(c-)AND(1): a new graph model - more than intersection, more than geometric

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Abstract

We introduce a new graph model. In this model, vertices are points spread in a metric space and each vertex is assigned to a box in the space. Two vertices are connected by an edge if and only if its respective boxes contain the opposite vertex. We focus our study to the simplest case in which vertices are spread in a one dimensional euclidean metric space. We give both, a combinatorial and a simple intersection characterization of the model. Based on these representations, we determine graph families that contain the model (e. g., boxicity 2 graphs) and others that the new model contains (e. g., interval graphs, outerplanar) for which we construct representation algorithms.

1 Introduction

A *geometric graph* is a graph where the set of vertices corresponds to a collection of points that belong to a metric space and an edge connects two vertices if and only if their corresponding points are at a distance of at most a parameter r. An important application of geometric graphs are *sensor networks*. Sensor networks are networks formed by sensor nodes, little devices deployed in a geographic area with monitor purposes. Sensors communicate with each other via a radio channel. Every sensor covers with its radio signal a communication area around it and two sensors communicate with each other when thy are placed within each other communication area. In an ideal world, the communication area of a sensor is a circle. Therefore, in the same ideal world, if every sensor covers equally sized communication areas, sensors form a geometric graph. That explains why researchers have used geometric graphs to represent sensor networks, particularly unit disk graphs [4] or some variations of it [7].

Nevertheless, such an ideal situation is difficult to find in a real deployment, mainly due to physical or geographical restrictions. For instance, when the deployment area is irregular, such as the case of sand dunes, the communication area of a sensor might be shrunken in one direction due to an obstacle,while, in the opposite direction, the area is free of any obstacle. On the other hand, some sensors may have directional antennas which produce communication areas that are far from be a circle, or that place the sensor location far from the *center* of it communication area. Therefore, the existence of a communication link between two sensors is not determined by the distance between them neither by the intersection of their communication areas. In fact, one has to be sure that communication areas *include* their fellow.

Consequently, we propose a new graph family which take into account such restrictions. Let us consider a *set* S and an element $p \in S$ as a *representative element* of S. For a graph of the family, each vertex corresponds to a pair (S, p) and an edge between two vertices exists if and only if the set associated with a vertex contains the representative element of its fellow and vice versa. Note that, according to this definition, nonempty intersection between two sets is not enough to guarantee the existence of their corresponding edge. Moreover, when the sets belong to a metric space, there is no positive distance between two representative elements that guarantees the existence of their corresponding edge. Therefore, this new definition differs from geometric graphs, as well as from intersection graphs.

1.1 Definitions

We start by defining the graph model that we consider in this document. All graphs considered in this document are finite, connected, undirected, loopless and without parallel edges. For a graph $G = (V, E)$, we denote by $V(G)$ and $E(G)$ the set of its vertices and edges, respectively[1]. The edge $\{u, v\}$ is denoted by uv. If $uv \in E(G)$ we say that v is a *neighbour* of u and vice versa. The set of neighbours of u is denoted by $\mathcal{N}(u)$. Additionally, the closed neighborhood of u is $\mathcal{N}[u] = \mathcal{N}(u) \cup \{u\}$.

A *box* in the d-dimensional Euclidean space is the product of d closed intervals. A box B is described as the set $B = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : L_i \leq x_i \leq R_i\}$, where L_i and R_i denote the extremes point of the interval in the i-th dimension. The *center* of a box is the equidistant point to every interval of a box. Namely, the center of the box B is the point $((L_1 + R_1)/2, (L_2 + R_2)/2, \ldots, (L_d + R_d)/2)$.

Définition 1 (AND-realization)**.** An AND-*realization* of a graph G in the d-dimensional Euclidean space is a collection of pairs $\{(B_v, p_v) : v \in V(G)\}$ where each vertex v is associated to a d-dimensional box B_v and to a representative element p_v in the box B_v, such that:

$$uv \in E(G) \Leftrightarrow (p_v \in B_u) \wedge (p_u \in B_v).$$

A *central* AND-realization or c-AND-realization of a graph is an AND-realization in which each representative element p_v is the center of its box B_v.

We denote by AND(d) the set of graphs admitting an AND-realization in the d-dimensional Euclidean space. The subset of AND(d) containing the graphs that admit a c-AND(d)-realization is denoted by c-AND(d). For simplicity, all along

[1]When the context makes clear the graph under consideration, we use V and E.

this document, we use notation (c-)AND when we say something that concerns to both classes c-AND and AND, respectively.

In this document, we focus in the study of sets AND(1) and c-AND(1). In this context, a box B_u becomes an interval in the Euclidean line that we denote $[L(u), R(u)]$. Note that, any (c-)AND(1)-realization $\mathcal{R} = \{([L(u), R(u)], p_u)\}_{u \in V(G)}$ can be modified without any modification in the graph that it represents. We define *d-translation* and *r-scaling* of $\mathcal{R}$ as the realizations $\{([L(u)+d, R(u)+d], p_u+d)\}_{u \in V(G)}$ and $\{([r \cdot L(u), r \cdot R(u)], r \cdot p_u)\}_{u \in V(G)}$, respectively. Note that d-translation and r-scaling do not affect the graph under consideration.

