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Research Article

Portfolio Selection with Subsistence Consumption Constraints and CARA Utility

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We consider the optimal consumption and portfolio choice problem with constant absolute risk aversion (CARA) utility and a subsistence consumption constraint. A subsistence consumption constraint means there exists a positive constant minimum level for the agent's optimal consumption. We use the dynamic programming approach to solve the optimization problem and also give the verification theorem. We illustrate the effects of the subsistence consumption constraint on the optimal consumption and portfolio choice rules by the numerical results.

1. Introduction

Following the seminal research works of Merton [1, 2], various problems of continuous-time optimal consumption and portfolio selection have been considered under various financial/economic constraints. One of the interesting research topics in a continuous-time portfolio selection problem is the optimization problem subject to a subsistence consumption constraint (or a downside consumption constraint). A subsistence constraint means that there exists a positive lower bound level for the agent's optimal consumption rate. Thus this constraint affects the agent's financial decision including her optimal portfolio.

Lakner and Nygren [3] have studied the portfolio optimization problem subject to a downside constraint for consumption and an insurance constraint for terminal wealth with a martingale approach. Gong and Li [4] have investigated the role of index bonds in the optimal consumption and portfolio selection problem with constant relative risk aversion (CRRA) utility and a real subsistence consumption constraint using the dynamic programming approach. Shin et al. [5] have also considered a similar problem to that of Gong and Li [4]. They have studied the portfolio selection problem with a general utility function and a downside consumption constraint using the martingale approach. Yuan and Hu [6] have investigated the optimal consumption and portfolio selection problem with a consumption habit constraint and a terminal wealth downside constraint using the martingale approach. In this paper we use the dynamic programming method based on Karatzas et al. [7] to derive the value function and the optimal policies in closed-form with a constant absolute risk aversion (CARA) utility function and a subsistence consumption constraint. Lim et al. [8] have considered a similar portfolio optimization problem combined with the voluntary retirement choice problem. Shin and Lim [9] have analyzed the effects of the subsistence consumption constraint for behavior of investment in the risky asset.

The rest of this paper proceeds as follows. Section 2 introduces the financial market. In Section 3 we consider the main optimization problem. We use the dynamic programming principle to derive the solutions in closed form with CARA utility and a subsistence consumption constraint. We also give some numerical results and the solutions derived by the martingale method. Section 4 concludes.

2. The Financial Market Setup

We assume that there are two assets in the financial market: one is a riskless asset with constant interest rate $r > 0$, and the other is a stock whose price process $\{S_t\}_{t\geq 0}$ evolves according to the stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad \text{for } t \geq 0, \tag{1}$$

where μ is the constant expected rate of return of the stock, $\sigma > 0$ is the constant volatility of the stock, and B_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is the augmentation under $\mathbb{P}$ of the natural filtration generated by the standard Brownian motion $\{B_t\}_{t\geq 0}$. We assume that $\mu \neq r$ so that the market price of risk θ is not zero:

$$\theta \triangleq \frac{\mu - r}{\sigma} \neq 0. \tag{2}$$

Let X_t be an economic agent's wealth at time t, π_t the amount of money invested in the stock at time t, and c_t the consumption rate at time t. The portfolio process $\{\pi_t\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfies, for all $t \geq 0$, almost surely (a.s.),

$$\int_0^t \pi_s^2 ds < \infty, \tag{3}$$

and the consumption rate process $\{c_t\}_{t\geq 0}$ is a nonnegative process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ such that, for all $t \geq 0$, a.s.,

$$\int_0^t c_s ds < \infty. \tag{4}$$

We assume that there is a subsistence consumption constraint which restricts the minimum consumption level. That is, the consumption process should satisfy

$$c_t \geq R, \quad \forall t \geq 0, \tag{5}$$

where $R > 0$ is a constant lower bound for the consumption rates. Thus the agent's wealth process $\{X_t\}_{t\geq 0}$ follows the SDE

$$dX_t = \left[rX_t + \pi_t\left(\mu - r\right) - c_t\right]dt + \sigma\pi_t dB_t, \tag{6}$$

with an initial endowment $X_0 = x > R/r$. (We need this restriction on the initial endowment for the positive consumption rate. See Lemma 3.1 of Gong and Li [4]). A consumption-portfolio plan $(\mathbf{c}, \boldsymbol{\pi}) := (\{c_t\}_{t\geq 0}, \{\pi_t\}_{t\geq 0})$ satisfying the above conditions is called admissible at $x > R/r$ if $X_t \geq R/r$, for all $t \geq 0$. We let $\mathcal{A}(x)$ denote the class of admissible controls at $x > R/r$.

