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An Approximation of the Analytic Solution of Some Nonlinear Heat Transfer Equations: A Survey by using Homotopy Analysis Method

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Abstract

In this letter, the approximate solution of nonlinear heat diffusion and heat transfer and also the energy balance for a differential fin element are developed via Homotopy Analysis Method HAM. This method is a strong and easy–to–use analytic tool for investigating nonlinear problems, which does not need small parameters. Homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series.

By suitable choice of the auxiliary parameter $\hbar$, we can obtain reasonable solutions for large modulus. In this study, we compare obtained results through HAM results, with those of homotopy perturbation method and the exact solutions. The first differential equation to be solved is a straight fin with a temperature–dependent thermal conductivity and the second one is the modeling equation of a cooling Lumped system with variable specific heat.

Keywords: Heat transfer; Homotopy analysis method; Fin Temperature Distribution; Cooling

1. Introduction

Most engineering problems, especially some diffusion and heat transfer equations are nonlinear, and in most cases it is difficult to solve them, especially analytically. Perturbation Method is one of the well–known methods to solve nonlinear problems. It is based on the existence of small/large parameters, the so

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called perturbation quantity. However, perturbation methods cannot provide us with a simple way to adjust and control the convergence region of the given approximate series.

Unlike analytical perturbation methods the homotopy perturbation method does not depend on a small parameter which is difficult to find. Comparing different methods show that, when the effect of the nonlinear term is negligible, homotopy perturbation method and the common perturbation method have nearly the same answers but when the nonlinear term in the heat equation is more effective, there will be a considerable difference between the results.

As the homotopy perturbation method does not need a small parameter, the answer will be closer to the exact solution and also to the numerical one.

HPM [1–5] is one of the most powerful methods which provides the user with acceptable analytical results of convenient convergence and stability. HPM is applied to a wide class of nonlinear differential equations including nonlinear heat transfer equations. In 1992, Liao employed the basic ideas of homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method HAM [6–16]. This method has been successfully applied to solve many types of nonlinear problems by others [17–24,28–30]. In this letter, the basic idea of HAM is introduced and then its applications in heat transfer are studied. Also a comparison is made with the exact solution and the HPM results. In this letter, the energy balance for a differential fin element is developed, the resulting nonlinear differential equation is solved by HAM to evaluate the temperature distribution within the fin. Using the temperature distribution, the efficiency of the fin is expressed through a term called thermo–geometric fin parameter (ψ), and thermal conductivity parameter (β), which describes the variation of thermal conductivity. Since the resulting analytical expression for the fin efficiency is too complicated, the data from the expression has been correlated for a wide range of thermo–geometric fin parameter and the thermal conductivity parameter. The correlation equations of compact from are useful for designing straight fins with variable thermal conductivities.

2. The Basic Concept of Homotopy Analysis Method

Let us assume the following nonlinear differential equation in form:

$$N(u(\tau)) = 0$$

where N is a nonlinear operator, τ is an independent variable and $u(\tau)$ is the solution of the equation, we define the function, $\phi(\tau, p)$ as follows:

$$\lim_{p \to 1} \phi(\tau, p) = u_0(\tau)$$

where, $p \in [0,1]$ and $u_0(\tau)$ is the initial guess which satisfies the initial or boundary condition and if

$$\lim_{p \to 1} \phi(\tau, p) = u(\tau)$$

Using the homatopy method generalization, Liao's so–called zero–order deformation equation will be:

$$(1-p)\ L[\phi(\tau,p)-u_0(\tau)]=phH(\tau)\ N[\phi(\tau,p)] \tag{1}$$

where $\hbar$ is the auxiliary parameter which increases the convergence of the result, $H(\tau)$ is the auxiliary function and L is the linear operator. It should be noted that there is a great freedom to choose the auxiliary parameter $\hbar$, the auxiliary function $H(\tau)$, the initial guess $u_0(\tau)$ and the auxiliary linear operator L. This freedom plays an important role in stablishing the keystone of validity and flexibility of HAM as shown in this paper.