Any (c-)AND(1)-realization of a graph induces a natural ordering of its vertices with respect to the position of its representative elements. Given an ordering π of the vertices of a graph G we denote by $<_\pi$ the total order induces by π. That is, $u <_\pi v$ if u appears before than v in π. The *extremes* vertices of an order $<_\pi$ are the vertices placed at the first and last position according to $<_\pi$. Given a vertex u, we denote by $\ell_\pi(u)$ and $\rho_\pi(u)$ the leftmost and rightmost neighbors of u in the order, i.e., $\ell_\pi(u) = \{v \in \mathcal{N}[u] : v <_\pi w \; \forall \, w \in \mathcal{N}[u], w \neq v\}$ and $\rho_\pi(u) = \{v \in \mathcal{N}[u] : w <_\pi v \; \forall \, w \in \mathcal{N}[u], w \neq v\}$.

1.2 Related work

We consider that there is an infinite amount of interesting literature related with geometric, outerplanar, interval, max-tolerance, boxicity 2, and all graphs we mention here. Nevertheless, since the graph family introduced in this document encloses a brand new definition, it is difficult to find literature directly related with (c-)AND graphs. Here, we mention the most relevant works for this research besides the literature we reference in the rest of the document.

First, we refer the reader to [3], an excellent survey authored by Brandstädt et al. that contains a description of innumerable graph families. This survey also presents containment relation between classes, graphs that separate one class form another, and priceless information in this area. Related with particular graph families, the notion of *boxicity* of a graph was introduced in [9]. On the other hand, the notion of *book embedding* of a graph was introduced in [1]. Finally, the book [5] surveys results related with (*max*-)*tolerance* graphs. Moreover, that book presents many results related with intersection graphs.

1.3 Our contributions

The main contribution of this document is the definition of two graph families: AND(d) and c-AND(d). We study the one dimensional version (c-)AND(1) of the family in which representative element positions induce an order of the vertices. We characterize the AND(1) family via a combinatorial characterization of the possible orders of its vertices in any AND(1)-realization. We also characterize both AND(1) and c-AND(1) families via an intersection model. These two characterizations help us to encounter a link between (c-)AND(1) definitions and other graph families, such as interval, outerplanar and boxicity 2 graphs. We also show clear differences with similar definitions such as tolerance and max-tolerance graphs. Finally, we algorithmically construct (c-)AND(1)-realizations for interval and outerplanar graphs.

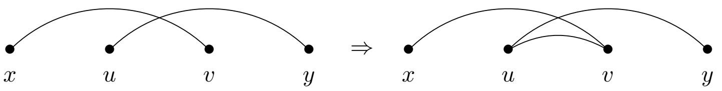

Figure 1: Graphic representation of the four point condition for AND(1).

2 A combinatorial characterization for AND(1) graphs

In this section, we give a combinatorial characterization for the graphs that admit an AND(1)-realization. We first point out the fact that any AND(1)-realization of a graph induces a natural ordering of its vertices by considering the position of their respective representative elements. This ordering needs to have representative elements assigned to different positions in order to be well defined. Nevertheless, it is easy to see that any (c-)AND(1)-realization can be modified to fulfill this property.

Définition 2. ($\mathcal{R}$-order) Given a graph G that belongs to AND(1) and an AND(1)-realization $\mathcal{R}$ of G such that all representative elements are embedded in different positions. The *$\mathcal{R}$-order* of the set V, denoted by $<_\mathcal{R}$, is the total order induced by the positions of the representative elements. That is, for any pair of vertices u and v:

$$u <_\mathcal{R} v \Leftrightarrow p_u < p_v.$$

Consider a $\mathcal{R}$-order of a graph G and two vertices $u <_\mathcal{R} v$ in V. If vertex u has a neighbor y after v ($v <_\mathcal{R} y$) and v has a neighbor x before u ($x <_\mathcal{R} u$) then, vertices u and v are mutually contained in its corresponding intervals. Thus, vertices u and v must be connected. Indeed, this property is crucial when recognizing a graph that belongs to the set AND(1). Therefore, we introduce the following definition for any ordering of the set of vertices of a graph.

Définition 3. Given a graph and an order $<_\pi$ of its set of vertices. We say that $<_\pi$ satisfies the *four point condition* for AND(1) if and only if for every quadruplet of vertices x, u, v, y, it holds:

$$x <_\pi u <_\pi v <_\pi y \text{ and } \{xv, uy\} \subseteq E \Rightarrow uv \in E.$$

Figure 1 shows a graphic representation of the four point condition for AND(1).

We prove that for any graph the existence of an ordering that satisfies the four point condition for AND(1) is necessary and sufficient to decide if it belongs to AND(1).

Théorème 4. *A graph G belongs to* AND(1) *if and only if there exists an ordering of its set of vertices that satisfies the four point condition for* AND(1).

Proof. As we have seen previously, the four point condition is necessary for any AND(1)-realization of G. For the converse, let $<_\pi$ be any ordering of the vertices of G which satisfies the four point condition.

Let $\mathcal{R}_\pi$ be a realization constructed in the following way: representative elements p_v are embedded in the Euclidean line arbitrarily but respecting the order $<_\pi$. For each $v \in V$, we define B_v as the interval covering from the leftmost to the rightmost neighbors of v according to $<_\pi$, that is $B_v = [\ell_\pi(v), \rho_\pi(v)]$.

In order to verify that $\mathcal{R}_\pi$ is an AND(1)-realization of G, consider an edge $uv \in E$ with $u <_\pi v$. By definition of $\mathcal{R}_\pi$, it holds that $u \in B_v$ and $v \in B_u$. On the other hand, if $u \in B_v$ and $v \in B_u$, then there exist vertices $y \in \mathcal{N}(u)$ and $x \in \mathcal{N}(v)$ such that $x <_\pi u <_\pi v <_\pi y$. Thus, vertices u and v are neighbors by the four point condition. □

The four point condition is an useful tool to recognize graph families that belong to the set AND(1) as well as families that do not belong to it. A first example are interval and outerplanar graphs.

