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Simultaneous Monotone Multiapproximation

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Abstract: Let f be a two variable continuously differentiable real-valued function of certain order on $L_p[-1,1]^2$ and let L be a linear differential operator involving mixed partial derivatives and suppose that $L(f) \geq 0$. Then there exists a sequence of two dimensional polynomials $Q_{m,n}(x,y)$ with $L(Q_{m,n}(x,y) \geq 0$, so that $f$is approximated simultaneously and in L_p by $Q_{m,n}$. This approximation is accomplished quantitatively the use of a suitable two dimensional first modulus of continuity.

1. Introduction

An essential topic of approximation theory is of monotone approximation, initiated by O.Shisha in 1965 (see [14]) .There the problem was: given a positive integer *r*, approximate with rates a given function whose *rth* derivative is ≥ 0 by polynomials having the same property. This initial problem was generalized by G.A. Anastassiou and O.Shisha in 1985 (see [2]) by replacing the *rth* derivative with a linear differential operator of order *r*. The rate of the related L_p convergence was given through the first modulus of continuity. During the last twenty –five years there was has been extensive research on monotone polynomial approximation, in particular, improving Shisha's initial result e.g .J.A. Roulier [13].Especially G.G. Lorentz and K. Zeller [9], G.G.Lorentz [8], and then R. DeVore [4] have obtained Jackson type estimates on the rate of L_p approximation of monotone functions by monotone polynomials. Furthermore E. Passow, L.Raymon, and J.A. Roulier [11, 12] have studied deeply the comonotone polynomial approximation of comonotone functions and D. J. Newman [10] was able to produce a Jackson type estimate related to comonotone approximation. More recently R. DeVore and X. Yu [5] have given a constructive proof of Timan –Teljackovski type pointwise estimates for monotone polynomial approximation involving the second modulus of smoothness ω_2 . Also D. Leviatan [6] presented pointwise estimates involving ω_2 and providing convex polynomial approximation, as well as simultaneous monotone and convex polynomial approximation. In addition, using a suitable Peetre functional, D. Leviatan [7] obtained estimates with respect to ω_2 of the Jackson type on the rate of the monotone polynomial approximation. Then he applied these results to get estimates on the degree of comonotone polynomial approximation. In this paper we deal with the following general two-dimensional problem (Theorem 3.2): let f be a two variable continuously differentiable real-valued function of given order and let L be a linear differential operator involving mixed partial derivative and suppose that $L(f) \geq 0$ .Then find a sequence of bivariate polynomials $Q_{m,n}(x,y)$ with the property $L(Q_{m,n}) \geq 0$ so that f is approximated simultaneously in $Q_{m,n}$ in the $L_p - quasi\ norm$ .This approximation is given with rates through inequalities involving the bivariate first modulus of continuity.

We would like to mention

$L_p - quasi\ normed\ space\ defined\ by$:

$$L_p[-1,1]^2 = \{f: f: [-1,1] \times [-1,1] \to R\, , \|f\|_p < \infty\}$$

$$such\ that \quad \|f\|_p = \left(\int_{-1}^{1}\int_{-1}^{1}|f(x_1,x_2)|^p\, dx_1 dx_2\right)^{\frac{1}{p}} \qquad 0 < p < 1$$

Definition 1.1: *Let* $f \in L_p[-1,1]^2, [-1,1]^2 = [-1,1] \times [-1,1]$. *The kth modulus of smoothness of* f *is defined as follows*:

$$\omega_k(f,\delta)_p := \omega_k(f,\delta,I^2)_p = sup_{|h_1|\leq\delta_1, |h_2|\leq\delta_2} \left\|\Delta_h^k(f,x)\right\|_p$$

Where $\Delta_h^k(f,x) := \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right)$, if $\left|x \mp \frac{kh}{2}\right| < 1$

$and \quad \delta = (\delta_1, \delta_2)\, , \quad \delta_1, \delta_2 > 0$. *And The Ditzian –Totik kth modulus of smoothness is*

$$\omega_k^{\varphi}(f,\delta)_p := \omega_k^{\varphi}(f,\delta,I^2)_p = sup_{|h_{1\varphi}|\le\delta_1,|h_{2\varphi}|\le\delta_2}\left\|\Delta_{h\varphi}^k(f,x)\right\|_p$$

Where $\varphi(x) = \sqrt{1-x^2}$.