Thus, when p increases from 0 to 1, the solution $\phi(\tau,\mathrm{p})$ changes between the initial guess $u_0(\tau)$ and the solution $\mathrm{u}(\tau)$, the taylor series expansion of $\phi(\tau,\mathrm{p})$ with respect to p is:

$$\phi(\tau,p)=u_0(\tau)+\sum_{m=1}^{\infty}\ u_m(\tau)\ p^m \tag{2}$$

$$u_0^{[m]}\ (\tau)=\frac{\partial^m\ \phi(\tau,p)}{\partial p^m}\Big|_{p=0} \tag{3}$$

where $u_0^{[m]}(\tau)$ for brevity is called the mth order of deformation derivation which reads:

$$u_m(\tau)=\frac{u_0^{[m]}}{m!}=\frac{1}{m!}\ \frac{\overset{m}{\partial}\ \phi(\tau;p)}{\partial\ p^m}\Big|_{p=0} \tag{4}$$

It's clear that if the auxiliary parameter $\hbar=-1$ and the auxiliary function is determined to be $H(\tau)=1$, Eq. (1) will be:

$$(1-p)\ \ L(Q(\tau,P))-u_0(\tau)]+P(\tau)\ N[Q(\tau,P)=0$$

The statement is commonly used in HPM procedure. Indeed, in HPM we solve the nonlinear differential equation by separating every Taylor expansion term. Now, we define the vector of $\vec{u}_m$ as follows:

$$\vec{u}_m=\{\vec{u}_1,\vec{u}_2,\vec{u}_3,...\vec{u}_n\}$$

According to Eq. (4), the governing equation and the corresponding initial condition of $u_m(\tau)$ can be deduced from zero–order deformation, Eq. (1). Differentiating Eq. (1) for *m* times with respect to the embedding parameter p and setting $p=0$ and finally dividing by $(m-1)!$, we will have the so called mth order deformation equation in the form:

$$L[u(\tau)-u_0(\tau)]=\hbar\, H(\tau)\, R_m(\vec{u}_m) \tag{5}$$

where,

$$R_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\ \frac{\partial^{m-1}N[\phi(\tau,p)]}{\partial p^{m-1}}\ \Big|_{p=0} \tag{6a}$$

and

$$\chi_m\ \begin{cases}0 & m\le 1\\ 1 & m>1\end{cases} \tag{6b}$$

Thus, applying the inverse operator to both sides of the linear equation, Eq. (5) we can easily solve the equation and compute the generated constants by applying the initial or boundary condition.

3. Applications

3.1 Example 1: Fin Temperature Distribution

Consider a straight fin with a temperature–dependent thermal conductivity, arbitrary constant cross–sectional area. *AC* is the perimeter and *b* is length (see [25, 27]). The fin is attached to a base surface of temperature T_b, extends into a fluid of temperature T_a, and its tip is insulated, the one–dimensional energy balance equation is given as:

$$AC\ \frac{d}{dx}[k(T)\frac{dT}{dx}] - ph(T_b - T_a) = 0$$

The thermal conductivity of the fin material is assumed to be a linear function of temperature as:

$$k(T) = k_a[1 + \lambda(T - Ta)\ ]$$

where k_a is the thermal conductivity at the ambient fluid temperature of the fin, and λ is a parameter describing the variation of the thermal conductivity. Employing the following dimensionless parameters:

$$\theta = \frac{T - Ta}{Tb - Ta} \qquad \xi = \frac{x}{b} \qquad \beta = \lambda(Ta - Tb)$$

$$\psi = (\frac{hpb^2}{k_a\ Ac})^{1/2}$$

the formulation of the problem reduces to:

$$\frac{d^2\theta}{d\xi^2} + \beta\frac{d^2\theta}{d\xi^2} + \beta\ (\frac{d\theta}{d\xi})^2 - \psi^2\theta = 0 \tag{7a}$$

$$\left.\begin{array}{lll} \frac{d\theta}{d\xi} = 0 & at & \xi = 0 \\ \theta = 1 & at & \xi = 1 \end{array}\right\} \tag{7b}$$