Corollaire 5. INTERVAL *and* OUTERPLANAR *graphs are subset of the set* AND(1)*.*

Proof. In [8], Oraliu proves that a graph G is an interval graph if and only if there exists a linear ordering on its vertices $<_\pi$ such that, for all triplet $u, v, w \in V$ with $u <_\pi v <_\pi w$ and $uw \in E$, it holds that $uv \in E$. Such an order $<_\pi$ satisfies the four point condition of AND(1)[2]. Thus, any interval graph G belongs to AND(1).

In order to prove that outerplanar graphs belongs to AND(1), let us recall the definition of *page embedding* of a graph (cf. [1]). A k-page embedding, or book embedding, of a graph G consists in an linear ordering of the vertices of G which are drawn on a line (the spine of the book) together with a partition of the edges into k pages such that two edges in the same page do not cross. The *pagenumber* of a graph is the smallest k for which the graph has a k-page embedding. In [2], Bilski proves that outerplanar graphs are exactly the graphs with pagenumber one. Therefore, for any outerplanar graph there exists an ordering of its vertices in which the edges do not cross. Such an ordering (by emptiness, since no two edges cross) satisfies the four point condition for AND(1). □

Later in this document, we prove stronger results with respect to interval and outerplanar graphs. In contrast to Corollary 5, four point condition allows to discard a graph from the AND(1) set.

Corollaire 6. *Let G be a graph such that all pairs of vertices $u, v \in V$ have at least two non adjacent common neighbors. Then G doest not belong to* AND*(1).*

An example of a graph with this property can be obtained by following the next two steps: (1) take any graph where each vertex has at least two non adjacent neighbors (for instance, a cycle with at least four vertices); (2) add two *universal* vertices that are connected to every vertex in the original graph, but which are not adjacent. These two steps construct a family of graphs that does not belong to AND(1). In section 4 we construct a different family of graphs which does not belong to AND(1).

3 Where is c-AND(1) in the graph world?

In this section we establish the relation of the c-AND(1) set with other well-known graph families. Particularly, we strengthen the result of Corollary 5 by

[2]The order $<_\pi$ can be obtained according to the left extreme of the intervals in an interval representation of G (ties are broken arbitrarily)

showing that interval and outerplanar graphs belong to c-AND(1). Both proofs are constructive and give an algorithm to construct c-AND(1)-realizations for these classes. We begin by showing the contention of interval graphs.

Théorème 7. *The set of* INTERVAL *graphs is a subset of* c-AND*(1).*

Proof. Let G be an interval graph. In [8] Olariu proves that for any interval graph there exists an order $<_\pi$ of its vertices such that for all triplet $u, v, w \in V$ with $u <_\pi v <_\pi w$ and $uw \in E$ then $uv \in E$. For the sake of simplicity we relabel vertices from 1 to n according to the ordering $<_\pi$.

We construct a c-AND(1)-realization of G greedily. At the i-th step we include the vertex i. The inclusion is made in such a way that, at the end of the step i, it holds, for all j, k, w in $\{1, \ldots, i\}$, that:

1. $p_{k-1} < p_k$
2. $\rho(j) <_\pi \rho(k) \Rightarrow R(j) < R(k)$.
3. $L(j) < p_{\ell(j)}$
4. $\rho(k) <_\pi j \Leftrightarrow R(k) < p_j$.

Condition 1 ensures that representative elements are placed according to order $<_\pi$. Condition 2 ensures that right extremes of intervals are in the same order than the values of $\rho(\cdot)$. Finally, conditions 3 and 4 guarantee that the partial realization at the end of step i corresponds to the subgraph induced by vertices $1, 2, \ldots, i$. Thus, at the end of the construction a c-AND(1)-realization of G is obtained.

At the first step all conditions are satisfied. Let suppose that conditions hold at the end of the step $i-1$. We add vertex i in two phases:

- First, we set the position of representative element p_i respecting conditions 1 and 4. That is, the representative element is placed after p_{i-1} and it is contained only by intervals associated to its previous neighbours.

- Second, we set the interval associated to i such that it contains all its previous neighbours, according to condition 3. Finally, we modify, if necessary, the interval of previous vertices in order to satisfy conditions 2.

Let j, k be two vertices with labels smaller than i such that $j \notin \mathcal{N}(i)$ and $k \in \mathcal{N}(i)$. Since vertices follow order $<_\pi$, then $\rho(j) < i \leq \rho(k)$. Thus, by condition 2, we have that $R(j) < R(k)$. Therefore, by defining auxiliary variables $L = \max\{R(j) : j \notin \mathcal{N}(i)\}$ and $R = \min\{R(k) : k \in \mathcal{N}(i)\}$, it holds $L < R$. Notice that in between L and R there might exist some representatives elements. Hence, by setting p_i as $(\max\{p_{i-1}, L\} + R)/2$, conditions 1 and 4 hold.