3. The Optimization Problem

Now the agent's optimization problem with initial wealth $X_0 = x > R/r$ is to choose $(\mathbf{c}, \boldsymbol{\pi}) \in \mathcal{A}(x)$ to maximize the following expected life-time utility:

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} u\left(c_t\right) dt\right]. \tag{7}$$

Here, $\beta > 0$ is the subjective discount factor and $u(\cdot)$ is a constant absolute risk aversion (CARA) utility function defined by

$$u\left(c\right) \triangleq -\frac{e^{-\gamma c}}{\gamma}, \tag{8}$$

where $\gamma > 0$ is the agent's coefficient of absolute risk aversion. Thus the agent's value function is given by

$$V^*\left(x\right) \triangleq \sup_{(\mathbf{c},\boldsymbol{\pi})\in\mathcal{A}(x)} \mathbb{E}\left[-\int_0^\infty \frac{e^{-\beta t - \gamma c_t}}{\gamma} dt\right]. \tag{9}$$

Bellman equation corresponding to the optimization problem for $x > R/r$ is

$$\max_{c\geq R,\pi}\left[\left\{rx + \pi\left(\mu - r\right) - c\right\}V'\left(x\right) + \frac{1}{2}\sigma^2\pi^2V''\left(x\right) - \beta V\left(x\right) - \frac{e^{-\gamma c}}{\gamma}\right] = 0. \tag{10}$$

We assume that the wealth process X_t must satisfy a transversality condition

$$\lim_{t\to\infty} e^{-\beta t} V\left(X_t\right) = 0. \tag{11}$$

We will find the solution $V(x)$, as the candidate value function, to Bellman equation (10) under the conditions that $V'(x) > 0$ and $V''(x) < 0$ for $x > R/r$ and $V'(\bar{x}) = u'(R) = e^{-\gamma R}$ for a real number $\bar{x} > R/r$. After obtaining the solution, we can check these conditions. Under these conditions, in particular, the first-order condition (FOC), $-V'(x)+u'(c) = 0$ with respect to $c \geq R$, is binding if $R/r < x < \bar{x}$ so that the maximizing $c \geq R$ in Bellman equation (10) is R in this case.

Thus, from the first-order conditions (FOCs) of Bellman equation (10), we derive the candidate optimal consumption and portfolio

$$c^* = \begin{cases} R, & \text{if } \dfrac{R}{r} < x < \bar{x} \\ -\dfrac{\log\left\{V'\left(x\right)\right\}}{\gamma}, & \text{if } x \geq \bar{x}, \end{cases}$$
$$\pi^* = -\frac{\theta V'\left(x\right)}{\sigma V''\left(x\right)}. \tag{12}$$

Remark 1. For later use, we consider two quadratic algebraic equations:

$$rm^2 - \left(r + \beta + \frac{\theta^2}{2}\right)m + \beta = 0, \tag{13}$$

with two roots m_1 $(0 < m_1 < 1)$ and $m_2 > 1$ and

$$\frac{\theta^2}{2\gamma}n^2 + \left(r - \beta - \frac{\theta^2}{2}\right)n - r\gamma = 0, \tag{14}$$

with two roots $n_1 < 0$ and $n_2 > \gamma$.

Theorem 2. *Let $V(x)$ be given by*

$$V(x)=\begin{cases} C_1\left(x-\frac{R}{r}\right)^{m_1}-\frac{e^{-\gamma R}}{\beta\gamma}, & \text{if } \frac{R}{r}<x<\overline{x} \\ \frac{1}{\beta}\left(r+\frac{\theta^2}{2\gamma}n_1\right)D_1e^{(n_1-\gamma)\xi} & \\ -\frac{1}{r\gamma}e^{-\gamma\xi}, & \text{if } x\geq\overline{x}, \end{cases} \tag{15}$$

where

$$D_1=-\frac{rm_1-\beta-\theta^2/2}{r^2\gamma e^{n_1R}\left(1-\left(\left(1-m_1\right)/\gamma\right)n_1\right)}>0, \tag{16}$$

$$\overline{x}=\frac{1-m_1}{\gamma}\left(n_1D_1e^{n_1R}+\frac{1}{r}\right)+\frac{R}{r}>\frac{R}{r}, \tag{17}$$

$$C_1=\frac{e^{-\gamma R}}{m_1}\left(\overline{x}-\frac{R}{r}\right)^{1-m_1}>0, \tag{18}$$

and ξ is determined from the algebraic equation

$$x=D_1e^{n_1\xi}+\frac{1}{r}\xi+\frac{1}{r^2\gamma}\left(r-\beta-\frac{\theta^2}{2}\right). \tag{19}$$

Then it satisfies Bellman equation (10).