Here, we choose the base function with an initial guess that satisfies the Boundary condition in form of:

$$\theta(\xi) = \sum_{n=0}^{\infty} a_n\,\xi^n \tag{8}$$

where a_n is a coefficient to be later determined, and Eq. (8) is called the solution expression. The initial guess is defined in the following form:

$$\theta_0(\xi) = c = 1 \tag{9}$$

Now, we verify the linear operator as:

$$L[\phi(\xi,p)]=\frac{d^2\phi(\xi,p)}{d\xi^2} \tag{10}$$

where

$$L(c_1\xi+c_2)=0 \tag{11}$$

and c_1 and c_2 are constants to be determined through the initial conditions. Now, we determine the nonlinear operator as:

$$N[\phi(\xi,p)]=\frac{d^2\phi(\xi,p)}{d\xi^2}+\beta\phi(\xi,p)\ \frac{d^2\phi(\xi,p)}{d\xi^2}+\beta\ (\frac{d\phi(\xi,p)}{d\xi})^2-\psi^2\phi(\xi,p) \tag{12}$$

According to Eqs. (6–a),(6–b) and (12), we have:

$$R_m(\vec{\theta}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}\left[\underbrace{\frac{d^2\phi(\xi,p)}{d\xi^2}}_{a}+\underbrace{\beta\phi\frac{d^2\phi(\xi,p)}{d\xi^2}}_{b}+\underbrace{\beta\left(\frac{d\phi(\xi,p)^2}{d\xi}\right)}_{c}-\underbrace{\psi^2\phi(\xi,p)}_{d}\right]\Big|_{p=0}$$

$$\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}a\,|_{p=0}=\theta''_{m-1}$$

$$\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}b\,|_{p=0}=\beta\sum_{n=\circ}^{m-1}\theta_n.\theta''_{m-1-n}$$

$$\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}c\,|_{p=0}=\beta\sum_{n=\circ}^{m-1}\theta'_n.\theta'_{m-1-n}$$

$$\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial p^{m-1}}d\,|_{p=0}=-\psi^2\theta_{m-1}$$

therefore,

$$R_m(\vec{\theta}_{m-1})=\theta''_{m-1}+\beta\sum_{n=0}^{m-1}\theta_n.\theta''_{m-1-n}+\beta\sum_{n=0}^{m-1}\theta'_n.\theta'_{m-1-n}-\psi^2\theta_{m-1} \tag{13}$$

where

$$\theta'=\frac{d\theta}{d\xi}\quad and\quad \theta''=\frac{d^2\theta}{d\xi^2}$$

Hence, in an overall form the m^th order of the deformation equation is:

$$\theta_{m-1}(\xi)=\theta_m(\xi)\chi_m+\hbar\int_\circ^\xi[\int_\circ^\xi H(\xi)R_m(\vec{\theta}_{m-1})d\xi]d\xi+c_1\xi+c_2 \tag{14}$$

where c_1 and c_2 can be determined through the boundary conditions. In accordance with the rule of solution expression, $H(\tau)$ must be in form of $d_n\xi^n$, where n is an integer and any $n\le -1$ is forbidden. For simplicity, let $H(\xi)=1$. Thus, we have:

$$\theta_1(\xi)=\hbar\int_0^\xi[\int_0^\xi(\beta\theta_0(\xi)\theta_0''(\xi)+\beta\theta_0'^{\,2}(\xi)+\theta_0''(\xi)-\tau^2\theta_0(\xi)d\xi]d\xi+c_1\xi+c_2$$

$$\theta_2(\xi)=\theta_1(\xi)+\hbar\int_0^\xi[(\beta\theta_0(\xi)\theta_1''(\xi)+\beta\theta_0''(\xi)\theta_1(\xi)$$

$$+2\beta\,\theta_0'(\xi)\theta_1'(\xi)+\theta_1''(\xi)-\psi^2\theta_1(\xi)d\xi]d\xi+c_3\xi+c\xi$$

where c_1, c_2, c_3 and c_4 can be determined through the boundary conditions.