In order to set the extremes of interval B_i, let define the set $P_i = \{j < i : \rho(j) < \rho(i)\}$. We recall that condition 2 imposes that $R(j) < R(i)$ for all vertex j in P_i. If R' denotes the $\max\{R(j) : j \in P_i\}$ then $R' < R(i)$. On the other hand, the interval B_i must contain $p_{\ell(i)}$ so that condition 3 is satisfied. Let define r_i as $\max\{p_i - p_{\ell(i)}, R' - p_i\} + 1$. We set $L(i) = p_i - r_i$ and $R(i) = p_i + r_i$. With this definition, all conditions are satisfied for vertices in P_i. However, condition 2, does not necessary hold for vertices that do not belong to P_i. To overcome this problem, we extend the intervals of those vertices by $2r_i$. That is, we re-define B_j as $[L(j) - r_i, R(j) + r_i]$ for all $j \notin P_i$. Thus, for all $j \notin P_i$, condition 2 is satisfied since $R(i) = p_i + r_i < R(j) + r_i$, and the Theorem holds. . □

The rest of the section is consecrated to prove that OUTERPLANAR graphs belong to c-AND(1). We first show that cycles belong to c-AND(1). Moreover, we show that any realization of a cycle has a well determined structure. Secondly, we construct a procedure to combine biconnected components and show how to glue two different cycles by an edge.

Lemme 8. *Let C_n be a cycle of length n, then C_n belongs to c-*AND*(1). Furthermore, let $\mathcal{R}$ be a* AND*(1)-realization of C_n and π the permutation induced by $<_{\mathcal{R}}$. Then there exists a clockwise (or anticlockwise) labeling of edges l such that:*

1. *Extreme vertices are adjacent and $\pi(l^{-1}(1)) = 1 \wedge \pi(l^{-1}(n)) = n$.*
2. *For all $u \in V$, $|l(u) - \pi(u)| \leq 1$.*
3. *If $\mathcal{R}$ is a c-*AND*(1)-realization then for all $u \in V$, $l(u) = \pi(u)$.*

Proof. Let C_n be a cycle. We prove that C_n belongs to c-AND(1) by constructing a particular realization. Let label the vertex set clockwise starting in an arbitrary vertex. Given $0 < \epsilon < 1$, we associate to each vertex $i \in \{2, \ldots, n-1\}$ the pair $([i-(1+\epsilon), i+(1+\epsilon)], i)$. Extreme vertices are assigned to pairs $([2-n-\epsilon, n+\epsilon], 1)$ and $([1-\epsilon, 2n-1+\epsilon], n)$, respectively. It is easy to check that previous defined set is actually a c-AND (1)-realization for C_n.

Let consider an AND(1)-realization $\mathcal{R}$ of the cycle C_n. If $n = 3$ the representative elements are always in a (anti-)clockwise order. Let assume that $n > 3$. We label the vertices clockwise (or anticlockwise) in such a way that: (1) the vertex with label 1 has the minimum value of p_u, i.e., $(\pi \circ l^{-1}(1) = 1)$ and, (2) the vertex with label 2 is the neighbour of 1 with the smaller position in the order: $\pi \circ l^{-1}(2) < \pi \circ l^{-1}(n)$.

In order to prove 1, let suppose by contradiction that the vertex with label n is not extreme. Define w as the vertex placed at the right of vertex with label n with the smallest label. By the definition of the labeling, it holds that $l(w) > 2$, moreover w has a neighbour placed between the vertices with labels 1 and n, which we denote by v. We conclude that quadruplet $l^{-1}(1) <_{\mathcal{R}} v <_{\mathcal{R}} l^{-1}(n) <_{\mathcal{R}} w$ violates four point condition, which is a contradiction.

We prove 2 and 3 greedily. Beforehand, let us introduce some definitions. We say that a vertex u satisfies the *pre-condition* if every vertex placed before u has a label smaller than $\pi(u)$. Clearly, extreme vertices satisfy the pre-condition. Let w be a vertex which satisfies the pre-condition but such that $l(w) \neq \pi(w)$, then $l(w) > \pi(w)$. Let denote by v the vertex with the biggest label at the left of w. By the definition of w it holds that $l(v) < n-1$. We denote by x the neighbour of v with label $l(v)+1$, thus $w <_{\mathcal{R}} x$. Let w' be the vertex in between v and x with the maximum label. Since $l(v) < l(x) < n$ then w' must have a neighbour y (with label $l(w')+1$) at the right of x. Thus, by the four point condition in the quadruplet $v <_{\mathcal{R}} w' <_{\mathcal{R}} x <_{\mathcal{R}} y$, vertices w' and x must be neighbours. We conclude that $w = w'$ and $l(w) = l(x)+1 = l(v)+2$. Additionally, the vertex immediately after x, that is, in the position $\pi(x)+1$ satisfies the pre-condition.

As we state before, extreme vertices satisfy the pre-condition. Let w be the first vertex according to $<_{\mathcal{R}}$ such that $l(w) \neq \pi(w)$. By its definition, w satisfies the pre-condition. Thus, by the previous discussion, $l(w) = \pi(w)+1$. Furthermore, the next vertex in the ordering, say x, has label $l(w)-1$ and then $l(w) - \pi(w) = 1 \wedge l(x) - \pi(x) = -1$. Furthermore, next vertex in the ordering must satisfy the pre-condition. By iterating over vertices according to the order $<_{\mathcal{R}}$, we verify that 2 holds. Finally, let consider the case when $\mathcal{R}$ is

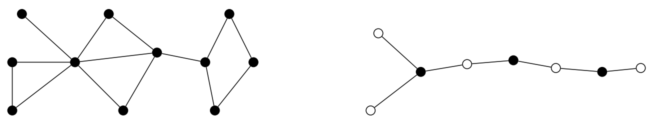

Figure 2: A graph and its block tree. White vertices represent maximal biconnected component while black vertices represent cutvertices.

a c-AND(1)-realization. Let $v <_{\mathcal{R}} w <_{\mathcal{R}} x <_{\mathcal{R}} y$ be the quadruplet previously constructed, where $l(y) = l(w) + 1$. If p_x is placed in the left half of the interval $[p_v, p_y]$ then $vw \in E$, otherwise $xy \in E$ which yields a contradiction. Thus, for all vertices in the c-AND(1)-realization $l(w) = \pi(w)$. □

Définition 9 (Safe vertex)**.** Let G be a graph in (c-)AND(1). We say that a vertex $v \in V$ is *safe* in G if there exists a (c-)AND(1)-realization $\mathcal{R} = \{(B_u, p_u)\}_{u \in V(G)}$ such that $v \in B_w$ if and only if $v = w \vee vw \in E(G)$.