The first modulus of continuity of f is defined as follows :

$$\omega_1(f,\delta)_p \coloneqq \omega_1(f,\delta,I^2)_p = sup_{|h_1|\le\delta_1,|h_2|\le\delta_2}\left\|f\left((x_1,x_2)+\left(\frac{h_1}{2},\frac{h_2}{2}\right)\right) - f\left((x_1,x_2)-\left(\frac{h_1}{2},\frac{h_2}{2}\right)\right)\right\|_p$$

$where\ \ \delta = (\delta_1,\delta_2)\ ,\ \ \delta_1,\ \delta_2 > 0$. *and*

The first modulus of continuity of f when $p = \infty$ *is defined as follows*:

$$\omega_1(f,\delta)_\infty \coloneqq \omega_1(f,\delta,I^2)_\infty = sup_{|h_1|\le\delta_1,|h_2|\le\delta_2}\left\|f\left((x_1,x_2)+\left(\frac{h_1}{2},\frac{h_2}{2}\right)\right) - f\left((x_1,x_2)-\left(\frac{h_1}{2},\frac{h_2}{2}\right)\right)\right\|_\infty$$

Definition 1.2

Let f be a real –valued function defined on $L_p[-1,1]^2$ *and m, n be two Positive integers .let* $B_{m,n}$ *be the Bernstein (polynomial) operator of order (m, n) given by*

$$B_{m,n}\,(f;x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} f\left(\frac{i}{m},\frac{j}{n}\right)\binom{m}{i}\binom{n}{j}\ x^i(1-x)^{m-i}\ y^j(1-y)^{n-j}$$

Definition 1.3

Let $f \in L_p[-1,1]^2$. *The local modulus of smoothness of f is defined as follows:*

$$\omega_1(f,x,\delta) = \sup_{|h_1|\le\delta_1\ ,|h_2|\le\delta_2}\{\left|f\left((y_1,y_2)+\left(\frac{h_1}{2},\frac{h_2}{2}\right)\right) - f\left((y_1,y_2) - f\left(\frac{h_1}{2},\frac{h_2}{2}\right)\right)\right|$$

$$y_1 \mp \frac{h_1}{2} \in \left[x_1-\frac{\delta_1}{2}, x_1+\frac{\delta_1}{2}\right]\ ,\ \ y_2 \mp \frac{h_2}{2} \in \left[x_2-\frac{\delta_2}{2}, x_2+\frac{\delta_2}{2}\right]\ \}$$

$where\ \ x = (x_1,x_2)$, $\delta = (\delta_1,\delta_2)$ *And* $\delta_1,\delta_2 > 0$

Note that $\omega_1(f,\delta)_\infty \equiv \omega_1(f,x,\delta,)$

Definition 1.4

let $f \in L_p[-1,1]^2$ *and* $\delta_1, \delta_2 > 0$ *. The* τ *−modulus of smoothness of f is defined as follows:* $\tau_1(f,\delta)_p := \tau_1(f,\delta,I^2)_p = \|\omega_1(f,,x,\delta)\|_p$

2. Auxiliary results

In this article we introduce the results that we need in our main theorem.

Lemma 2.1

let $\{\hat{z}_i := (z_i, z_i), -1 =: z_0 < z_1 < \cdots z_{n+1} := 1\}$ *be a partition of the square* I^2 *into n+1 subintervals .using the notation* $I_i^2 = (z_{i-1}, z_i)^2$, $\delta_i{}^2 = (z_{i+1} - z_{i-1})^2$, $i = 1,2, \dots n$

$$\sum_{i=1}^{n} \omega_1 (f, \hat{z}_i, \hat{h})^p \ \delta_i^2 \le C(p)\, \tau_1(f, \hat{h})_p$$

where $\hat{z}_i = (z_i, z_i)$, $\hat{h} = (h, h)$

Proof:

$$\sum_{i=1}^{n} \omega_1 (f, \hat{z}_i, \hat{h})^p \ \delta_i^2 \le C(p) \sum_{i=1}^{n} \iint_{I_i} \omega_1(f, z, \hat{h})^p \ dzdz$$

$$\sum_{i=1}^{n} \omega_1 (f, \hat{z}_i, \hat{h})^p \ \delta_i^2 \ \le C(p) \int_{-1}^{1} \int_{-1}^{1} \omega_1(f, z, \hat{h})^p \, dzdz$$

