Part 2

THEORY OF ELASTICITY

CHAPTER 5

FINITE STRAIN ELASTICITY

5.1. Green-Elasticity

Elastic deformation does not cause irreversible rearrangement of internal structure, and the corresponding Helmholtz free energy is a function of stress and temperature only. Restricting consideration to isothermal elastic deformation ($\dot{\theta} = 0$), Eqs. (4.3.7) and (4.3.9) give

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} = \frac{1}{\rho^0}\, \mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)}, \tag{5.1.1}$$

i.e.,

$$\mathbf{T}_{(n)} = \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}}\,, \quad \Psi = \rho^0\, \psi\left(\mathbf{E}_{(n)}\right). \tag{5.1.2}$$

Alternatively, Eq. (5.1.2) can be deduced by adopting an experimentally observed property that there is no net work left in a body upon any closed cycle of elastic strain, i.e.,

$$\oint \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = 0. \tag{5.1.3}$$

This implies that

$$\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = \mathrm{d}\Psi \tag{5.1.4}$$

is a total differential, which leads to Eq. (5.1.2). The function $\Psi = \Psi\left(\mathbf{E}_{(n)}\right)$ is the strain energy function per unit initial volume. It represents the work done to isothermally deform a unit of initial volume to the state of strain $\mathbf{E}_{(n)}$. The explicit representation of the function $\Psi\left(\mathbf{E}_{(n)}\right)$ depends on the selected strain measure $\mathbf{E}_{(n)}$ and the material properties.

Since the material and spatial strain tensors (see Section 2.3) are related by

$$\mathbf{E}_{(n)} = \hat{\boldsymbol{\mathcal{E}}}_{(n)} = \mathbf{R}^T \cdot \boldsymbol{\mathcal{E}}_{(n)} \cdot \mathbf{R}, \tag{5.1.5}$$

the strain energy per unit mass can be written as

$$\psi = \psi\left(\mathbf{E}_{(n)}\right) = \psi\left(\hat{\boldsymbol{\mathcal{E}}}_{(n)}\right). \tag{5.1.6}$$

It can be easily verified that

$$\frac{\partial \psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}} = \mathbf{R} \cdot \frac{\partial \psi}{\partial \mathbf{E}_{(n)}} \cdot \mathbf{R}^T, \quad \dot{\hat{\boldsymbol{\mathcal{E}}}}_{(n)} = \mathbf{R}^T \cdot \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} \cdot \mathbf{R}, \tag{5.1.7}$$

and the rate of ψ becomes

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} = \frac{\partial \psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}. \tag{5.1.8}$$

The stress tensor $\boldsymbol{\mathcal{T}}_{(n)}$ conjugate to spatial strain tensor $\boldsymbol{\mathcal{E}}_{(n)}$ is defined in Section 3.6 by

$$\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \boldsymbol{\mathcal{T}}_{(n)} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}, \quad \boldsymbol{\mathcal{T}}_{(n)} = \mathbf{R} \cdot \mathbf{T}_{(n)} \cdot \mathbf{R}^T. \tag{5.1.9}$$

Consequently, in addition to (5.1.2), from Eq. (5.1.7) we deduce that

$$\boldsymbol{\mathcal{T}}_{(n)} = \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}}, \quad \Psi = \rho^0 \, \psi \left(\hat{\boldsymbol{\mathcal{E}}}_{(n)} \right). \tag{5.1.10}$$

In view of the expressions for the conjugate stress and strain tensors corresponding to $n = \pm 1$, given in Section 3.6, the following expressions for the Kirchhoff stress $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ are obtained from Eqs. (5.1.2) and (5.1.10)

$$\boldsymbol{\tau} = \mathbf{F} \cdot \frac{\partial \Psi}{\partial \mathbf{E}_{(1)}} \cdot \mathbf{F}^T = \mathbf{F}^{-T} \cdot \frac{\partial \Psi}{\partial \mathbf{E}_{(-1)}} \cdot \mathbf{F}^{-1}, \tag{5.1.11}$$

$$\boldsymbol{\tau} = \mathbf{V} \cdot \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(1)}} \cdot \mathbf{V} = \mathbf{V}^{-1} \cdot \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(-1)}} \cdot \mathbf{V}^{-1}. \tag{5.1.12}$$

If the conjugate pair associated with $n = 1/2$ is used, from Eq. (3.6.3) and Eq. (5.1.2) there follows

$$\hat{\boldsymbol{\tau}} = \frac{1}{2} \left(\mathbf{U} \cdot \frac{\partial \Psi}{\partial \mathbf{E}_{(1/2)}} + \frac{\partial \Psi}{\partial \mathbf{E}_{(1/2)}} \cdot \mathbf{U} \right) - \frac{1}{2} \left(\mathbf{U} \cdot \hat{\mathbf{K}} - \hat{\mathbf{K}} \cdot \mathbf{U} \right). \tag{5.1.13}$$

Here, $\hat{\boldsymbol{\tau}} = \mathbf{R}^T \cdot \boldsymbol{\tau} \cdot \mathbf{R}$ and

$$\hat{\mathbf{K}} = \frac{1}{J_1 J_2 + J_3} \left(J_1 \mathbf{I}^0 - \mathbf{U}^{-1} \right) \cdot \left(\mathbf{U}^{-1} \cdot \frac{\partial \Psi}{\partial \mathbf{E}_{(1/2)}} - \frac{\partial \Psi}{\partial \mathbf{E}_{(1/2)}} \cdot \mathbf{U}^{-1} \right) \cdot \left(J_1 \mathbf{I}^0 - \mathbf{U}^{-1} \right). \tag{5.1.14}$$

The invariants of $\mathbf{U}^{-1}$ are denoted by J_i. In the derivation, the results from Subsection 1.12.1 were used to solve the matrix equation of the type $\mathbf{A} \cdot \mathbf{X} + \mathbf{X} \cdot \mathbf{A} = \mathbf{B}$.

If Eq. (3.6.25) is used, Eq. (5.1.10) gives

$$\boldsymbol{\tau} = \frac{1}{2} \left(\mathbf{V} \cdot \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(1/2)}} + \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(1/2)}} \cdot \mathbf{V} \right) - \frac{1}{2} \left(\mathbf{V} \cdot \mathbf{K} - \mathbf{K} \cdot \mathbf{V} \right), \tag{5.1.15}$$

where

$$\mathbf{K} = \frac{1}{J_1 J_2 + J_3} \left(J_1 \mathbf{I} - \mathbf{V}^{-1} \right) \cdot \left(\mathbf{V}^{-1} \cdot \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(1/2)}} - \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(1/2)}} \cdot \mathbf{V}^{-1} \right) \cdot \left(J_1 \mathbf{I} - \mathbf{V}^{-1} \right). \tag{5.1.16}$$

The invariants of $\mathbf{V}^{-1}$ are equal to those of $\mathbf{U}^{-1}$ and are again denoted by J_i. The transition from Eq. (5.1.13) to (5.1.15) is straightforward by noting that

$$\hat{\mathbf{K}} = \mathbf{R}^T \cdot \mathbf{K} \cdot \mathbf{R}. \tag{5.1.17}$$

For elastically isotropic materials, considered in the next section, the tensors $\mathbf{V}^{-1}$ and $\partial \Psi / \partial \boldsymbol{\mathcal{E}}_{(1/2)}$ are coaxial, hence commutative, and $\mathbf{K} = 0$. Similar expressions are obtained when Eqs. (3.6.6) and (3.6.26) are used to specify the conjugate stress and strain measures corresponding to $n = -1/2$.

With a properly specified strain energy function $\Psi \left(\mathbf{E}_{(n)} \right)$ for a given material, Eqs. (5.1.11) and (5.1.12), or (5.1.13) and (5.1.15), define the stress response at any state of finite elastic deformation. Since stress is derived from the strain energy function, the equations are referred to as the constitutive equations of hyperelasticity or Green-elasticity (Doyle and Ericksen, 1956; Truesdell and Noll, 1965).

The nominal stress is

$$\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}}, \tag{5.1.18}$$

which follows from

$$\dot{\Psi} = \mathbf{P} \cdot \; \cdot \dot{\mathbf{F}}, \tag{5.1.19}$$

and $\Psi = \Psi(\mathbf{F})$. Since Ψ is unaffected by rotation of the deformed configuration,

$$\Psi(\mathbf{F}) = \Psi(\mathbf{Q} \cdot \mathbf{F}). \tag{5.1.20}$$

By choosing $\mathbf{Q} = \mathbf{R}^T$, it follows that Ψ depends on $\mathbf{F}$ only through $\mathbf{U}$, or $\mathbf{C} = \mathbf{U}^2$, i.e.,

$$\Psi = \Psi(\mathbf{C}), \quad \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}. \tag{5.1.21}$$

The functional dependences of Ψ on different tensor arguments such as $\mathbf{F}$, $\mathbf{U}$ or $\mathbf{C}$ are, of course, different.