A safe vertex allows the union of two different biconnected components. This important property comes from the fact that in a realization where a vertex v is safe, the interval B_v can be enlarged as much as required without modify the original graph. The following lemma, whose detailed proof can be found in Appendix A, states the safe vertex utility.

Lemme 10. *Consider two graphs $G_1, G_2 \in$ (c-)*AND*(1) and two vertices $w_1 \in V(G_1)$ and $w_2 \in V(G_2)$. Let G be the graph obtained by identifying w_1 and w_2. If w_2 is safe in G_2, then it holds that $G \in$ (c-)*AND*(1).*

Given a graph G, the *block tree* of G is the graph having two types of vertices : blocks and cutvertices. A block vertex represents a maximal biconnected component of G while cutvertices are the articulation points between blocks. The edges of the block tree join blocks with cutvertices. A block is adjacent to a cutvertex if the block contains the cutvertex. Figure 2 shows an example of a graph and its block tree.

Théorème 11. *Let G be a connected graph and T its block tree. If all maximal biconnected components of G belong to (c-)*AND*(1) and T can be rooted such that every cutvertex is safe in its descendants, then G belongs to (c-)*AND*(1).*

Proof. The proof follows directly from Lemma 10 by adding biconnected components of G in a bread-first traversal order. □

The previous result allows us to constructively obtain a realization of a graph by gluing the realization of its biconnected components. As a consequence, BLOCK GRAPHS, graphs in which all biconnected components induce a clique, belong to c-AND(1).

An analogous result to Lemma 10 can be obtained to identify edges in two different cycles:

Lemme 12. *Given two cycles C_n, C'_m and two edges $uv \in E(C_n)$ and $u'v' \in E(C'_m)$, let G be the graph obtained by identifying uv and $u'v'$. Then, $G \in$ c-*AND*(1).*

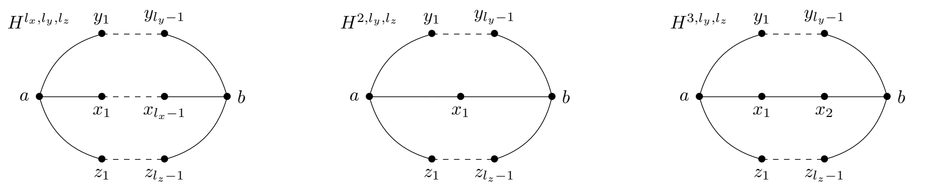

Figure 3: This figure shows, from left to right, a graphic representation of (a general) H^{l_x,l_y,l_z}, and the particular cases of H^{2,l_y,l_z} and H^{3,l_y,l_z}.

Proof. For $0 < \epsilon < 1$ we construct two c-AND(1)-realizations $\mathcal{R}$ and $\mathcal{R}'$ of C_n and C'_m respectively according to the procedure described in the proof of Theorem 8. Furthermore, we suppose that u and v are the extreme vertices of realization $\mathcal{R}$ but $u'$and v' are not the extreme vertices of $\mathcal{R}'$. We perform a $1/(m-1)$-scaling and a translation of $\mathcal{R}$ so that the positions of representative elements of u and v equal those of u' and v' in $\mathcal{R}'$. The realization $\mathcal{R} \cup \mathcal{R}' \setminus \{(B_{u'}, u'), (B_{v'}, v')\}$ is a c-AND(1)-realization for the graph G. Notice that this realization can be done with any vertex as an extreme (safe) vertex. □

Théorème 13. *The set of* OUTERPLANAR *graphs is a subset of* c-AND*(1).*

Proof. Maximal biconnected components of an outerplanar graph are dissections of a convex polygon, which belong to c-AND(1) by Lemma 12. The proof follows by gluing biconnected components according to Theorem 11. □

4 Differences between AND(1) and c-AND(1)

In this section we show the difference between AND(1) and c-AND(1) via a family of graphs that belongs to AND(1) but which does not belong to c-AND(1). We start with a remark upon the fact that the property of being part of AND(1) or c-AND(1) is hereditary, i.e., if a graph G belongs to (c-)AND(1) then every induced subgraph of G also belongs to (c-)AND(1). Indeed, if a graph G has a (c-)AND(1)-realization then the same realization is also a (c-)AND(1)-realization for every induced subgraph of G when the corresponding vertices are deleted. From the hereditary property, we define a graph G that does not belong to (c-)AND(1) as *minimal* with respect to (c-)AND(1) if and only if every proper induced subgraph of G does belong to (c-)AND(1). All graphs in the family separating AND(1) and c-AND(1) are minimal with respect to c-AND(1) and they are based in the following definition.