Proof. By using Remark 1, we can check the inequalities in (16) and (17). The inequality in (18) holds by (17). Define the function $X(c)$ of c on $[R,\infty)$ by

$$X(c)=D_1e^{n_1c}+\frac{1}{r}c+\frac{1}{r^2\gamma}\left(r-\beta-\frac{\theta^2}{2}\right). \tag{20}$$

By using (16) and (17), one can check

$$X(R)=\overline{x}. \tag{21}$$

Since the function $X(c)$ is increasing in c, it has the inverse function. Let $C(x)$ for $x\geq\overline{x}$ be the inverse function of $X(c)$. In particular, we have

$$C(\overline{x})=R. \tag{22}$$

By (19), we have

$$\xi=C(x)\quad\text{for } x\geq\overline{x}. \tag{23}$$

By using Remark 1, (16), (18), (22), and (23), we can show that the function $V(x)$ defined by (15) is continuous. By using Remark 1, (20), (23), and the inverse relationship between X and C, we can obtain

$$V'(x)=e^{-\gamma C(x)},\qquad V''(x)=-\frac{\gamma e^{-\gamma C(x)}}{X'(C(x))},\quad\text{for } x>\overline{x}. \tag{24}$$

By (15), (18), (20), (22), and (24), we get the smooth-pasting (C^1) condition

$$V'(\overline{x}-)=m_1C_1\left(\overline{x}-\frac{R}{r}\right)^{m_1-1}=e^{-\gamma R}=V'(\overline{x}+), \tag{25}$$

and the high-contact (C^2) condition

$$\begin{aligned} V''(\overline{x}-)&=m_1\left(m_1-1\right)C_1\left(\overline{x}-\frac{R}{r}\right)^{m_1-2} \\ &=-\frac{\gamma e^{-\gamma R}}{X'(R)}=V''(\overline{x}+). \end{aligned} \tag{26}$$

Thus, the function $V(x)$ is twice continuously differentiable. Furthermore, $V'(x)>0$ and $V''(x)<0$ for $x>R/r$ and $V'(\overline{x})=u'(R)=e^{-\gamma R}$.

For $R/r<x<\overline{x}$, if we substitute FOCs (12) into Bellman equation (10), we obtain the changed Bellman equation

$$(rx-R)V'(x)-\frac{1}{2}\theta^2\frac{\left(V'(x)\right)^2}{V''(x)}-\beta V(x)-\frac{e^{-\gamma R}}{\gamma}=0. \tag{27}$$

We can easily check that $V(x)$ is the solution to (27) for $R/r<x<\overline{x}$.

For $x\geq\overline{x}$, we also obtain the changed Bellman equation from (10):

$$\begin{aligned} &rxV'(x)-\frac{1}{2}\theta^2\frac{\left(V'(x)\right)^2}{V''(x)}-\beta V(x) \\ &\quad+\frac{V'(x)}{\gamma}\left(\log V'(x)-1\right)=0. \end{aligned} \tag{28}$$

If we substitute (24) into Bellman equation (28), then we obtain the equation

$$\begin{aligned} &rX(c)e^{-\gamma c}+\frac{\theta^2}{2\gamma}X'(c)e^{-\gamma c}-\beta V(X(c)) \\ &\quad-\frac{e^{-\gamma c}}{\gamma}(\gamma c+1)=0. \end{aligned} \tag{29}$$

By using (15) and (23), we can check that (29) holds. □

Now we can derive the candidate optimal policies with the function $V(\cdot)$ in Theorem 2.