5.2. Cauchy-Elasticity

Constitutive equations of finite elasticity can be derived without assuming the existence of the strain energy function. Suppose that at any state of elastic deformation, the stress is a single-valued function of strain, regardless of the history or deformation path along which the state has been reached. Since no strain energy is assumed to exist, the work done by the stress could in general be different for different deformation paths. This type of elasticity is known as Cauchy-elasticity, although experimental evidence does not indicate existence of any Cauchy-elastic material that is also not Green-elastic. In any case, we write

$$\mathbf{T}_{(n)} = \mathbf{f}\left(\mathbf{E}_{(n)}\right), \tag{5.2.1}$$

where $\mathbf{f}$ is a second-order tensor function, whose representation depends on the selected strain measure $\mathbf{E}_{(n)}$ (relative to an undeformed configuration and its orientation), and on elastic properties of the material. In terms of the spatial stress and strain measures, Eq. (5.2.1) can be rewritten as

$$\hat{\boldsymbol{\mathcal{T}}}_{(n)} = \mathbf{f}(\hat{\boldsymbol{\mathcal{E}}}_{(n)}), \quad \hat{\boldsymbol{\mathcal{T}}}_{(n)} = \mathbf{R}^T \cdot \boldsymbol{\mathcal{T}}_{(n)} \cdot \mathbf{R}. \tag{5.2.2}$$

The rotated Kirchhoff stress can be expressed from these equations by using any of the conjugate stress and strain measures. For example,

$$\begin{gathered} \hat{\boldsymbol{\tau}} = \mathbf{g}\left(\mathbf{E}_{(1)}\right), \\ \mathbf{g}\left(\mathbf{E}_{(1)}\right) = \left(\mathbf{I}^0 + 2\mathbf{E}_{(1)}\right)^{1/2} \cdot \mathbf{f}\left(\mathbf{E}_{(1)}\right) \cdot \left(\mathbf{I}^0 + 2\mathbf{E}_{(1)}\right)^{1/2}, \end{gathered} \tag{5.2.3}$$

or

$$\begin{gathered} \hat{\boldsymbol{\tau}} = \mathbf{g}\left(\mathbf{E}_{(-1)}\right), \\ \mathbf{g}\left(\mathbf{E}_{(-1)}\right) = \left(\mathbf{I}^0 - 2\mathbf{E}_{(-1)}\right)^{1/2} \cdot \mathbf{f}\left(\mathbf{E}_{(-1)}\right) \cdot \left(\mathbf{I}^0 - 2\mathbf{E}_{(-1)}\right)^{1/2}. \end{gathered} \tag{5.2.4}$$

Note that $(\det \mathbf{F})$ can be cast in terms of the invariants of $\mathbf{E}_{(n)}$, since

$$(\det \mathbf{F})^{2n} = 1 + 2nI_E - 4n^2 II_E + 8n^3 III_E. \tag{5.2.5}$$

Thus, Eqs. (5.2.3) and (5.2.4) also define $\hat{\boldsymbol{\sigma}} = \mathbf{R}^T \cdot \boldsymbol{\sigma} \cdot \mathbf{R}$ in terms of $\mathbf{E}_{(1)}$ and $\mathbf{E}_{(-1)}$.

All constitutive equations given in this section are objective under rigid-body rotation of the deformed configuration. The material tensors are unaffected by the transformation $\mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F}$, since $\mathbf{E}^*_{(n)} = \mathbf{E}_{(n)}$ and $\mathbf{T}^*_{(n)} = \mathbf{T}_{(n)}$. The spatial tensors transform according to $\boldsymbol{\mathcal{E}}^*_{(n)} = \mathbf{Q} \cdot \boldsymbol{\mathcal{E}}_{(n)} \cdot \mathbf{Q}^T$ and

$\boldsymbol{\mathcal{T}}^*_{(n)} = \mathbf{Q} \cdot \boldsymbol{\mathcal{T}}_{(n)} \cdot \mathbf{Q}^T$, preserving the physical structure of the constitutive equations such as Eq. (5.2.2).

5.3. Isotropic Green-Elasticity

If the strain energy does not depend along which material directions the principal strains are applied, so that

$$\Psi\left(\mathbf{Q}_0 \cdot \mathbf{E}_{(n)} \cdot \mathbf{Q}_0^T\right) = \Psi\left(\mathbf{E}_{(n)}\right) \tag{5.3.1}$$

for any rotation tensor $\mathbf{Q}_0$, the material is elastically isotropic. A scalar function which satisfies Eq. (5.3.1) is said to be an isotropic function of its second-order tensor argument. Such a function can be expressed in terms of the principal invariants of the strain tensor $\mathbf{E}_{(n)}$, defined according to Eqs. (1.3.3)–(1.3.5), i.e.,

$$\Psi = \Psi\left(I_E, II_E, III_E\right). \tag{5.3.2}$$

Since

$$\begin{gathered}\frac{\partial I_E}{\partial \mathbf{E}_{(n)}} = \mathbf{I}^0, \quad \frac{\partial II_E}{\partial \mathbf{E}_{(n)}} = \mathbf{E}_{(n)} - I_E \mathbf{I}^0, \\ \frac{\partial III_E}{\partial \mathbf{E}_{(n)}} = \mathbf{E}^2_{(n)} - I_E \mathbf{E}_{(n)} - II_E \mathbf{I}^0,\end{gathered} \tag{5.3.3}$$

Equation (5.1.2) yields, by partial differentiation,

$$\mathbf{T}_{(n)} = c_0 \mathbf{I}^0 + c_1 \mathbf{E}_{(n)} + c_2 \mathbf{E}^2_{(n)}. \tag{5.3.4}$$

The parameters are

$$\begin{gathered}c_0 = \frac{\partial \Psi}{\partial I_E} - I_E \frac{\partial \Psi}{\partial II_E} - II_E \frac{\partial \Psi}{\partial III_E}, \quad c_1 = \frac{\partial \Psi}{\partial II_E} - I_E \frac{\partial \Psi}{\partial III_E}, \\ c_2 = \frac{\partial \Psi}{\partial III_E}.\end{gathered} \tag{5.3.5}$$

For example, if it is assumed that (Saint-Venant–Kirchhoff assumption)

$$\Psi = \frac{1}{2}(\lambda + 2\mu) I_E^2 + 2\mu II_E, \tag{5.3.6}$$

a generalized Hooke's law for finite strain is obtained as

$$\mathbf{T}_{(n)} = \lambda I_E \mathbf{I}^0 + 2\mu \mathbf{E}_{(n)}. \tag{5.3.7}$$

The Lamé material constants λ and μ should be specified for each selected strain measure $\mathbf{E}_{(n)}$. If a cubic representation of Ψ is assumed (Murnaghan,

1951), i.e.,

$$\Psi = \frac{1}{2}(\lambda + 2\mu)I_E^2 + 2\mu II_E + \frac{l+2m}{3} I_E^3 + 2mI_E II_E + nIII_E, \qquad (5.3.8)$$

the stress response is

$$\begin{aligned} \mathbf{T}_{(n)} = &[\lambda I_E + lI_E^2 + (2m-n)II_E]\mathbf{I}^0 \\ &+ [2\mu + (2m-n)I_E]\mathbf{E}_{(n)} + n\mathbf{E}_{(n)}^2. \end{aligned} \qquad (5.3.9)$$

The constants l, m, and n are the Murnaghan's constants.

By choosing $\mathbf{Q}_0 = \mathbf{R}$, Eq. (5.3.1) gives

$$\Psi\left(\boldsymbol{\mathcal{E}}_{(n)}\right) = \Psi\left(\mathbf{E}_{(n)}\right), \qquad (5.3.10)$$

so that Ψ is also an isotropic function of $\boldsymbol{\mathcal{E}}_{(n)}$. Since $\boldsymbol{\mathcal{E}}_{(n)}$ and $\mathbf{E}_{(n)} = \hat{\boldsymbol{\mathcal{E}}}_{(n)}$ share the same invariants, from Eqs. (5.1.10) and (5.3.10) it follows that

$$\boldsymbol{\mathcal{T}}_{(n)} = c_0\mathbf{I} + c_1\boldsymbol{\mathcal{E}}_{(n)} + c_2\boldsymbol{\mathcal{E}}_{(n)}^2. \qquad (5.3.11)$$

The parameters c_i are defined by Eq. (5.3.5), with $I_E = I_{\mathcal{E}}$, $II_E = II_{\mathcal{E}}$, and $III_E = III_{\mathcal{E}}$. Equation (5.3.11) shows that, for elastic deformation of isotropic materials, the tensors $\boldsymbol{\mathcal{T}}_{(n)}$ and $\boldsymbol{\mathcal{E}}_{(n)}$ have principal directions parallel. Likewise, $\mathbf{T}_{(n)}$ and $\mathbf{E}_{(n)}$ have parallel their principal directions.

The conjugate stress to logarithmic strain $\mathbf{E}_{(0)}$ for an elastically isotropic material is $\mathbf{T}_{(0)} = \hat{\boldsymbol{\tau}}$. The corresponding constitutive structures are

$$\begin{aligned} \hat{\boldsymbol{\tau}} &= \frac{\partial\Psi}{\partial\mathbf{E}_{(0)}} = c_0\mathbf{I}^0 + c_1\mathbf{E}_{(0)} + c_2\mathbf{E}_{(0)}^2, \\ \boldsymbol{\tau} &= \frac{\partial\Psi}{\partial\boldsymbol{\mathcal{E}}_{(0)}} = c_0\mathbf{I} + c_1\boldsymbol{\mathcal{E}}_{(0)} + c_2\boldsymbol{\mathcal{E}}_{(0)}^2, \end{aligned} \qquad (5.3.12)$$

where c_i are given by Eq. (5.3.5), in which the invariants of the logarithmic strain are appropriately used. Recall that the invariants of $\boldsymbol{\mathcal{E}}_{(0)} = \ln\mathbf{V}$ are equal to those of $\mathbf{E}_{(0)} = \ln\mathbf{U}$.