So,

$$\theta_0(\xi) = 1$$

$$\theta_1(\xi) = -\tfrac{1}{2}\hbar\psi^2\xi^2 + \tfrac{1}{2}\hbar\psi^2$$

$$\theta_2(\xi) = -\tfrac{1}{2}\hbar\psi^2\xi^2 + \tfrac{1}{2}\hbar\psi^2 + h\left\{\frac{1}{24}\psi^4\xi^4 + \tfrac{1}{2}(-\beta\hbar\psi^2 - \hbar\psi^2 - \tfrac{1}{2}\psi^4\hbar)\xi^2\right\}$$

$$+\tfrac{5}{24}\psi^4\hbar^2 + \tfrac{1}{2}\beta\hbar^2\psi^2 + 1/2\,\hbar^2\psi^2$$

Now we can compare the results of HAM with the exact solution and that of HPM method:

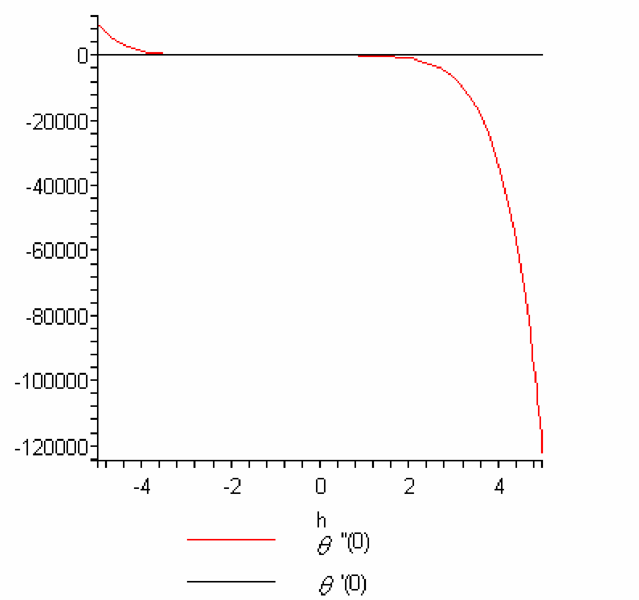

Figure 1. Heat transfer coefficint for $(\beta = 0, \psi = 0{,}5)$, by 7th order approximation.

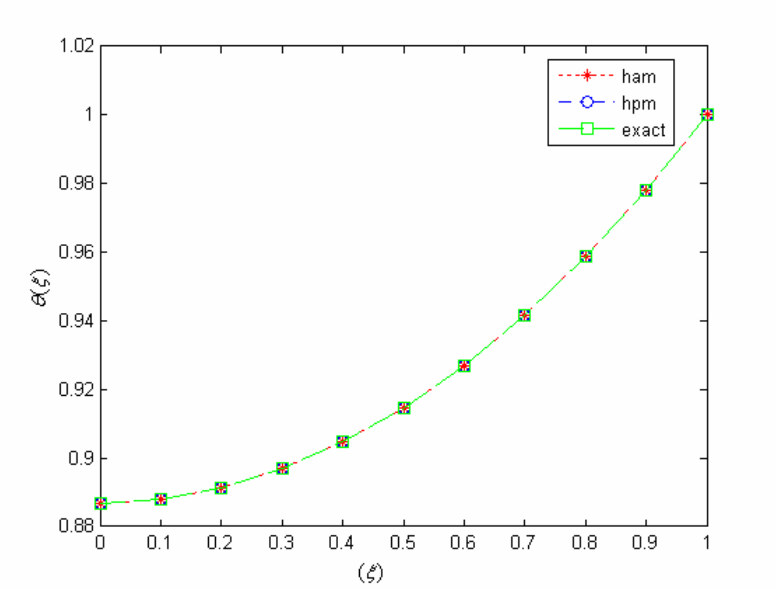

Figure 2. The comparison of the results of HAM for $(\beta = 0\ ,\ \psi = 0.5, \hbar = -0.9)$ *with those of HPM and the exact solution.*

3.2 Example 2: Cooling of A Lumped System with Variable Specific Heat.

Consider the cooling of a lumped system [26,27]; let the system have volume *V*, surface area *A*, density ρ, specific heat *c* and initial temperature, T_i. At time $t = 0$, the system is exposed to a convective environment at temperature, T_a, with convective heat transfer coefficient *h*.