Theorem 3. *The candidate optimal policies are given by (c^*,π^*) such that*

$$c_t^*=\begin{cases} R, & \text{if } \frac{R}{r}<X_t<\overline{x}, \\ \xi_t, & \text{if } X_t\geq\overline{x}, \end{cases} \tag{30}$$

$$\pi_t^*=\begin{cases} \frac{\theta}{\sigma\left(1-m_1\right)}\left(X_t-\frac{R}{r}\right), & \text{if } \frac{R}{r}<X_t<\overline{x}, \\ \frac{\theta}{\sigma\gamma}\left(n_1D_1e^{n_1\xi_t}+\frac{1}{r}\right), & \text{if } X_t\geq\overline{x}, \end{cases} \tag{31}$$

where ξ_t is determined from the optimal wealth process

$$X_t=D_1e^{n_1\xi_t}+\frac{1}{r}\xi_t+\frac{1}{r^2\gamma}\left(r-\beta-\frac{\theta^2}{2}\right). \tag{32}$$

Theorem 4 (verification theorem). *The value function of the optimization problem* (9) *is equal to* $V(x)$ *in Theorem 2. That is,* $V^*(x) = V(x)$. *Consequently the candidate consumption and portfolio in Theorem 3 are the optimal policies of problem* (9).

Proof. For arbitrary given consumption and portfolio plan $(\mathbf{c}, \boldsymbol{\pi}) \in \mathscr{A}(x)$ and $T \in (0, \infty)$, we have

$$\begin{aligned}
&\mathbb{E}\left[-\int_0^T \frac{e^{-\beta t-\gamma c_t}}{\gamma}dt\right] \\
&\quad \leq \mathbb{E}\left[-\int_0^T e^{-\beta t}\left\{\left(rX_t+\pi_t(\mu-r)-c_t\right)V'(X_t)\right.\right. \\
&\qquad\qquad \left.\left.+\frac{1}{2}\sigma^2\pi_t^2V''(X_t)-\beta V(X_t)\right\}dt\right] \\
&\quad = \mathbb{E}\left[-\int_0^T d\left(e^{-\beta t}V(X_t)\right)\right] \\
&\qquad + \mathbb{E}\left[\int_0^T e^{-\beta t}\sigma\pi_t V'(X_t)\,dB_t\right] \\
&\quad = V(x)-\mathbb{E}\left[e^{-\beta T}V(X_T)\right],
\end{aligned} \tag{33}$$

where the inequality is obtained from Bellman equation (10), the first equality from applying Itô's formula to $e^{-\beta t}V(X_t)$, and the second equality from $\mathbb{E}[\int_0^T e^{-\beta t}\sigma\pi_t V'(X_t)dB_t] = 0$. Taking $T \uparrow \infty$ and using transversality condition (11), we have

$$V(x) \geq \mathbb{E}\left[-\int_0^\infty \frac{e^{-\beta t-\gamma c_t}}{\gamma}dt\right], \tag{34}$$

for arbitrary given consumption and portfolio plan $(\mathbf{c}, \boldsymbol{\pi}) \in \mathscr{A}(x)$; that is,

$$V(x) \geq \sup_{(\mathbf{c},\boldsymbol{\pi})\in\mathscr{A}(x)} \mathbb{E}\left[-\int_0^\infty \frac{e^{-\beta t-\gamma c_t}}{\gamma}dt\right]. \tag{35}$$

Now we consider the candidate optimal consumption and portfolio plan $(\mathbf{c}^*, \boldsymbol{\pi}^*) \in \mathscr{A}(x)$ in Theorem 3. For $T \in (0, \infty)$, we have

$$\begin{aligned}
&\mathbb{E}\left[-\int_0^T \frac{e^{-\beta t-\gamma c_t^*}}{\gamma}dt\right] \\
&\quad = \mathbb{E}\left[-\int_0^T e^{-\beta t}\left\{\left(rX_t+\pi_t^*(\mu-r)-c_t^*\right)V'(X_t)\right.\right. \\
&\qquad\qquad \left.\left.+\frac{1}{2}\sigma^2(\pi_t^*)^2V''(X_t)-\beta V(X_t)\right\}dt\right] \\
&\quad = \mathbb{E}\left[-\int_0^T d\left(e^{-\beta t}V(X_t)\right)\right] \\
&\qquad + \mathbb{E}\left[\int_0^T e^{-\beta t}\sigma\pi_t^* V'(X_t)\,dB_t\right] \\
&\quad = V(x)-\mathbb{E}\left[e^{-\beta T}V(X_T)\right],
\end{aligned} \tag{36}$$

where the first equality is obtained from Bellman equation (10), the second from applying Itô's formula to $e^{-\beta t}V(X_t)$, and the third from $\mathbb{E}[\int_0^T e^{-\beta t}\sigma\pi_t^* V'(X_t)dB_t] = 0$. Taking $T \uparrow \infty$ and using transversality condition (11), we have

$$V(x) = \mathbb{E}\left[-\int_0^\infty \frac{e^{-\beta t-\gamma c_t^*}}{\gamma}dt\right]. \tag{37}$$