5.4. Further Expressions for Isotropic Green-Elasticity

Using Eq. (3.6.12) to express $\mathbf{T}_{(n)}$ in terms of $\mathbf{T}_{(1/2)}$, we have

$$\mathbf{T}_{(n)} = \mathbf{U}^{1-2n}\cdot\mathbf{T}_{(1/2)} = \mathbf{U}^{-2n}\cdot\hat{\boldsymbol{\tau}}. \qquad (5.4.1)$$

Substituting this into Eq. (5.1.2), carrying in mind that $\mathbf{U}^{2n} = \mathbf{I}^0 + 2n\mathbf{E}_{(n)}$, gives

$$\hat{\boldsymbol{\tau}} = \frac{\partial\Psi}{\partial\mathbf{E}_{(n)}} + n\left(\mathbf{E}_{(n)}\cdot\frac{\partial\Psi}{\partial\mathbf{E}_{(n)}} + \frac{\partial\Psi}{\partial\mathbf{E}_{(n)}}\cdot\mathbf{E}_{(n)}\right), \qquad (5.4.2)$$

written in a symmetrized form. Equation (5.4.2) applies for either positive or negative n. A dual representation, employing the spatial stress and strain tensors, is

$$\boldsymbol{\tau} = \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}} + n \left(\boldsymbol{\mathcal{E}}_{(n)} \cdot \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}} + \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}} \cdot \boldsymbol{\mathcal{E}}_{(n)} \right). \tag{5.4.3}$$

Since Ψ is an isotropic function, it follows that all material strain tensors $\mathbf{E}_{(n)}$ are coaxial with $\hat{\boldsymbol{\tau}}$, and all spatial strain tensors $\boldsymbol{\mathcal{E}}_{(n)}$ are coaxial with $\boldsymbol{\tau}$.

When the strain energy Ψ is represented in terms of the strain invariants, Eqs. (5.4.2) and (5.4.3) give, upon partial differentiation,

$$\hat{\boldsymbol{\tau}} = b_0 \mathbf{I}^0 + b_1 \mathbf{E}_{(n)} + b_2 \mathbf{E}_{(n)}^2, \tag{5.4.4}$$

$$\boldsymbol{\tau} = b_0 \mathbf{I} + b_1 \boldsymbol{\mathcal{E}}_{(n)} + b_2 \boldsymbol{\mathcal{E}}_{(n)}^2, \tag{5.4.5}$$

with the parameters

$$\begin{gathered} b_0 = c_0 + 2nc_2 III_E, \quad b_1 = c_1 + 2n\left(c_0 + c_2 II_E\right), \\ b_2 = c_2 + 2n\left(c_1 + c_2 I_E\right). \end{gathered} \tag{5.4.6}$$

More specifically, these are

$$b_0 = \frac{\partial \Psi}{\partial I_E} - I_E \frac{\partial \Psi}{\partial II_E} - \left(II_E - 2n III_E\right) \frac{\partial \Psi}{\partial III_E}, \tag{5.4.7}$$

$$b_1 = 2n \frac{\partial \Psi}{\partial I_E} + \left(1 - 2n I_E\right) \frac{\partial \Psi}{\partial II_E} - I_E \frac{\partial \Psi}{\partial III_E}, \tag{5.4.8}$$

$$b_2 = 2n \frac{\partial \Psi}{\partial II_E} + \frac{\partial \Psi}{\partial III_E}. \tag{5.4.9}$$

5.5. Constitutive Equations in Terms of B

The finite strain constitutive equations of isotropic elasticity are often expressed in terms of the left Cauchy–Green deformation tensor $\mathbf{B} = \mathbf{V}^2$. Since $\boldsymbol{\mathcal{E}}_{(1)} = (\mathbf{B} - \mathbf{I})/2$, from Eq. (5.4.3) it follows that

$$\boldsymbol{\tau} = \mathbf{B} \cdot \frac{\partial \Psi}{\partial \mathbf{B}} + \frac{\partial \Psi}{\partial \mathbf{B}} \cdot \mathbf{B}, \tag{5.5.1}$$

written in a symmetrized form. Alternatively, this follows directly from

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \mathbf{B}} : \overset{\circ}{\mathbf{B}} = \boldsymbol{\tau} : \mathbf{D}, \tag{5.5.2}$$

and the connection

$$\overset{\circ}{\mathbf{B}} = \mathbf{B}\cdot\mathbf{D} + \mathbf{D}\cdot\mathbf{B}. \tag{5.5.3}$$

The function $\Psi(\mathbf{B})$ is an isotropic function of $\mathbf{B}$. Introducing the strain energy representation

$$\Psi = \Psi\left(I_B, II_B, III_B\right), \tag{5.5.4}$$

Equation (5.5.1) gives (Rivlin, 1960)

$$\boldsymbol{\tau} = 2\left[\left(III_B\,\frac{\partial\Psi}{\partial III_B}\right)\mathbf{I} + \left(\frac{\partial\Psi}{\partial I_B} - I_B\,\frac{\partial\Psi}{\partial II_B}\right)\mathbf{B} + \left(\frac{\partial\Psi}{\partial II_B}\right)\mathbf{B}^2\right]. \tag{5.5.5}$$

If $\mathbf{B}^2$ is eliminated by using the Cayley–Hamilton theorem, Eq. (5.5.5) can be restructured as

$$\boldsymbol{\tau} = 2\left[\left(III_B\,\frac{\partial\Psi}{\partial III_B} + II_B\,\frac{\partial\Psi}{\partial II_B}\right)\mathbf{I} + \left(\frac{\partial\Psi}{\partial I_B}\right)\mathbf{B} + \left(III_B\,\frac{\partial\Psi}{\partial II_B}\right)\mathbf{B}^{-1}\right]. \tag{5.5.6}$$

These are in accord with Eq. (5.4.5), which can be verified by inspection. In the transition, the following relationships between the invariants of $\mathbf{E}_{(1)}$ or $\boldsymbol{\mathcal{E}}_{(1)}$, and $\mathbf{B}$ are noted

$$\begin{aligned} I_E &= \frac{1}{2}\left(I_B - 3\right), \quad II_E = \frac{1}{4}\,II_B + \frac{1}{2}\,I_B - \frac{3}{4}, \\ III_E &= \frac{1}{8}\left(III_B + II_B + I_B - 1\right), \end{aligned} \tag{5.5.7}$$

$$\begin{aligned} I_B &= 2I_E + 3, \quad II_B = 4II_E - 4I_E - 3, \\ III_B &= 8III_E - 4II_E + 2I_E + 1. \end{aligned} \tag{5.5.8}$$

The constitutive equation of isotropic elastic material in terms of the nominal stress is

$$\mathbf{P} = \mathbf{F}^{-1}\cdot\boldsymbol{\tau} = \mathbf{F}^T\cdot\left(\frac{\partial\Psi}{\partial\mathbf{B}} + \mathbf{B}^{-1}\cdot\frac{\partial\Psi}{\partial\mathbf{B}}\cdot\mathbf{B}\right). \tag{5.5.9}$$

By using the strain energy representation of Eq. (5.5.4), this becomes

$$\mathbf{P} = 2\mathbf{F}^T\cdot\left[\left(\frac{\partial\Psi}{\partial I_B} - I_B\,\frac{\partial\Psi}{\partial II_B}\right)\mathbf{I} + \left(\frac{\partial\Psi}{\partial II_B}\right)\mathbf{B} + \left(III_B\,\frac{\partial\Psi}{\partial III_B}\right)\mathbf{B}^{-1}\right]. \tag{5.5.10}$$

Different specific forms of the strain energy function were used in the literature. For example, Ogden (1984) constructed a strain energy function

$$\Psi = \frac{a}{2}\left(I_B - 3 - \ln\,III_B\right) + c\left(III_B^{1/2} - 1\right)^2, \tag{5.5.11}$$

where a and c are the material parameters. Based on their theoretical analysis and experimental data Blatz and Ko (1962) proposed an expression for the strain energy for compressible foamed elastomers. Other representations can be found in Blatz, Sharda, and Tschoegl (1974), Morman (1986), Ciarlet (1988), Beatty (1996), and Holzapfel (2000).

5.6. Constitutive Equations in Terms of Principal Stretches

The strain energy of an isotropic material can be often conveniently expressed in terms of the principal stretches λ_i (the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$, which are invariant quantities), i.e.,

$$\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3). \tag{5.6.1}$$

Suppose that all principal stretches are different, and that $\mathbf{N}_i$ and $\mathbf{n}_i$ are the principal directions of the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$, respectively, so that

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \, \mathbf{N}_i \otimes \mathbf{N}_i, \quad \mathbf{E}_{(n)} = \sum_{i=1}^{3} \frac{1}{2n} \left(\lambda_i^{2n} - 1\right) \mathbf{N}_i \otimes \mathbf{N}_i, \tag{5.6.2}$$

and

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i, \quad \mathbf{F} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{N}_i. \tag{5.6.3}$$

For an isotropic elastic material, the principal directions of the strain tensor $\mathbf{E}_{(n)}$ are parallel to those of its conjugate stress tensor $\mathbf{T}_{(n)}$, and we can write

$$\mathbf{T}_{(n)} = \sum_{i=1}^{3} T_i^{(n)} \, \mathbf{N}_i \otimes \mathbf{N}_i. \tag{5.6.4}$$

The principal stresses are here

$$T_i^{(n)} = \frac{\partial \Psi}{\partial E_i^{(n)}} = \lambda_i^{1-2n} \, \frac{\partial \Psi}{\partial \lambda_i} \,, \tag{5.6.5}$$

with no sum on i. Recall that $\lambda_i^{2n} = 1 + 2nE_i^{(n)}$. For example, for $n = 1$ we obtain the principal components of the symmetric Piola–Kirchhoff stress,

$$T_i^{(1)} = \frac{\partial \Psi}{\partial E_i^{(1)}} = \frac{1}{\lambda_i} \frac{\partial \Psi}{\partial \lambda_i} \,. \tag{5.6.6}$$

The principal directions of the Kirchhoff stress $\boldsymbol{\tau}$ of an isotropic elastic material are parallel to those of $\mathbf{V}$, so that

$$\boldsymbol{\tau} = \sum_{i=1}^{3} \tau_i \, \mathbf{n}_i \otimes \mathbf{n}_i. \tag{5.6.7}$$

