation*, J. Wiley, New York, 1985.

[6] M. Rabbani, *New homotopy perturbation method to solve non-linear problems*, J. Math. Comput. Sci. 7 (2013) 272–275.

[7] M. Rabbani and K. Maleknejad, *A modification for solving Fredholm-Hammerstein integral equation by Using wavelet basis*, Kybernetes: The international journal of systems, Cyb. Manag. 38 (2009) 615–620.

[8] A.M. Wazwaz, *A reliable modification of Adomian decomposition method*, Appl. Math. Comput. 102 (1999) 77–86.