Thus, from (35) and (37), we show that $V(x)$ which is the solution to Bellman equation (10) is a real value function of the optimization problem (9). □

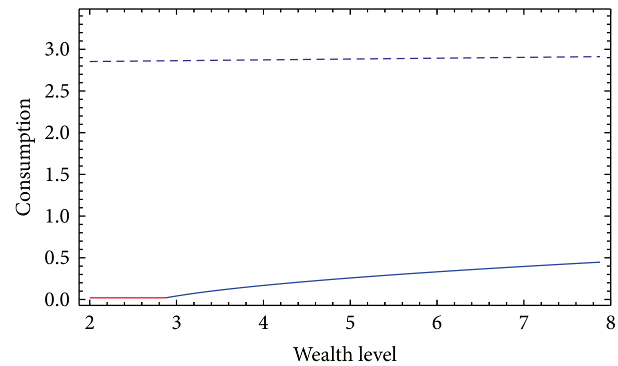

FIGURE 1: This figure is the optimal consumption rate c_t^* when $\beta = 0.05$, $r = 0.01$, $\mu = 0.07$, $\sigma = 0.2$, $R = 0.02$, and $\gamma = 3$. Solid line gives the optimal consumption rate with a subsistence consumption constraint and dotted line gives the optimal consumption rate without the constraint.

Now we compare our solution in Theorem 3 with the Merton's solution with CARA utility. The optimal consumption and portfolio policies without the subsistence constraint are given by

$$c_t^M = rX_t + \frac{1}{r\gamma}\left(\beta - r + \frac{1}{2}\theta^2\right), \qquad \pi_t^M = \frac{\theta}{\sigma r\gamma}, \tag{38}$$

respectively. Figures 1 and 2 give the numerical results for the optimal consumption and portfolio.

Remark 5. Dynamic programming principle can be also applied to the CRRA utility function with a subsistence consumption constraint following our approach. See Gong and Li [4] and Lee and Shin [10].

Remark 6. Following Shin et al. [5] we can use the martingale method to derive a similar solution with CARA utility. We will give (rough) sketch of the derivation.

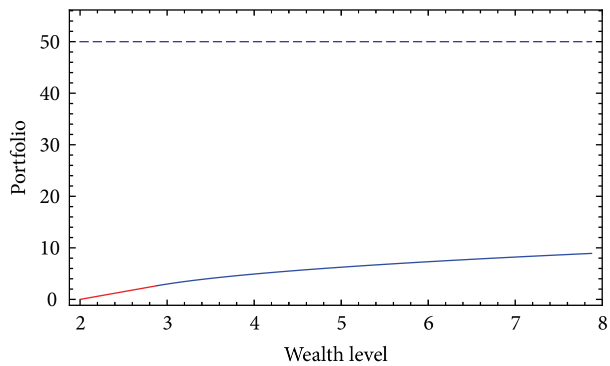

Figure 2: This figure is the optimal portfolio π_t^* when $\beta = 0.05$, $r = 0.01$, $\mu = 0.07$, $\sigma = 0.2$, $R = 0.02$, and $\gamma = 3$. Solid line gives the optimal portfolio with a subsistence consumption constraint and dotted line gives the optimal portfolio without the constraint.

From the agent's wealth dynamics X_t in (6), we obtain the budget constraint

$$\mathbb{E}\left[\int_0^\infty c_t H_t dt\right] \le x, \tag{39}$$

where the state price density H_t is defined by

$$H_t \triangleq e^{-(r+(1/2)\theta^2)t-\theta B_t}. \tag{40}$$

A dual utility function $\tilde{u}(y)$ of the CARA utility function $u(c) = -e^{-\gamma c}/\gamma$ is derived by

$$\tilde{u}(y) = \sup_{c \ge R}\left[-\frac{e^{-\gamma c}}{\gamma} - cy\right] = \left[\frac{1}{\gamma} y \log y - \frac{1}{\gamma} y\right] \mathbf{1}_{\{0<y\le\bar{y}\}} + \left[-\frac{e^{-\gamma R}}{\gamma} - Ry\right] \mathbf{1}_{\{y>\bar{y}\}}, \tag{41}$$