The corresponding principal components are

$$\tau_i = \lambda_i^2 T_i^{(1)} = \lambda_i \frac{\partial \Psi}{\partial \lambda_i} \,. \tag{5.6.8}$$

Finally, decomposing the nominal stress as

$$\mathbf{P} = \sum_{i=1}^{3} P_i \, \mathbf{n}_i \otimes \mathbf{N}_i, \tag{5.6.9}$$

we have

$$P_i = \lambda_i T_i^{(1)} = \frac{\partial \Psi}{\partial \lambda_i} \,. \tag{5.6.10}$$

5.7. Incompressible Isotropic Elastic Materials

For an incompressible material the deformation is necessarily isochoric, so that $\det \mathbf{F} = 1$. Only two invariants of $\mathbf{E}_{(n)}$ are independent, since

$$III_E = -\frac{1}{4n^2} \left(I_E - 2n II_E \right). \tag{5.7.1}$$

Thus, the strain energy can be expressed as

$$\Psi = \Psi \left(I_E, II_E \right), \tag{5.7.2}$$

and we obtain

$$\boldsymbol{\sigma} = (b_0 - p)\mathbf{I} + b_1 \boldsymbol{\mathcal{E}}_{(n)} + b_2 \boldsymbol{\mathcal{E}}_{(n)}^2. \tag{5.7.3}$$

Here, p is an arbitrary pressure, and b_i are defined by Eqs. (5.4.7)–(5.4.9), without terms proportional to $\partial \Psi / \partial III_E$. Alternatively, if Eqs. (5.5.5) and (5.5.6) are specialized to incompressible materials, there follows

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2 \left[\left(\frac{\partial \Psi}{\partial I_B} - I_B \frac{\partial \Psi}{\partial II_B} \right) \mathbf{B} + \left(\frac{\partial \Psi}{\partial II_B} \right) \mathbf{B}^2 \right], \tag{5.7.4}$$

and

$$\boldsymbol{\sigma} = -p_0\mathbf{I} + 2 \left[\left(\frac{\partial \Psi}{\partial I_B} \right) \mathbf{B} + \left(\frac{\partial \Psi}{\partial II_B} \right) \mathbf{B}^{-1} \right]. \tag{5.7.5}$$

In Eq. (5.7.5), all terms proportional to $\mathbf{I}$ are absorbed in p_0.

Equation (5.7.4) can also be derived by viewing an incompressible material as a material with internal constraint

$$III_B - 1 = 0. \tag{5.7.6}$$

A Lagrangian multiplier $-p/2$ can then be introduced, such that

$$\Psi = \Psi\left(I_B, II_B\right) - \frac{p}{2}\left(III_B - 1\right), \tag{5.7.7}$$

and Eq. (5.5.1) directly leads to Eq. (5.7.4).

For the Mooney–Rivlin material (rubber model; see Treloar, 1975), the strain energy is

$$\Psi = aI_E + bII_E = \frac{a+b}{2}\left(I_B - 3\right) + \frac{b}{4}\left(II_B + 3\right), \tag{5.7.8}$$

and for the neo-Hookean material

$$\Psi = aI_E = \frac{a}{2}\left(I_B - 3\right). \tag{5.7.9}$$

The strain energy representation, suggested by Ogden (1972,1982),

$$\Psi = \sum_{n=1}^{N} a_n \mathrm{tr}\,\mathbf{E}_{(n)} = \sum_{n=1}^{N} \frac{a_n}{\alpha_n}\left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right) \tag{5.7.10}$$

may be used in some applications, where N is positive integer, but α_n need not be integers (the tensors $\mathbf{E}_{(n)}$ are here defined by Eq. (2.3.1) with α_n replacing $2n$; Hill, 1978). The material parameters are a_n and α_n. Incompressibility constraint is $\lambda_1\lambda_2\lambda_3 = 1$. Other representations in terms of principal stretches λ_i have also been explored (Valanis and Landel, 1967; Rivlin and Sawyers, 1976; Anand, 1986; Arruda and Boyce, 1993).

5.8. Isotropic Cauchy-Elasticity

For isotropic elastic material the tensor function $\mathbf{f}$ in Eq. (5.2.1) is an isotropic function of strain,

$$\mathbf{f}\left(\mathbf{Q}_0 \cdot \mathbf{E}_{(n)} \cdot \mathbf{Q}_0^T\right) = \mathbf{Q}_0 \cdot \mathbf{f}\left(\mathbf{E}_{(n)}\right) \cdot \mathbf{Q}_0^T, \tag{5.8.1}$$

and, by the representation theorem from Section 1.11, the stress response can be written as

$$\mathbf{T}_{(n)} = c_0\mathbf{I}^0 + c_1\mathbf{E}_{(n)} + c_2\mathbf{E}_{(n)}^2. \tag{5.8.2}$$

The parameters c_i are scalar functions of the invariants of $\mathbf{E}_{(n)}$. Similarly, from Eq. (5.2.2) it follows that

$$\boldsymbol{\mathcal{T}}_{(n)} = c_0\mathbf{I} + c_1\boldsymbol{\mathcal{E}}_{(n)} + c_2\boldsymbol{\mathcal{E}}_{(n)}^2. \tag{5.8.3}$$

In view of the isotropic elasticity relationships

$$\hat{\boldsymbol{\tau}} = \mathbf{U}^{2n} \cdot \mathbf{T}_{(n)}, \quad \boldsymbol{\tau} = \mathbf{V}^{2n} \cdot \boldsymbol{\mathcal{T}}_{(n)}, \tag{5.8.4}$$

equations (5.8.2) and (5.8.3) can be rephrased as

$$\hat{\boldsymbol{\tau}} = b_0\mathbf{I}^0 + b_1\mathbf{E}_{(n)} + b_2\mathbf{E}^2_{(n)}, \quad \boldsymbol{\tau} = b_0\mathbf{I} + b_1\boldsymbol{\mathcal{E}}_{(n)} + b_2\boldsymbol{\mathcal{E}}^2_{(n)}, \tag{5.8.5}$$

where b_i are given by Eq. (5.4.6). The constitutive equations of Green-elasticity are recovered if the strain energy function exists, so that the constants c_i in Eq. (5.4.6) are specified by Eq. (5.3.5).

Finally, it is noted that Eqs. (5.8.5) can be recast in terms of $\mathbf{C} = \mathbf{U}^2$ and $\mathbf{B} = \mathbf{V}^2$, with the results

$$\hat{\boldsymbol{\tau}} = a_0\mathbf{I}^0 + a_1\mathbf{C} + a_2\mathbf{C}^2, \quad \boldsymbol{\tau} = a_0\mathbf{I} + a_1\mathbf{B} + a_2\mathbf{B}^2. \tag{5.8.6}$$

The scalar parameters a_i depend on the invariants of $\mathbf{C}$ or $\mathbf{B}$. The last expression can also be deduced directly from $\mathbf{T}_{(1)} = \mathbf{f}\left(\mathbf{E}_{(1)}\right)$ by the representation theorem for the isotropic function $\mathbf{f}$, dependent on the Lagrangian strain $\mathbf{E}_{(1)} = \left(\mathbf{C} - \mathbf{I}^0\right)/2$. Furthermore, since $(\det \mathbf{F}) = III_C^{1/2}$, Eqs. (5.8.6) define the stress tensors $\hat{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}$, as well ($\boldsymbol{\sigma}$ being the Cauchy stress). For incompressible materials

$$\boldsymbol{\sigma} = -p_1\mathbf{I} + b_1\boldsymbol{\mathcal{E}}_{(n)} + b_2\boldsymbol{\mathcal{E}}^2_{(n)} = -p_2\mathbf{I} + a_1\mathbf{B} + a_2\mathbf{B}^2, \tag{5.8.7}$$

where p_1 and p_2 are arbitrary pressures. Additional discussion can be found in the books by Leigh (1968) and Malvern (1969).

5.9. Transversely Isotropic Materials

For an elastically isotropic material, elastic properties are equal in all directions. Any rotation of the undeformed reference configuration before the application of a given stress has no effect on the subsequent stress-deformation response. The material symmetry group is the full orthogonal group. If the symmetry group of the material is less than the full orthogonal group, the material is anisotropic (aelotropic). For the most general anisotropy, the isotropy group consists only of identity transformation 1 and the central inversion transformation $\bar{1}$. Any rotation of the reference configuration prior to application of stress will change the elastic response of such a material.

The material is said to have a plane of elastic symmetry if the reference configuration obtained from the undeformed configuration by reflection in the plane of symmetry is indistinguishable from the undeformed configuration (in the sense of elastic response).