Définition 14. Let H^{l_x,l_y,l_z} be a finite graph that consists of two not neighboring vertices, say vertices a and b, together with three vertex disjoint paths that connect a with b. The three paths that connect vertex a with vertex b follow: path $x_1, x_2, \ldots, x_{l_x-1}$, path $y_1, y_2, \ldots, y_{l_y-1}$ and path $z_1, z_2, \ldots, z_{l_z-1}$, where the edge-length of the paths, denoted by l_x, l_y, and l_z, are larger or equal than 2. A graphic representation of H^{l_x,l_y,l_z} is shown in Figure 3.

Lemme 15. *Any H^{l_x,l_y,l_z} graph, where l_x, l_y and l_z are strictly larger than 3, does not belong to* AND(1).

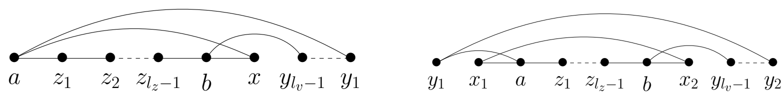

Figure 4: A graphic representation of H^{2,l_y,l_z} (left) and H^{3,l_y,l_z}(right) where vertices are ordered respecting the four point condition for AND(1).

Proof. The proof is by contradiction. Let $\mathcal{R}$ be an AND(1)-realization of H^{l_x,l_y,l_z}. For the sake of simplicity, let denote by X the path $x_0, x_1, x_2, \ldots, x_{l_x-1}, x_{l_x}$ where $x_0 = a$ and $x_{l_x} = b$. We define Y and Z in the same way. By Lemma 8.1, the extreme vertices of the realization must be neighbours. Then, both extremes belong to the same path. Without lost of generality, we assume that vertex a is placed before than b in the realization ($a <_{\mathcal{R}} b$) and that both extremes belong to path X, say x_k and x_{k+1} with $k \in \{0, \ldots, l_x - 1\}$. Therefore, according to Lemma 8, the induced cycles $X \cup Y$ and $X \cup Z$ must be oriented clockwise $(x_k, \ldots, x_0, y_1, \ldots, y_{l_y-1}, b, \ldots, x_{l_x-1}, \ldots, x_{k-1})$ and anti-clockwise ($x_k, \ldots, x_0, z_1, \ldots, z_{l_y-1}, b, \ldots, x_{l_x-1}, \ldots, x_{k-1})$ respectively.

On the other hand, from Lemma 8.2 it holds that a labeling l of an induced cycle satisfies the following property : if for two vertices u, v it holds $l(u) < l(v-1)$ then $\pi(u) < \pi(v)$, i.e., $u <_{\mathcal{R}} v$. Thus, in the cycle $X \cup Y$, $y_2 <_{\mathcal{R}}$ and $a <_{\mathcal{R}} y_j$ for all $j > 2$. Symmetrically, for the cycle $X \cup Z$, $a <_{\mathcal{R}} z_j$ for all $j > 2$ and $z_2 <_{\mathcal{R}} b$.

Finally, in the induced realization of cycle $Y \cup Z$. The only possible pairs of extreme vertices are $(a, y_1), (y_1, y_2)$ and $(a, z_1), (z_1, z_2)$. If the extremes are (a, y_1) or (y_1, y_2), then the cycle $Y \cup Z$ is oriented anti-clockwise and $b <_{\mathcal{R}} y_2$ which is a contradiction. Otherwise, if (a, z_1) or (z_1, z_2) are the extremes, cycle $Y \cup Z$ is oriented clockwise and $b <_{\mathcal{R}} z_2$ which also yields a contradiction. $\square$

We shall see now that, indeed, the smaller cases H^{2,l_y,l_z} and H^{3,l_y,l_z} are minimal graphs that separate AND(1) from c-AND(1).

Lemme 16. *Any H^{l_x,l_y,l_z} graph, where l_x, l_y and l_z are equal or larger than 2, does not belong to c-*AND(1).

The proof of this lemma follows the same ideas of the proof of Lemma 15. Therefore, we omit the proof here, but it can be found in the Appendix A.

With the previous Lemma we have presented an infinite family of graphs that do not belong c-AND(1). Nevertheless, some of these graphs do belong to AND(1).

Lemme 17. *Graphs H^{2,l_y,l_z} and H^{3,l_y,l_z} belong to* AND(1) *for any l_y and $l_z \geq 2$ and l_y and $l_z \geq 3$, respectively.*

The proof of this lemma follows by giving orderings of the set of vertices of H^{2,l_y,l_z} and H^{3,l_y,l_z} that satisfy the four point condition for AND(1). Figure 4 shows graphically such orders. A detailed proof of this lemma can be found in the Appendix A.

We consider important to stress the complete bipartite graph $K_{2,3}$ as a particular case of Lemma 16 and Lemma 17, i.e., $K_{2,3}$ belongs to AND(1) but it does not belong to c-AND(1). Such an importance comes from the fact that $K_{2,3}$ is the smallest complete bipartite graph that does not belong to c-AND(1).

As a consequence of Lemma 16 and the fact that the property of belonging to c-AND(1) is hereditary, we can say that any graph that contains a H^{l_x,l_y,l_z} as an induced subgraph does not belong to c-AND(1). On the other hand, from Lemma 17 we know that some of these graphs do belong to AND(1).