Assume that specific heat c is a linear function temperature of the form:

$$c = c_a[1 + \beta(T - Ta)] \tag{15}$$

where c_a is the specific heat, at temperature T_a and β is a constant, the cooling equation and the initial condition are:

$$\rho VC\,\frac{dT}{dt} + h\,A(T - T_a) = 0 \qquad\qquad T(0) = T_i \tag{16}$$

Introducing. Eq.(13) and using the following dimensionless parameters:

$$\theta = \frac{T - Ta}{Ti - Ta} \qquad\qquad \tau = \frac{t}{pvc_a/hA} \qquad\qquad \varepsilon = \beta(T - Ta)$$

Eq. (16) is transformed to:

$$(1 + \varepsilon\theta)\frac{d\theta}{d\tau} + \theta = 0 \qquad\qquad \theta(0) = 1 \tag{17}$$

Due to the physics of the problem that is a decaying function by time, we assume the solution in the form of an exponential function as follows:

$$u(\tau) = \sum_{n=1}^{\infty} d_n e^{-n\tau} \tag{18}$$

where d_n is a coefficient to be determined afterwards. Eq. (18), the so–called rule of solution expression, guides us to the selection of an auxiliary function which is denoted by $H(\tau)$.

According to Eq. (18), other expressions such as $\tau^m e^{-n\tau}$ or $(d_n Ln(\tau))$ must be avoided, and according to Eqs. (7–a), (7–b) and (18), we choose the linear operator as the following term:

$$L[\phi(\tau,p)] = \frac{d\phi(\tau,p)}{d\tau} + \phi(\tau,p) \tag{19}$$

with the property

$$L(c_1 e^{-\tau}) = 0$$

where c_1 is a constant to be determined through the initial conditions. Now, we can verify the nonlinear operator as:

$$N[\phi(\tau,P)] = \frac{d\phi(\tau,p)}{d\tau} + \phi(\tau,P) + \varepsilon\phi(\tau,p) + \frac{d\phi(\tau,P)}{d\tau}$$

According to Eqs. (17–a) , (17–b) and (18), the initial guess should be in the form $u_0(\tau) = e^{-\tau}$ to satisfy the initial condition and the rule of solution expression. From Eqs. (6) and (19) we have:

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial^{m-1}}\left[\underbrace{\frac{d\phi(\tau,p)}{d\tau}}_{a} + \underbrace{\phi(\tau,p)}_{b} + \underbrace{\varepsilon\phi(\tau,p)\frac{d\phi(\tau,p)}{d\tau}}_{c}\right]$$

$$\frac{1}{(m-1)!}\ \frac{\partial^{m-1}}{\partial p^{m-1}}\ a\Big|_{p=0} = \theta'_{m-1}$$

$$\frac{1}{(m-1)!}\ \frac{\partial^{m-1}}{\partial p^{m-1}}\ b\Big|_{p=0} = \theta_{m-1}$$

$$\frac{1}{(m-1)!}\ \frac{\partial^{m-1}}{\partial p^{m-1}}\ c\Big|_{p=0} = \varepsilon\sum_{n=0}^{m-1}\theta_n.\theta'_{m-1-n}$$

so

$$R_m(\vec{u}_{m-1}) = \theta'_{m-1} + \theta_{m-1} + \varepsilon\sum_{n=0}^{m-1}\ \theta_n\ \theta'_{m-1-n} \tag{20}$$

where $\theta' = \dfrac{d\theta}{d\tau}$, hence we have the mth oeder:

as follows.

$$\left(\theta'_m(\tau) + \theta(\tau)\right) = (\theta'_{m-1}(\tau) + \theta_{m-1}(\tau))\,\chi_m + \hbar H(\tau)\,R_m\,(\vec{\theta}_{m-1}(\tau)) \tag{21}$$

Due to the rule of solution expression, the appearance of expression is form of $d_n\tau$ should be avoided, Hence, $H(\tau)$ must be in form of $e^{-k\tau}$ where k is an integer and every $k \geq 0$ is forbidden, we let $H(\tau) = e^{-\tau}$.