Transversely isotropic material has a single preferred direction (axis of isotropy). Its symmetry group consists of arbitrary rotations about the axis of isotropy, say $\mathbf{m}^0$, and rotations that carry $\mathbf{m}^0$ into $-\mathbf{m}^0$. Every plane containing $\mathbf{m}^0$ is a plane of elastic symmetry, so that reflections in these planes also belong to the symmetry group. The elastic strain energy function can be consequently written as

$$\Psi = \Psi\left(I_E, II_E, III_E, E_{33}, E_{31}^2 + E_{32}^2\right), \tag{5.9.1}$$

provided that the coordinate axes are selected so that $\mathbf{m}^0$ is in the coordinate direction $\mathbf{e}_3$. The arguments in Eq. (5.9.1) are invariant under the transformations from the symmetry group of transverse isotropy. This can be derived as follows. For transversely isotropic material, the strain energy is a scalar function of the strain tensor $\mathbf{E}_{(n)}$ and the unit vector $\mathbf{m}^0$,

$$\Psi = \Psi\left(\mathbf{E}_{(n)}, \mathbf{m}^0\right). \tag{5.9.2}$$

The function Ψ is invariant under all orthogonal transformations of the reference configuration that carry both $\mathbf{E}_{(n)}$ and $\mathbf{m}^0$, i.e.,

$$\Psi\left(\mathbf{Q}_0 \cdot \mathbf{E}_{(n)} \cdot \mathbf{Q}_0^T, \mathbf{Q}_0 \cdot \mathbf{m}^0\right) = \Psi\left(\mathbf{E}_{(n)}, \mathbf{m}^0\right). \tag{5.9.3}$$

Such a function Ψ is said to be an isotropic function of both $\mathbf{E}_{(n)}$ and $\mathbf{m}^0$, simultaneously. Physically, the rotated strain $\mathbf{Q}_0 \cdot \mathbf{E}_{(n)} \cdot \mathbf{Q}_0^T$, applied relative to the rotated axis of isotropy $\mathbf{Q}_0 \cdot \mathbf{m}^0$, gives the same strain energy as the strain $\mathbf{E}_{(n)}$ applied relative to the original axis of isotropy $\mathbf{m}^0$. Of course, Ψ is not an isotropic function of the strain alone, i.e.,

$$\Psi\left(\mathbf{Q}_0 \cdot \mathbf{E}_{(n)} \cdot \mathbf{Q}_0^T, \mathbf{m}^0\right) \neq \Psi\left(\mathbf{E}_{(n)}, \mathbf{m}^0\right) \tag{5.9.4}$$

in general, although the equality sign holds for those $\mathbf{Q}_0$ that belong to the symmetry group of transverse isotropy.

Representation of isotropic scalar functions of second-order tensors and vectors is well-known (e.g., Boehler, 1987). The function $\Psi\left(\mathbf{E}_{(n)}, \mathbf{m}^0\right)$ can be expressed in terms of individual and joint invariants of $\mathbf{E}_{(n)}$ and $\mathbf{m}^0$, i.e.,

$$\Psi = \Psi\left(I_E, II_E, III_E, \mathbf{m}^0 \cdot \mathbf{E}_{(n)} \cdot \mathbf{m}^0, \mathbf{m}^0 \cdot \mathbf{E}_{(n)}^2 \cdot \mathbf{m}^0\right). \tag{5.9.5}$$

It is convenient to introduce the second-order tensor

$$\mathbf{M}^0 = \mathbf{m}^0 \otimes \mathbf{m}^0. \tag{5.9.6}$$

This is an idempotent tensor, for which

$$\mathbf{M}^0 \cdot \mathbf{M}^0 = \mathbf{M}^0, \quad I_M = 1, \quad II_M = III_M = 0. \tag{5.9.7}$$

When applied to an arbitrary vector $\mathbf{a}_0$, the tensor $\mathbf{M}^0$ projects it on $\mathbf{m}^0$,

$$\mathbf{M}^0 \cdot \mathbf{a}_0 = (\mathbf{m}^0 \cdot \mathbf{a}_0)\mathbf{m}^0. \tag{5.9.8}$$

The joint invariants of $\mathbf{E}_{(n)}$ and $\mathbf{m}^0$ in Eq. (5.9.5) can thus be written as

$$\begin{aligned} K_1 &= \mathbf{m}^0 \cdot \mathbf{E}_{(n)} \cdot \mathbf{m}^0 = \mathrm{tr}\left(\mathbf{M}^0 \cdot \mathbf{E}_{(n)}\right), \\ K_2 &= \mathbf{m}^0 \cdot \mathbf{E}^2_{(n)} \cdot \mathbf{m}^0 = \mathrm{tr}\left(\mathbf{M}^0 \cdot \mathbf{E}^2_{(n)}\right), \end{aligned} \tag{5.9.9}$$

and the strain energy becomes

$$\Psi = \Psi\left(I_E, II_E, III_E, K_1, K_2\right). \tag{5.9.10}$$

The stress response is accordingly

$$\mathbf{T}_{(n)} = c_0\mathbf{I}^0 + c_1\mathbf{E}_{(n)} + c_2\mathbf{E}^2_{(n)} + c_3\mathbf{M}^0 + c_4\left(\mathbf{M}^0 \cdot \mathbf{E}_{(n)} + \mathbf{E}_{(n)} \cdot \mathbf{M}^0\right). \tag{5.9.11}$$

The parameters c_0, c_1 and c_3 are defined by Eqs. (5.3.5), and

$$c_3 = \frac{\partial \Psi}{\partial K_1}, \quad c_4 = \frac{1}{2}\frac{\partial \Psi}{\partial K_2}. \tag{5.9.12}$$

If we choose $\mathbf{Q}_0 = \mathbf{R}$ (rotation tensor from the polar decomposition of deformation gradient), from Eq. (5.9.3) it follows that

$$\Psi\left(\boldsymbol{\mathcal{E}}_{(n)}, \overline{\mathbf{m}}\right) = \Psi\left(\mathbf{E}_{(n)}, \mathbf{m}^0\right), \tag{5.9.13}$$

where

$$\overline{\mathbf{m}} = \mathbf{R} \cdot \mathbf{m}^0. \tag{5.9.14}$$

Thus, Ψ is also an isotropic function of the spatial strain $\boldsymbol{\mathcal{E}}_{(n)}$ and the vector $\overline{\mathbf{m}}$. A dual equation to Eq. (5.9.11), expressed relative to the deformed configuration, is consequently

$$\boldsymbol{\mathcal{T}}_{(n)} = c_0\mathbf{I} + c_1\boldsymbol{\mathcal{E}}_{(n)} + c_2\boldsymbol{\mathcal{E}}^2_{(n)} + c_3\overline{\mathbf{M}} + c_4\left(\overline{\mathbf{M}} \cdot \boldsymbol{\mathcal{E}}_{(n)} + \boldsymbol{\mathcal{E}}_{(n)} \cdot \overline{\mathbf{M}}\right). \tag{5.9.15}$$

The tensor $\overline{\mathbf{M}}$ is defined by

$$\overline{\mathbf{M}} = \overline{\mathbf{m}} \otimes \overline{\mathbf{m}} = \mathbf{R} \cdot \mathbf{M}^0 \cdot \mathbf{R}^T. \tag{5.9.16}$$

For example, if $n = 1$, Eq. (5.9.15) gives the Kirchhoff stress

$$\boldsymbol{\tau} = b_0\mathbf{I} + b_1\boldsymbol{\mathcal{E}}_{(1)} + b_2\boldsymbol{\mathcal{E}}^2_{(1)} + c_3\mathbf{M} + c_4\left(\mathbf{M} \cdot \boldsymbol{\mathcal{E}}_{(1)} + \boldsymbol{\mathcal{E}}_{(1)} \cdot \mathbf{M}\right). \tag{5.9.17}$$

The coefficients b_i are written in terms of c_i by Eqs. (5.4.6), and

$$\mathbf{M} = \mathbf{m} \otimes \mathbf{m} = \mathbf{F} \cdot \mathbf{M}^0 \cdot \mathbf{F}^T, \quad \mathbf{m} = \mathbf{V} \cdot \overline{\mathbf{m}} = \mathbf{F} \cdot \mathbf{m}^0. \tag{5.9.18}$$

The vector $\mathbf{m}$ in the deformed configuration is obtained by deformation $\mathbf{F}$ from the vector $\mathbf{m}^0$ in the undeformed configuration. However, while $\mathbf{m}^0$ and $\overline{\mathbf{m}}$ are the unit vectors, the (embedded) vector $\mathbf{m}$ is not. The tensor $\mathbf{M}^0 = \mathbf{F}^{-1} \cdot \mathbf{M} \cdot \mathbf{F}^{-T}$ is induced from $\mathbf{M}$ by a transformation of the contravariant type.

If transversely isotropic material is inextensible in the direction of the axis of isotropy, so that there exists a deformation constraint

$$\mathbf{m}^0 \cdot \mathbf{C} \cdot \mathbf{m}^0 = \overline{\mathbf{m}} \cdot \mathbf{B} \cdot \overline{\mathbf{m}} = 1, \quad \text{or} \quad \mathbf{m}^0 \cdot \mathbf{E}_{(1)} \cdot \mathbf{m}^0 = \overline{\mathbf{m}} \cdot \boldsymbol{\mathcal{E}}_{(1)} \cdot \overline{\mathbf{m}} = 0, \tag{5.9.19}$$

the strain energy can be written by using the Lagrangian multiplier as

$$\Psi = \Psi\left(I_E, II_E, III_E, K_1, K_2\right) + (\det \mathbf{F})\sigma_m \, \mathbf{m}^0 \cdot \mathbf{E}_{(1)} \cdot \mathbf{m}^0. \tag{5.9.20}$$

Thus, we add to the right-hand side of Eq. (5.9.11) the term $(\det \mathbf{F})\sigma_m \mathbf{M}^0$, and to the right-hand side of Eq. (5.9.17) the term $(\det \mathbf{F})\sigma_m \mathbf{M}$, where the Lagrangian multiplier σ_m is an arbitrary tension in the direction $\mathbf{m}$.