where $\bar{y} = e^{-\gamma R}$. The dual value function $v(y)$ is given by

$$\begin{aligned} v(y) &= \mathbb{E}^{y_0=y}\left[\int_0^\infty e^{-\beta t}\left[\left\{\frac{1}{\gamma} y_t \log(y_t) - \frac{1}{\gamma} y_t\right\} \mathbf{1}_{\{0<y_t\le\bar{y}\}} + \left\{-\frac{e^{-\gamma R}}{\gamma} - Ry_t\right\} \mathbf{1}_{\{y_t>\bar{y}\}}\right] dt\right] \\ &= \begin{cases} c_1 y^{p_+} + \dfrac{1}{r\gamma} y \log y + \dfrac{1}{r^2\gamma}\left(\beta - 2r + \dfrac{1}{2}\theta^2\right) y, & \text{if } 0 < y \le \bar{y}, \\ d_2 y^{p_-} - \dfrac{R}{r} y - \dfrac{e^{-\gamma R}}{\beta\gamma}, & \text{if } y > \bar{y}, \end{cases} \end{aligned} \tag{42}$$

where $y_t \triangleq ye^{\beta t}H_t$ and $p_+ > 1$ and $p_- < 0$ are two real roots of the quadratic algebraic equation

$$\frac{1}{2}\theta^2 p^2 + \left(\beta - r - \frac{1}{2}\theta^2\right) p - \beta = 0. \tag{43}$$

The undetermined coefficients of $v(y)$ are given by

$$\begin{aligned} c_1 &= \frac{\left((p_- - 1)/r^2\right)\left(\beta - 2r + (1/2)\theta^2\right) + p_-/\beta - 1/r}{\gamma(p_+ - p_-)} \times e^{-\gamma R(1-p_+)}, \\ d_2 &= \frac{\left((p_+ - 1)/r^2\right)\left(\beta - 2r + (1/2)\theta^2\right) + p_+/\beta - 1/r}{\gamma(p_+ - p_-)} \times e^{-\gamma R(1-p_-)}. \end{aligned} \tag{44}$$

Then we use the Legendre transform inverse formula to obtain the value function $V_m(x)$ as follows:

$$\begin{aligned} V_m(x) &= \inf_{y>0}\left[v(y) + xy\right] \\ &= \begin{cases} \dfrac{p_- - 1}{p_-}\left(-\dfrac{1}{p_- d_2}\right)^{1/(p_- - 1)} \times \left(x - \dfrac{R}{r}\right)^{p_-/(p_- - 1)} - \dfrac{e^{-\gamma R}}{\beta\gamma}, & \text{if } \dfrac{R}{r} < x < \bar{x}_m, \\ (1 - p_+) c_1 (y^*)^{p_+} - \dfrac{1}{r\gamma}(y^*), & \text{if } x \ge \bar{x}_m, \end{cases} \end{aligned} \tag{45}$$

where

$$\bar{x}_m = -p_+ c_1 e^{-\gamma R(p_+ - 1)} + \frac{R}{r} + \frac{1}{r^2\gamma}\left(r - \beta - \frac{1}{2}\theta^2\right), \tag{46}$$

and y^* is determined from the following algebraic equation:

$$x = -p_+ c_1 (y^*)^{p_+ - 1} - \frac{1}{r\gamma}\log(y^*) + \frac{1}{r^2\gamma}\left(r - \beta - \frac{1}{2}\theta^2\right). \tag{47}$$

Actually we can show that $V_m(x)$ in (45) and $\bar{x}_m$ in (46) agree with $V(x)$ in (15) and $\bar{x}$ in (17), respectively, if we set $y^* = e^{-\gamma\xi}$ and show that

$$m_1 C_1 = (-p_- d_2)^{1/(1-p_-)}, \qquad D_1 = -p_+ c_1. \tag{48}$$

Refer to Lee and Shin [10].

Remark 7. We simplify the calculation by using the dynamic programming approach, where, instead of Legendre transformation in the martingale approach, we introduce the function $X(c)$ in (20) whose inverse is $C(x)$ in (23). The link between the two methods is the relation $y^* = e^{-\gamma\xi} = e^{-\gamma C(x)} = u'(C(x)) = V'(x)$ for $x \ge \bar{x}$, where the last equality is in (24).

4. Concluding Remarks

We have considered the optimal consumption and portfolio choice problem with constant absolute risk aversion (CARA) utility and a subsistence consumption constraint. The existence of a subsistence consumption constraint is realistic since one needs a minimum level of consumption to live. For example, we cannot live without necessities. We have obtained the closed form solution to optimization problem by using the dynamic programming approach. We have illustrated the effects of the subsistence consumption constraint on the optimal consumption and portfolio by the numerical results. Furthermore one can consider the optimization problem under regime switching as future research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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