Therefor Eqs. (5), (7–a) , (7–b) we have:

$$\theta_1(\tau) + \theta'_1(\tau) = \hbar\left\{\theta_0(\tau)e^{-\tau} + \theta'_0(\tau)e^{-\tau} - \varepsilon(e^{-\tau})^3\right\} \tag{22}$$

After substituting $\theta_0(\tau)$ in Eq. (22) we have:

$$\theta_1(\tau)+\theta_1'(\tau)=-\hbar\varepsilon(e^{-\tau})^3 \tag{23}$$

So, after solving Eq. (23), we have:

$$\theta_1(\tau)=(1/2\,\hbar\varepsilon e^{-2\tau}+c_1)\,e^{-\tau}$$

where c_1 is computed through the initial condition so , $c_1=-1/2\hbar\varepsilon$ and

$$\theta_1(\tau)=(1/2\,\hbar\varepsilon e^{-2\tau}-1/2\,\hbar\varepsilon).e^{-\tau} \tag{24}$$

also

$$\theta_2(\tau)+\theta_2'(\tau)=\theta_1(\tau)+\theta_1'(\tau)+\left\{\hbar\theta_1(\tau).e^{-\tau}+\hbar\theta_1'(\tau).e^{-\tau}-2\hbar^2\varepsilon^2(e^{-5\tau})+\hbar^2\varepsilon^2(e^{-3\tau})\right\}$$

$$\theta_2(\tau)=\{-\hbar\varepsilon\{1/2\,\hbar\varepsilon\, e^{-2\tau}-1/2\,\varepsilon\hbar e^{-4\tau}-1/2e^{-2\tau}-1/3\hbar e^{-3\tau}\}+c_2\}e^{-\tau}$$

where c_2 is computed through the initial condition

$so, c_2=-1/2\hbar\varepsilon-1/3\hbar^2\varepsilon$

thus we can obtain $\theta_2(\tau)$ asfollows:

$$\theta_2(\tau)=\{-\hbar\varepsilon(\tfrac{1}{2}\hbar\varepsilon\, e^{-2\tau}-\tfrac{1}{2}\varepsilon\hbar e^{-4\tau}-\tfrac{1}{2}e^{-2\tau}-\tfrac{1}{3}\hbar e^{-3\tau})-\tfrac{1}{2}\hbar\varepsilon-\tfrac{1}{3}\hbar^2\varepsilon\}.e^{-\tau}$$

we can determine $\theta_3(\tau),\theta_4(\tau),\ldots$ with the same way as illustrated. So,

$$\theta(\tau)=\theta_0(\tau)+\theta_1(\tau)+\theta_2(\tau)+\theta_3(\tau)+\ldots$$

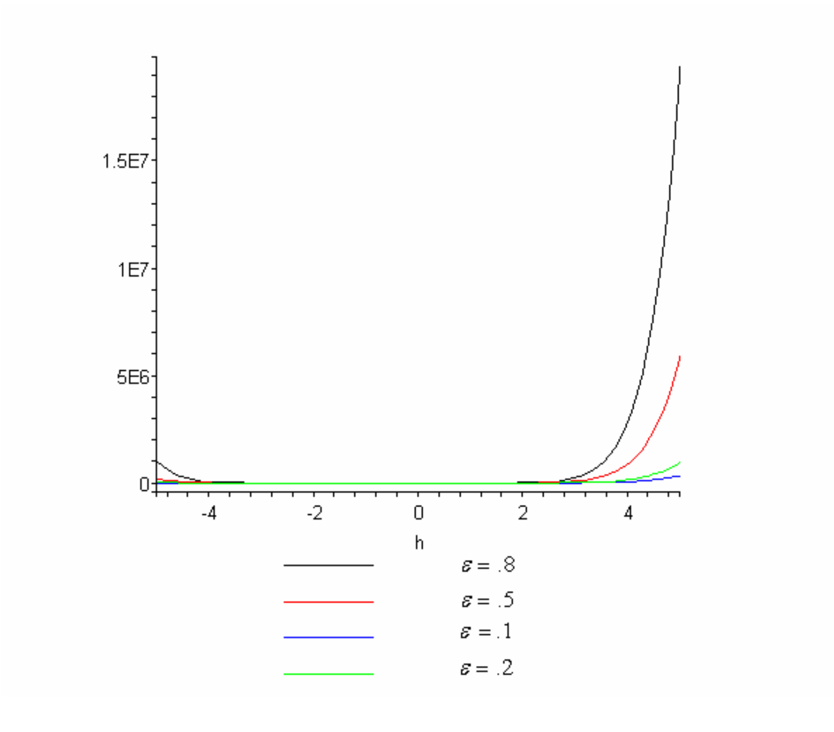

Figure 3. Heat transfer coefficint for $(\varepsilon=0.1,\varepsilon=0.2,\varepsilon=0.3,\varepsilon=0.8)$*, by 10th order approximation.*