5.9.1. Transversely Isotropic Cauchy-Elasticity

In this case, the stress is assumed to be a function of $\mathbf{E}_{(n)}$ and $\mathbf{M}^0$ at the outset,

$$\mathbf{T}_{(n)} = \mathbf{f}\left(\mathbf{E}_{(n)}, \mathbf{M}^0\right). \tag{5.9.21}$$

This must be an isotropic tensor function of both $\mathbf{E}_{(n)}$ and $\mathbf{M}^0$, so that

$$\mathbf{Q}_0 \cdot \mathbf{T}_{(n)} \cdot \mathbf{Q}_0^T = \mathbf{f}\left(\mathbf{Q}_0 \cdot \mathbf{E}_{(n)} \cdot \mathbf{Q}_0^T, \mathbf{Q}_0 \cdot \mathbf{M}^0 \cdot \mathbf{Q}_0^T\right). \tag{5.9.22}$$

Representation of isotropic second-order tensor functions of two symmetric second-order tensor arguments is well-known. The set of generating tensors is given in Eq. (1.11.10). Indeed, consider the most general isotropic invariant of $\mathbf{E}_{(n)}$, $\mathbf{M}^0$ and a symmetric tensor $\mathbf{H}$, which is linear in $\mathbf{H}$. Since $\mathbf{M}^0$ is idempotent, this invariant is

$$\begin{aligned} g = {} & c_0 \operatorname{tr} \mathbf{H} + c_1 \operatorname{tr}\left(\mathbf{E}_{(n)} \cdot \mathbf{H}\right) + c_2 \operatorname{tr}\left(\mathbf{E}_{(n)}^2 \cdot \mathbf{H}\right) + c_3 \operatorname{tr}\left(\mathbf{M}^0 \cdot \mathbf{H}\right) \\ & + c_4 \operatorname{tr}\left[\left(\mathbf{M}^0 \cdot \mathbf{E}_{(n)} + \mathbf{E}_{(n)} \cdot \mathbf{M}^0\right) \cdot \mathbf{H}\right] \\ & + c_5 \operatorname{tr}\left[\left(\mathbf{M}^0 \cdot \mathbf{E}_{(n)}^2 + \mathbf{E}_{(n)}^2 \cdot \mathbf{M}^0\right) \cdot \mathbf{H}\right]. \end{aligned} \tag{5.9.23}$$

The parameters c_i are scalar invariants of $\mathbf{E}_{(n)}$ and $\mathbf{M}^0$. The stress tensor is derived as the gradient of g with respect to $\mathbf{H}$, which gives

$$\mathbf{T}_{(n)} = c_0\mathbf{I}^0 + c_1\mathbf{E}_{(n)} + c_2\mathbf{E}_{(n)}^2 + c_3\mathbf{M}^0 + c_4\left(\mathbf{M}^0\cdot\mathbf{E}_{(n)} + \mathbf{E}_{(n)}\cdot\mathbf{M}^0\right) + c_5\left(\mathbf{M}^0\cdot\mathbf{E}_{(n)}^2 + \mathbf{E}_{(n)}^2\cdot\mathbf{M}^0\right). \tag{5.9.24}$$

The term proportional to c_5 in Eq. (5.9.24) for transversely isotropic Cauchy-elasticity is absent in the case of transversely isotropic Green-elasticity, cf. Eq. (5.9.11). Also, it is noted that in the transition to linearized theory (retaining linear terms in strain $\mathbf{E}_{(n)}$ only), the Cauchy-elasticity of transversely isotropic materials involves six independent material parameters, while the Green-elasticity involves only five of them.

5.10. Orthotropic Materials

Elastic material is orthotropic in its reference configuration if it possesses three mutually orthogonal planes of elastic symmetry. Its symmetry group consists of reflections in these planes. Therefore, we introduce two second -order tensors

$$\mathbf{M}^0 = \mathbf{m}^0\otimes\mathbf{m}^0, \quad \mathbf{N}^0 = \mathbf{n}^0\otimes\mathbf{n}^0, \tag{5.10.1}$$

which are associated with the unit vectors $\mathbf{m}^0$ and $\mathbf{n}^0$, normal to two of the planes of elastic symmetry in the undeformed configuration. The tensor associated with the third plane of symmetry is $\mathbf{I}^0 - \mathbf{M}^0 - \mathbf{N}^0$, and need not be considered. The strain energy is then

$$\Psi = \Psi\left(\mathbf{E}_{(n)}, \mathbf{M}^0, \mathbf{N}^0\right). \tag{5.10.2}$$

This must be an isotropic function of all three tensor arguments,

$$\Psi\left(\mathbf{Q}_0\cdot\mathbf{E}_{(n)}\cdot\mathbf{Q}_0^T, \mathbf{Q}_0\cdot\mathbf{M}^0\cdot\mathbf{Q}_0^T, \mathbf{Q}_0\cdot\mathbf{N}^0\cdot\mathbf{Q}_0^T\right) = \Psi\left(\mathbf{E}_{(n)}, \mathbf{M}^0, \mathbf{N}^0\right), \tag{5.10.3}$$

and thus dependent on individual and joint invariants of its tensor arguments. Since $\mathbf{M}^0\cdot\mathbf{N}^0 = 0$, by the orthogonality of $\mathbf{m}^0$ and $\mathbf{n}^0$, it follows that

$$\Psi = \Psi\left(I_E, II_E, III_E, K_1, K_2, K_3, K_4\right). \tag{5.10.4}$$

The invariants K_1 and K_2 are defined by Eq. (5.9.9), and K_3 and K_4 by the corresponding expressions in which $\mathbf{M}^0$ is replaced with $\mathbf{N}^0$. The stress

response is

$$\mathbf{T}_{(n)} = c_0\mathbf{I}^0 + c_1\mathbf{E}_{(n)} + c_2\mathbf{E}_{(n)}^2 + c_3\mathbf{M}^0 + c_4\left(\mathbf{M}^0\cdot\mathbf{E}_{(n)} + \mathbf{E}_{(n)}\cdot\mathbf{M}^0\right) + c_5\mathbf{N}^0 + c_6\left(\mathbf{N}^0\cdot\mathbf{E}_{(n)} + \mathbf{E}_{(n)}\cdot\mathbf{N}^0\right). \tag{5.10.5}$$

The coefficients c_0 to c_4 are specified by Eqs. (5.3.5) and (5.9.12), and c_5 and c_6 by equations (5.9.12) in which the derivatives are taken with respect to K_3 and K_4.

Equation (5.10.5) has a dual equation in the deformed configuration

$$\boldsymbol{\mathcal{T}}_{(n)} = c_0\mathbf{I} + c_1\boldsymbol{\mathcal{E}}_{(n)} + c_2\boldsymbol{\mathcal{E}}_{(n)}^2 + c_3\overline{\mathbf{M}} + c_4\left(\overline{\mathbf{M}}\cdot\boldsymbol{\mathcal{E}}_{(n)} + \boldsymbol{\mathcal{E}}_{(n)}\cdot\overline{\mathbf{M}}\right) + c_5\overline{\mathbf{N}} + c_6\left(\overline{\mathbf{N}}\cdot\boldsymbol{\mathcal{E}}_{(n)} + \boldsymbol{\mathcal{E}}_{(n)}\cdot\overline{\mathbf{N}}\right), \tag{5.10.6}$$

where

$$\overline{\mathbf{M}} = \overline{\mathbf{m}}\otimes\overline{\mathbf{m}}, \quad \overline{\mathbf{N}} = \overline{\mathbf{n}}\otimes\overline{\mathbf{n}}, \tag{5.10.7}$$

and

$$\overline{\mathbf{m}} = \mathbf{R}\cdot\mathbf{m}^0, \quad \overline{\mathbf{n}} = \mathbf{R}\cdot\mathbf{n}^0. \tag{5.10.8}$$

In particular, for $n = 1$, Eq. (5.10.6) gives

$$\boldsymbol{\tau} = b_0\mathbf{I} + b_1\boldsymbol{\mathcal{E}}_{(1)} + b_2\boldsymbol{\mathcal{E}}_{(1)}^2 + c_3\mathbf{M} + c_4\left(\mathbf{M}\cdot\boldsymbol{\mathcal{E}}_{(1)} + \boldsymbol{\mathcal{E}}_{(1)}\cdot\mathbf{M}\right) + c_5\mathbf{N} + c_6\left(\mathbf{N}\cdot\boldsymbol{\mathcal{E}}_{(1)} + \boldsymbol{\mathcal{E}}_{(1)}\cdot\mathbf{N}\right). \tag{5.10.9}$$

The coefficients b_i are expressed in terms of c_i by Eqs. (5.4.6), and

$$\mathbf{M} = \mathbf{m}\otimes\mathbf{m}, \quad \mathbf{N} = \mathbf{n}\otimes\mathbf{n}. \tag{5.10.10}$$

The vectors $\mathbf{m}$ and $\mathbf{n}$ are

$$\mathbf{m} = \mathbf{V}\cdot\overline{\mathbf{m}} = \mathbf{F}\cdot\mathbf{m}^0, \quad \mathbf{n} = \mathbf{V}\cdot\overline{\mathbf{n}} = \mathbf{F}\cdot\mathbf{n}^0. \tag{5.10.11}$$

5.10.1. Orthotropic Cauchy-Elasticity

The stress is here assumed to be a function of three tensor arguments, such that

$$\mathbf{T}_{(n)} = \mathbf{f}\left(\mathbf{E}_{(n)}, \mathbf{M}^0, \mathbf{N}^0\right). \tag{5.10.12}$$

If the undeformed configuration is rotated by $\mathbf{Q}_0$, we have

$$\mathbf{Q}_0\cdot\mathbf{T}_{(n)}\cdot\mathbf{Q}_0^T = \mathbf{f}\left(\mathbf{Q}_0\cdot\mathbf{E}_{(n)}\cdot\mathbf{Q}_0^T, \mathbf{Q}_0\cdot\mathbf{M}^0\cdot\mathbf{Q}_0^T, \mathbf{Q}_0\cdot\mathbf{N}^0\cdot\mathbf{Q}_0^T\right), \tag{5.10.13}$$

which implies that $\mathbf{f}$ must be an isotropic tensor function of all three of its tensor arguments. The most general form of this function is

$$\begin{aligned}\mathbf{T}_{(n)} &= c_0\mathbf{I}^0 + c_1\mathbf{E}_{(n)} + c_2\mathbf{E}^2_{(n)} + c_3\mathbf{M}^0 + c_6\mathbf{N}^0 \\ &+ c_4\left(\mathbf{M}^0\cdot\mathbf{E}_{(n)} + \mathbf{E}_{(n)}\cdot\mathbf{M}^0\right) + c_5\left(\mathbf{M}^0\cdot\mathbf{E}^2_{(n)} + \mathbf{E}^2_{(n)}\cdot\mathbf{M}^0\right) \\ &+ c_7\left(\mathbf{N}^0\cdot\mathbf{E}_{(n)} + \mathbf{E}_{(n)}\cdot\mathbf{N}^0\right) + c_8\left(\mathbf{N}^0\cdot\mathbf{E}^2_{(n)} + \mathbf{E}^2_{(n)}\cdot\mathbf{N}^0\right).\end{aligned} \tag{5.10.14}$$

The terms proportional to c_5 and c_8 in Eq. (5.10.14) are absent in the case of orthotropic Green-elasticity, cf. Eq. (5.10.5). In the transition to linearized theory (retaining linear terms in strain $\mathbf{E}_{(n)}$ only), the Cauchy-elasticity of orthotropic materials involves twelve independent material parameters, while the Green-elasticity involves only nine of them.