Théorème 18. *There exist an infinite amount of minimal graphs that do not belong to c-*AND(1) *but which do belong to* AND(1)*.*

Proof. The proof follows almost directly form lemmas 16 and 17. Indeed, from those lemmas we know that every H^{2,l_y,l_z} or H^{3,l_y,l_z} graph separates families c-AND(1) and AND(1). In order to prove minimality, we need to see that any proper subgraph of a graph in H^{2,l_y,l_z} or H^{3,l_y,l_z} belongs to c-AND(1). Now, any proper subgraph of H^{2,l_y,l_z} or H^{3,l_y,l_z} is outerplanar. As we have seen in previous sections, we know that outerplanar graphs do belong to c-AND(1). $\square$

5 Intersection graph characterization for (c-)AND(1)

In this section we give a characterization of AND(1) and c-AND(1) classes as intersection graphs. We consider interesting the study of the BOXICITY of those classes. We prove that (c)-AND(1) set can be identified with a subset of BOXICITY(2) class.

Théorème 19. *A graph G belongs to* AND(1) *if and only if G is the intersection graph of boxes in the upper half-plane and tangent to the diagonal $\mathcal{L}_0 : x + y = 0$.*

Proof. Let $G = (V, E)$ be a graph in (c-)AND(1) and $\{([L(v), R(v)], p_v)\}_{v\in V}$ a realization. For each $v \in V$ we define the box B'_v in the 2-dimensional euclidean space as the product of its right interval with the negative scaling of its left interval (Figure 5). That is to say:

$$B'_v = [p_v, R(v)]\times[-p_v, -L(v)] = \{(x, y) \in \mathbb{R}^2, p_v \leq x \leq R(v)\wedge -p_v \leq y \leq -L(v)\}.$$

Let consider two vertices $u, v \in V$ with $p_u \leq p_v$. Then, $uv \in E(G)$ if and only if $p_v - p_u \leq \min\{R(u) - p_u, p_v - L(v)\}$ or equivalently $B'_u \cap B'_v \neq \emptyset$.

The previous construction can be made using any non-axis parallel reference line $\mathcal{L} : Ax+By+C = 0$. For this scenario we can apply a reflexion, a translation and a scaling to the boxes in order to obtain an equivalent representation which use $\mathcal{L}_0$. $\square$

Remarque 20. A direct consequence of the previous result is the fact that AND(1) $\subset$ BOXICITY(2). This consequence can be generalized. Consider any graph $G = (V, E)$ in (c-)AND(d). The graph G can be expressed as the intersection of d graphs $\bigcap_{i=1}^{d} G_i$ with $G_i = (V, E_i) \in$ (c-)AND(1). Given a realization of G, the graph G_i is determined by the projection of each box and representative elements on the ith axis of $\mathbb{R}^d$. This observation together with Theorem 19 show that for $d \geq 1$, AND(d) $\subset$ BOXICITY($2d$).

This new characterization of graphs in (c-)AND(1) allows us to establish a relation with other graph models. Kaufmann et al. in [6] proved that a graph is a max-tolerance graph if and only if it can be represented as the intersection graph of isosceles, axis parallel, right triangles or (lower halves). A slightly different

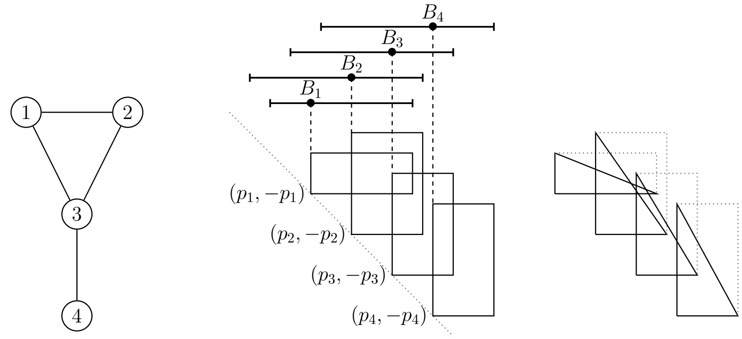

Figure 5: An example of a graph in AND(1) (left) with its intersection model with boxes (center) and triangles (right).

representation of (c-)AND(1) graphs can be obtained by keeping the left lower half of boxes in the intersection model (Figure 5 right). Therefore, the following corollary holds.

Corollaire 21. c-AND*(1)* $\subset$ MAX-TOLERANCE.

Remarque 22. A family that is closely related with max-tolerance graphs is the family of tolerance graphs (cf. [5]). Though, definitions are similar, tolerance graphs do not contain cycles of length longer than 4. Therefore, as consequence of Lemma 8, we know that the family of tolerance graphs does not contain neither AND(1) nor c-AND(1). Therefore, we know that a family of graphs that a priori might be similar to AND(1) differs from it.

6 Conclusions and Future work

In this document we introduced a new graph family named AND(d) and a subfamily c-AND(d). We studied the one dimensional version of the family. We gave a combinatorial characterization for the AND(1) and an intersection characterization for both AND(1) and c-AND(1) families. These characterizations allowed us to show the relation with other well known graph classes. These results are graphical expressed in Figure 6. Moreover, we provided algorithms to construct realizations for INTERVAL and OUTERPLANAR graphs. We also provided an infinite set of minimal graphs that separates AND(1) and c-AND(1).

Our work suggests several directions for future research. Perhaps the most natural question is to find a combinatorial characterization for the c-AND(1) family. Another interesting open problem concerns to determine the complexity of the recognition problem for both AND(1) and c-AND(1) families. The study of higher dimensions of the families is an alternative way to continue this research. Another interesting question is the study of the family of graphs generated when points are embedded in a different metric space, like the d-dimensional torus.

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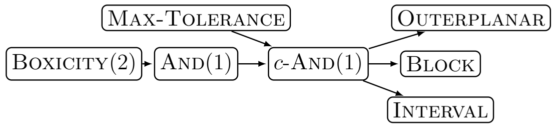

Figure 6: Relation between the graph classes in the document. Arrows point from the superclass to the subclass.