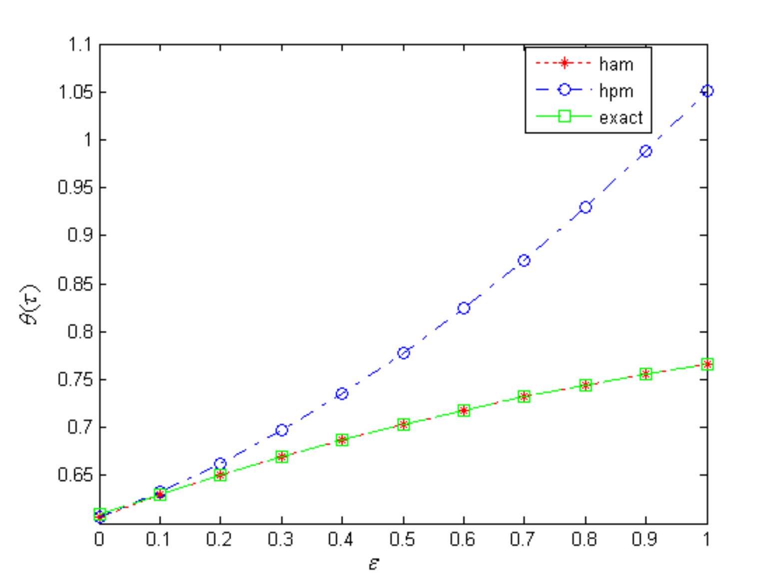

Figure 4. The comparison of the results of HAM for $(\tau = 0.5)$ *and* $(\hbar = -1)$ *with those of HPM and the exact solution.*

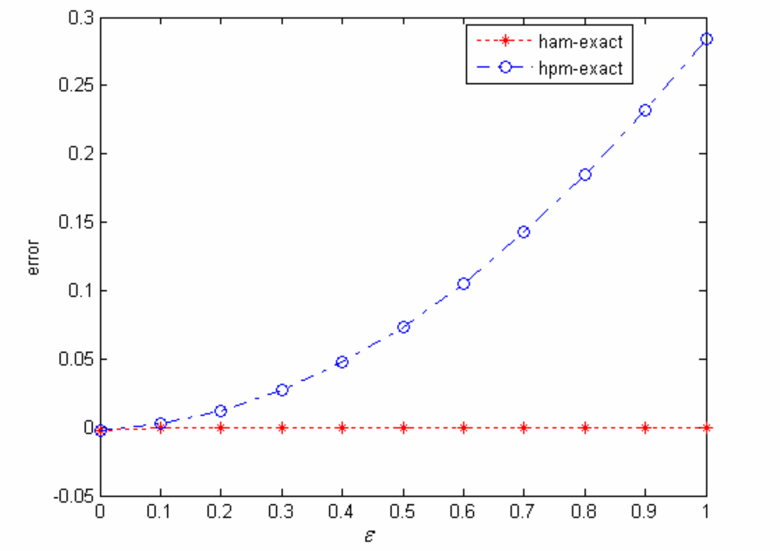

Figure 5. The error of the two methods (HAM and HPM) in comparison with the exact solution.

4. Conclusion

In this study, convective straight fins with temperature–dependent thermal conductivity were analyzed using HAM.This method provides highly accurate numerical solutions for nonlinear problems in comparison with other methods.

The comparison of the method reveals that the approximations obtained by HAM and HPM converge to the exact solution quite fast. Moreover, HAM is faster than HPM, (see figures 2, 4). The auxiliary parameter $\hbar$ provides us with a convenient way to adjust and control the convergence and its rate for the solutions series. When small parameter of ε is increased the error of HAM is less than HPM in comparison with the exact solution. The nonlinear differential equation, which is expressed in terms of suitable dimensionless parameters and is presented in terms of regression equations is obtained by standard statistical techniques; these results can be used for designing straight fins with variable thermal conductivities.

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