5.11. Crystal Elasticity

5.11.1. Crystal Classes

Anisotropic materials known as crystal classes possess three preferred directions, defined by unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. There are thirty two crystal classes (point groups). Each class is characterized by a group of orthogonal transformations which carry the reference undeformed configuration into an equivalent configuration, indistinguishable from the original configuration. Since elastic properties of crystals are centrosymmetric, the eleven Laue groups can be identified. All point groups belonging to the same Laue group have common polynomial representation of the strain energy function in terms of the corresponding polynomial strain invariants. Crystal classes are grouped into seven crystal systems. In describing them, the following convention will be used. By $\frac{n}{m}$ is meant the rotation by an angle $2\pi/n$, followed by a reflection in the plane normal to the axis of rotation. By $\bar{n}$ is meant the rotation by an angle $2\pi/n$, followed by an inversion.

i) Triclinic System (Laue group N). For this crystal system there is no restriction on the orientation of the vectors $\mathbf{a}_i$. Two point groups of this system are $(1, \bar{1})$. Since components of the strain tensor $\mathbf{E}_{(n)}$ are unaltered by identity and central inversion transformations, no restriction is placed on the form of the polynomial representation of the strain energy in terms of

the six strain components, i.e.,

$$\Psi = \Psi\left(E_{11}, E_{22}, E_{33}, E_{12}, E_{31}, E_{32}\right). \tag{5.11.1}$$

Any rectangular Cartesian coordinate system may be chosen as a reference system.

ii) Monoclinic System (Laue group M). The preferred directions $\mathbf{a}_1$ and $\mathbf{a}_2$ are not orthogonal, and the direction $\mathbf{a}_3$ is perpendicular to the plane $(\mathbf{a}_1, \mathbf{a}_2)$. There are three point groups of the monoclinic system. They are $\left(2,\, m,\, \frac{2}{m}\right)$. The symmetry transformation of the first point group is the rotation $\mathcal{Q}_3$ about X_3 axis through 180°, for the second it is reflection $\mathcal{R}_3$ in the plane normal to X_3 axis, and for the third it is the rotation $\mathcal{Q}_3$ followed by the reflection $\mathcal{R}_3$. For each point group, the strain energy is a polynomial of the seven polynomial strain invariants of this system, i.e.,

$$\Psi = \Psi\left(E_{11}, E_{22}, E_{33}, E_{12}, E_{31}^2, E_{32}^2, E_{31}E_{32}\right). \tag{5.11.2}$$

The rectangular Cartesian system is used with the axis X_3 parallel to $\mathbf{a}_3$, and with the axes X_1 and X_2 in any two orthogonal directions within $(\mathbf{a}_1, \mathbf{a}_2)$ plane.

iii) Orthorombic System (Laue group O). The preferred directions $\mathbf{a}_i$ are mutually perpendicular. There are three point groups of this system. They are $\left(222,\, mm2,\, \frac{2}{m}\frac{2}{m}\frac{2}{m}\right)$. For each point group, the strain energy is a polynomial of the seven polynomial strain invariants,

$$\Psi = \Psi\left(E_{11}, E_{22}, E_{33}, E_{12}^2, E_{31}^2, E_{32}^2, E_{12}E_{31}E_{32}\right). \tag{5.11.3}$$

The axes of the reference coordinate system are parallel to $\mathbf{a}_i$.

iv) Tetragonal System (Laue groups TII and TI). The vectors $\mathbf{a}_i$ are mutually perpendicular, but the direction $\mathbf{a}_3$ has a special significance and is called the principal axis of symmetry. The Laue group TII contains three point groups $\left(4,\, \bar{4},\, \frac{4}{m}\right)$. The corresponding strain energy is expressible as a polynomial in the twelve polynomial strain invariants. These are

$$\begin{aligned}
&E_{11} + E_{22}, \quad E_{33}, \quad E_{31}^2 + E_{32}^2, \quad E_{12}^2, \quad E_{11}E_{22},\\
&E_{12}(E_{11} - E_{22}), \quad E_{31}E_{32}(E_{11} - E_{22}), \quad E_{12}E_{31}E_{32},\\
&E_{12}\left(E_{31}^2 - E_{32}^2\right), \quad E_{11}E_{32}^2 + E_{22}E_{31}^2, \quad E_{31}^2E_{32}^2,\\
&E_{31}E_{32}\left(E_{31}^2 - E_{32}^2\right).
\end{aligned} \tag{5.11.4}$$

The Laue group TI contains four point groups $\left(422,\ 4mm,\ \bar{4}2m,\ \frac{4}{m}\frac{2}{m}\frac{2}{m}\right)$. The corresponding strain energy can be expressed as a polynomial in the eight polynomial strain invariants,

$$\begin{aligned} &E_{11}+E_{22}, \quad E_{33}, \quad E_{31}^2+E_{32}^2, \quad E_{12}^2, \quad E_{11}E_{22}, \\ &E_{12}E_{31}E_{32}, \quad E_{11}E_{32}^2+E_{22}E_{31}^2, \quad E_{31}^2E_{32}^2. \end{aligned} \tag{5.11.5}$$

The axes of the reference coordinate system are parallel to $\mathbf{a}_i$.

v) Cubic System (Laue groups CII and CI). The vectors $\mathbf{a}_i$ are mutually perpendicular. The Laue group CII contains two point groups $\left(23,\ \frac{2}{m}\bar{3}\right)$. The corresponding strain energy is a polynomial in the fourteen polynomial strain invariants. They are listed by Green and Adkins (1960), Eq. (1.11.2). The Laue group CI contains three point groups $\left(432,\ \bar{4}3m,\ \frac{4}{m}\bar{3}\frac{2}{m}\right)$. The corresponding strain energy is a polynomial in the nine polynomial strain invariants, which are listed in *op. cit.*, Eq. (1.11.4).

vi) Rhombohedral System (Laue groups RII and RI). The vector $\mathbf{a}_3$ is perpendicular to the basal plane defined by vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, where $\mathbf{a}_2$ is at 120° from $\mathbf{a}_1$. The Laue group RII contains two point groups $(3,\ \bar{3})$. The corresponding strain energy is a polynomial in the fourteen polynomial strain invariants. They are listed in *op. cit.*, Eq. (1.12.5). The Laue group RI contains three point groups $\left(32,\ 3m,\ \bar{3}\frac{2}{m}\right)$. The corresponding strain energy is a polynomial in the nine polynomial strain invariants, listed by Green and Adkins (1960) in Eq. (1.12.8) (rhombohedral system is there considered to be hexagonal).

vii) Hexagonal System (Laue groups HII and HI). The vector $\mathbf{a}_3$ is perpendicular to the basal plane defined by vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, where $\mathbf{a}_2$ is at 120° from $\mathbf{a}_1$. The Laue group HII contains three point groups $\left(6,\ \bar{6},\ \frac{6}{m}\right)$. The corresponding strain energy is a polynomial in the fourteen polynomial strain invariants; Eq. (1.12.11) of *op. cit.* The Laue group HI contains four point groups $\left(622,\ 6mm,\ \bar{6}m2,\ \frac{6}{m}\frac{2}{m}\frac{2}{m}\right)$. The corresponding strain energy is a polynomial in the nine polynomial strain invariants. These are given by Eq. (1.12.13) of *op. cit.*

In the remaining two subsections we consider the general strain energy representation, with a particular attention given to cubic crystals and their elastic constants.