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A Appendix

Proof of Lemma 10. Let G be a graph obtained by the identification of vertices w_1 and w_2 of two different graphs G_1 and G_2. Consider two (c-)AND(1)-realizations $\mathcal{R}_1 = \{(B_u, p_u)\}_{u\in V(G_1)}$ and $\mathcal{R}_2 = \{(B_u, p_u)\}_{u\in V(G_2)}$ of G_1 and G_2, respectively, such that w_2 is safe in $\mathcal{R}_2$. We denote by l the minimum distance between p_{w_1} and the representative elements of its neighbours, that is $l = \min_{u\in\mathcal{N}(w_1)}\{|p_{w_1} - p_u|\}$. Let B be an interval such that $\cup_{v\in V(G_2)}\{B_v\} \subseteq B$ and denote by L its length. We construct the realization $\mathcal{R}'_2 = \{(B'_v, p'_v)\}_{v\in V(G_2)}$ from $\mathcal{R}_2$ by the following procedure:

- apply a $(-p_{w_2})$-translation,
- scale the realization by a factor $l/(2L)$,
- perform a (p_{w_1})-translation in order to equals the position of representatives elements of w_1 and w_2.

Let B_w be the interval with center in p_{w_1} and of length equal to the maximum between B_{w_1} and B'_{w_2}. Then, let us define $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}'_2 \setminus \{(B_{w_1}, p_{w_1}), (B'_{w_2}, p'_{w_2})\} \cup (B_w, p_w)$. We see that $\mathcal{R}$ is a (c-)AND (1) realization for G. In fact, all edges $uv \in E(G_1) \cup E(G_2)$ are induced by $\mathcal{R}$. Furthermore by the definition of $\mathcal{R}'_2$ and the fact that w_2 is safe, no new edges are generated by $\mathcal{R}$. □

Proof of Lemma 16. The proof is by contradiction. Let $\mathcal{R}$ be an c-AND(1)-realization of H^{l_x,l_y,l_z}. Let denote by X the path $x_0, x_1, x_2, \ldots, x_{l_x-1}, x_{l_x}$ where $x_0 = a$ and $x_{l_x} = b$. We define Y and Z in the same way. Since the extreme vertices of the realization must be neighbours (Lemma 8.1), then both extremes belong to the same path. W.l.o.g., we assume that vertex a is placed before than b in the realization ($a <_{\mathcal{R}} b$) and that both extremes belong to X, says x_k and x_{k+1} with $k \in \{0, \ldots, l_x - 1\}$. Therefore, according to Lemma 8, the induced cycles $X \cup Y$ and $X \cup Z$ must be oriented clockwise and anti-clockwise, respectively. That is: $x_k, \ldots, x_0, y_1, \ldots, y_{l_y-1}, b, \ldots, x_{l_x-1}, \ldots, x_{k-1}$ and $x_k, \ldots, x_0, z_1, \ldots, z_{l_y-1}, b, \ldots, x_{l_x-1}, \ldots, x_{k-1}$, respectively. Thus, $y_1 <_{\mathcal{R}} b$ and $z_1 <_{\mathcal{R}} b$.

Let consider the induced realization of cycle $Y \cup Z$. By the previous discussion we conclude that a is the left extreme vertex of the induced realization. Thus, right extreme have to be y_1 or z_1. Then, either $b <_{\mathcal{R}} y_1$ or $b <_{\mathcal{R}} z_1$ which is a contradiction. □

Proof of Lemma 17. Consider any H^{2,l_y,l_z} graph. In this case, the first path has length 2, therefore, we denote its vertex by x without subindex. In order to prove the Lemma, we give an ordering of the vertices of H^{2,l_y,l_z} that satisfies the four point condition for AND(1). Consider the following ordering for $V(H^{2,l_y,l_z})$:

$$a, z_1, z_2, \ldots, z_{l_z-1}, b, x, y_{l_y-1}, y_{l_y-2} \ldots, y_1.$$

A graphic representation of H^{2,l_y,l_z} where the vertices are ordered according the above described ordering is shown in Figure 4.

As it can be seen in Figure 4, according to this ordering of the vertices the only pair of edges that crosses one to another are edges ax and by_{l_y-1}. Since vertices b and x are connected by an edge in H^{2,l_y,l_z}, the four point condition is satisfied. Therefore, the graph H^{2,l_y,l_z} belongs to AND(1).

Consider any H^{3,l_y,l_z} graph, we give an ordering of $V(H^{3,l_y,l_z})$ that satisfies the four point condition for AND(1). Consider the following order for $V(H^{3,l_y,l_z})$:

$$y_1, x_1, a, z_1, z_2, \ldots, z_{l_z-1}, b, x_2, y_{l_y-1}, y_{l_y-2}, \ldots, y_2.$$

A graphic representation of H^{3,l_y,l_z} where the vertices are ordered according the above described ordering is shown in Figure 4.

In order to finish the proof, we have to check that this ordering satisfies the four point condition for AND(1). As it can be seen in Figure 4, there are two pair of edges that crosses one to each other. One pair is composed by edges $y_1 a$ and $x_1 x_2$. Since vertices a and x_1 are neighbors, the condition holds. The second pair is composed by edges $x_1 x_2$ and $b y_{l_y-1}$. Since vertices x_2 and b are neighbors, the condition holds. Therefore, the graph H^{3,l_y,l_z} belongs to AND(1). $\square$