5.11.2. Strain Energy Representation

For each Laue group, the strain energy can be expanded in a Taylor series about the state of zero strain and stress as

$$\Psi = \frac{1}{2!}\, C_{ijkl}E_{ij}E_{kl} + \frac{1}{3!}\, C_{ijklmn}E_{ij}E_{kl}E_{mn} + \cdots . \tag{5.11.6}$$

The E_{ij} are the rectangular Cartesian components of the strain tensor $\mathbf{E}_{(n)}$, and $C_{ijklmn\ldots}$ are the corresponding elastic stiffness constants or elastic moduli. For simplicity, we omit the label (n). The components of the conjugate stress are

$$T_{ij} = \frac{\partial \Psi}{\partial E_{ij}} = C_{ijkl}E_{kl} + \frac{1}{2}\, C_{ijklmn}E_{kl}E_{mn} + \cdots . \tag{5.11.7}$$

Elastic constants of the k^{th} order are the components of the tensor of the order $2k$. Since they are the strain gradients of Ψ evaluated at zero strain,

$$C_{ijkl} = \left(\frac{\partial^2 \Psi}{\partial E_{ij}\partial E_{kl}}\right)_0, \quad C_{ijklmn} = \left(\frac{\partial^3 \Psi}{\partial E_{ij}\partial E_{kl}\partial E_{mn}}\right)_0, \ldots, \tag{5.11.8}$$

they possess the obvious basic symmetries. For example, the third-order elastic constants satisfy

$$C_{ijklmn} = C_{jiklmn}, \quad C_{ijklmn} = C_{klijmn} = C_{mnklij}. \tag{5.11.9}$$

Following the Voigt notation

$$11 \sim 1, \quad 22 \sim 2, \quad 33 \sim 3, \quad 23 \sim 4, \quad 13 \sim 5, \quad 12 \sim 6, \tag{5.11.10}$$

and the recipe

$$E_{ij} = \frac{1}{2}(1 + \delta_{ij})\eta_\vartheta, \quad \vartheta = 1, 2, ..., 6, \tag{5.11.11}$$

Equation (5.11.6) can be rewritten as (Brugger, 1964)

$$\begin{aligned}\Psi = \frac{1}{2}\sum_i c_{ii}\eta_i^2 + \sum_{i<j} c_{ij}\eta_i\eta_j + \frac{1}{6}\sum_i c_{iii}\eta_i^3 \\ + \frac{1}{2}\sum_{i\neq j} c_{iij}\eta_i^2\eta_j + \sum_{i<j<k} c_{ijk}\eta_i\eta_j\eta_k + \cdots .\end{aligned} \tag{5.11.12}$$

For triclinic crystals, whose symmetry group consists solely of the identity transformation, there are $\binom{5+k}{k}$ independent k^{th} order elastic constants (Toupin and Bernstein, 1961), i.e., there are at most 21 independent second-order elastic constants c_{ij}, and at most 56 independent third-order elastic constants c_{ijk}. For other crystal systems, fewer independent constants are involved, since they must be invariant under the group of transformations

defining the material symmetry. This requires certain constants to vanish and supplies relations among some of the remaining ones. The tables for the second- and third-order independent elastic constants in crystals for all crystallographic groups can be found in Brugger (1965) and Thurston (1984). An analysis of eigenvalues and eigentensors of the elastic constants C_{ijkl} of an anisotropic material is given by Ting (1987), Mehrabadi and Cowin (1990), and Sutcliffe (1992).

5.11.3. Elastic Constants of Cubic Crystals

For cubic crystals belonging to the Laue group CI, there are at most three independent second-order and six independent third-order elastic constants. Written with respect to principal cubic axes, the strain energy can be expressed as (Birch, 1947)

$$\begin{aligned}\Psi = {} & \frac{1}{2}\,c_{11}\left(\eta_1^2+\eta_2^2+\eta_3^2\right)+\frac{1}{2}\,c_{44}\left(\eta_4^2+\eta_5^2+\eta_6^2\right)\\ & + c_{12}\left(\eta_1\eta_2+\eta_2\eta_3+\eta_3\eta_1\right)+\frac{1}{6}\,c_{111}\left(\eta_1^3+\eta_2^3+\eta_3^3\right)\\ & + \frac{1}{2}\,c_{112}\left[\eta_1^2\left(\eta_2+\eta_3\right)+\eta_2^2(\eta_3+\eta_1)+\eta_3^2\left(\eta_1+\eta_2\right)\right]\\ & + \frac{1}{2}\,c_{144}\left(\eta_4^2\eta_1+\eta_5^2\eta_2+\eta_6^2\eta_3\right)+\frac{1}{2}\,c_{244}\left[\eta_4^2\left(\eta_2+\eta_3\right)\right.\\ & \left. + \eta_5^2\left(\eta_3+\eta_1\right)+\eta_6^2\left(\eta_1+\eta_2\right)\right]+c_{123}\,\eta_1\eta_2\eta_3+c_{456}\,\eta_4\eta_5\eta_6,\end{aligned} \tag{5.11.13}$$

to third-order terms in strain. The corresponding components of the fourth-order tensor of the second-order elastic moduli, written with respect to an arbitrary rectangular Cartesian basis, are

$$C_{ijkl} = c_{12}\delta_{ij}\delta_{kl}+2c_{44}I_{ijkl}+\left(c_{11}-c_{12}-2c_{44}\right)A_{ijkl}. \tag{5.11.14}$$

The components of the symmetric fourth-order unit tensor are again denoted by I_{ijkl}, and

$$A_{ijkl} = a_ia_ja_ka_l+b_ib_jb_kb_l+c_ic_jc_kc_l. \tag{5.11.15}$$

The vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are the orthogonal unit vectors along the principal cubic axes (previously denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$).

Two independent linear invariants of the elastic moduli tensor C_{ijkl} are

$$C_{iijj} = 3\left(c_{11}+2c_{12}\right), \quad C_{ijij} = 3\left(c_{11}+2c_{44}\right). \tag{5.11.16}$$

In the case when $c_{11}-c_{12}=2c_{44}$, the components C_{ijkl} are the components of an isotropic fourth-order tensor,

$$C_{ijkl} = c_{12}\delta_{ij}\delta_{kl} + 2c_{44}I_{ijkl}. \tag{5.11.17}$$

If the Cauchy symmetry

$$C_{ijkl} = C_{ikjl} \tag{5.11.18}$$

applies, then $c_{12} = c_{44}$. For example, in atomistic calculations the Cauchy symmetry is an inevitable consequence whenever the atomic interactions are modeled by pairwise central forces.

The sixth-order tensor of the third-order elastic moduli has the Cartesian components

$$\begin{aligned} C_{ijklmn} &= c_1\delta_{ij}\delta_{kl}\delta_{mn} + c_2\delta_{(ij}I_{klmn)} + c_3\delta_{(ik}\delta_{lm}\delta_{nj)} \\ &\quad + c_4\delta_{(ij}A_{klmn)} + c_5a_{(i}a_jb_kb_lc_mc_{n)} + c_6a_{(i}b_jc_ka_lb_mc_{n)}. \end{aligned} \tag{5.11.19}$$

The following constants are conveniently introduced

$$\begin{aligned} c_1 &= -\frac{1}{2}\left(c_{111} - 3c_{112} + 4c_{144} - 4c_{244}\right), \\ c_2 &= 6c_{144}, \quad c_3 = 4\left(c_{244} - c_{144}\right), \\ c_4 &= -\frac{3}{2}\left(c_{112} - c_{111} + 4c_{244}\right), \quad c_5 = 6\left(c_{123} - c_1\right), \\ c_6 &= 24\left(c_{144} - c_{244} + 2c_{456}\right). \end{aligned} \tag{5.11.20}$$

The notation such as $\delta_{(ij}A_{klmn)}$ designates the symmetrization. For example, we have

$$\begin{aligned} \delta_{(ij}I_{klmn)} &= \frac{1}{3}\left(\delta_{ij}I_{klmn} + \delta_{kl}I_{mnij} + \delta_{mn}I_{ijkl}\right), \\ \delta_{(ik}\delta_{lm}\delta_{nj)} &= \frac{1}{4}\left(\delta_{ik}I_{jlmn} + \delta_{il}I_{jkmn} + \delta_{im}I_{klnj} + \delta_{in}I_{klmj}\right). \end{aligned} \tag{5.11.21}$$

The tensors $\delta_{ij}\delta_{kl}\delta_{mn}$, $\delta_{(ij}I_{klmn)}$, and $\delta_{(ik}\delta_{lm}\delta_{nj)}$ constitute an integrity basis for the sixth-order isotropic tensors (Spencer, 1982). The tensors appearing on the right-hand side of Eq. (5.11.19) are the base tensors for the sixth-order elastic stiffness tensor with cubic symmetry. Other base tensors could also be constructed. The tensor representations of the second- and third-order elastic compliances are given by Lubarda (1997,1999).

Three independent linear invariants of the sixth-order tensor in Eq. (5.11.19) are

$$\begin{aligned} C_{iijjkk} &= 3\left(c_{111} + 6c_{112} + 2c_{123}\right), \\ C_{iiklkl} &= 3\left(c_{111} + 2c_{112} + 2c_{144} + 4c_{244}\right), \\ C_{ijjkki} &= 3\left(c_{111} + 6c_{244} + 2c_{456}\right). \end{aligned} \tag{5.11.22}$$

For isotropic materials

$$\begin{aligned} c_{111} &= c_{123} + 6c_{144} + 8c_{456}, \\ c_{112} &= c_{123} + 2c_{144}, \\ c_{244} &= c_{144} + 2c_{456}, \end{aligned} \tag{5.11.23}$$

so that C_{ijklmn} becomes an isotropic sixth-order tensor

$$C_{ijklmn} = c_{123}\delta_{ij}\delta_{kl}\delta_{mn} + 6c_{144}\delta_{(ij}I_{klmn)} + 8c_{456}\delta_{(ik}\delta_{lm}\delta_{nj)}. \tag{5.11.24}$$

If the Milder symmetry

$$C_{ijklmn} = C_{ikjlmn} \tag{5.11.25}$$

applies, then $c_{123} = c_{144} = c_{456}$.

The three independent third-order elastic constants of an isotropic material (c_{123}, c_{144}, and c_{456}) are related to Murnaghan's constants l, m, and n, which appear in the strain energy representation (5.3.8), by

$$l = c_{144} + \frac{1}{2}\,c_{123}, \quad m = c_{144} + 2c_{456}, \quad n = 4c_{456}. \tag{5.11.26}$$

Toupin and Bernstein (1961) used the notation $\nu_1 = c_{123}$, $\nu_2 = c_{144}$, and $\nu_3 = c_{456}$, referring to them as the third-order Lamé constants.

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