Where $z = (z_1, z_2)$

$$\sum_{i=1}^{n} \omega_1 (f, \hat{z}_i, \hat{h})^p \ \delta_i^2 \ \le \ C\, \tau_1(f, \hat{h})_p \qquad \blacksquare$$

We prove our main result with the help of a bivariate simultaneous approximation theorem

Lemma 1.2.2 [3] *it holds* that

$$\left\|f^{(k,l)} - (B_{m,n}f)^{(k,l)}\right\|_\infty \le t(k,l).\, \omega_1\left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}}\right)_\infty$$

$$+\ max\ \{\frac{k(k-1)}{m},\frac{l(l-1)}{n}\}.\left\|f^{(k,l)}\right\|_\infty$$

Where $m > k \geq 0,\ n > l \geq 0$ *are integers, f is a real –valued function on* $[0,1]^2$ *such that* $f^{(k,l)}$ *is continuous, and t is a positive real-valued function on* N^2*, N= {0, 1, 2…}, Here* $\|.\|_\infty$ *is the supremum norm*

i.e.$\|f\|_\infty = sup_{x_1,x_2\in[-1,1]}\,|f(x_1,x_2)|$

3 .The main results

In this section we introduce our main theorem.

Theorem 3.1

$$\left\|f^{(k,l)}-(B_{m,n}f)^{(k,l)}\right\|_p \leq C(k,l,p).\,\omega_1\left(f^{(k,l)};\left(\frac{1}{\sqrt{m-k}},\frac{1}{\sqrt{n-l}}\right)\right)_p$$

$$+\max\ \{\frac{k(k-1)}{m},\frac{l(l-1)}{n}\}.\ \left\|f^{(k,l)}\right\|_p$$

Where $m > k \geq 0, n > l \geq 0$ *are integers, f is a real – valued function on* $[-1,1]^2$ *such that* $f^{(k,l)}$ *is continuous, and C is a positive constant depending on k , l and p, (0<p<1).*

Proof: Using lemma 1.2.2, we have

$$\left\|f^{(k,l)}-(B_{m,n}f)^{(k,l)}\right\|_p \leq C\,(k,l,p)\left\|\left\|f^{(k,l)}-(B_{m,n}f)^{(k,l)}\right\|_\infty\right\|_p \leq$$

$$C(k,l,p)\ \left\|\omega_1\left(f^{(k,l)};\left(\frac{1}{\sqrt{m-k}},\frac{1}{\sqrt{n-l}}\right)\right)_\infty + \max\left\{\frac{k(k-1)}{m},\frac{l(l-1)}{n}\right\}\left\|f^{(k,l)}\right\|_\infty\right\|_p$$

Let

$$\left\|f^{(k,l)}-(B_{m,n}f)^{(k,l)}\right\|_p \leq C(k,l,p)\,\Psi + C(k,l,p)\max\left\{\frac{k(k-1)}{m},\frac{l(l-1)}{n}\right\}.\left\|f^{(k,l)}\right\|_p$$

Where $\Psi = \left\|\omega_1\left(f^{(k,l)};\left(\frac{1}{\sqrt{m-k}},\frac{1}{\sqrt{n-l}}\right)\right)_\infty\right\|_p$

By definition of $\omega_1(f,\delta)_\infty$ and definition of $\omega_1(f,x,\delta)$ we get

$$\Psi = \left(\int_{-1}^{1}\int_{-1}^{1}\left|\omega_1\left(f^{(k,l)};\left(\frac{1}{\sqrt{m-k}},\frac{1}{\sqrt{n-l}}\right)\right)_\infty\right|^p dxdx\right)^{\frac{1}{p}}$$

$$\Psi = \left(\sum_{i=1}^{n}\iint_{I_i}\left|\omega_1\left(f^{(k,l)};\left(\frac{1}{\sqrt{m-k}},\frac{1}{\sqrt{n-l}}\right),I_i^2\right)_\infty\right|^p dxdx\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\sum_{i=1}^{n}\iint_{I_i}\left|\omega_1\left(f^{(k,l)},\hat{x}_i,\hat{h}_i\right)\right|^p dxdx\right)^{\frac{1}{p}}$$

where $\hat{x}_i = (x_i, x_i)$, $\hat{h}_i = (h_i, h_i)$

Then using lemma 1.2.1 we get

$$\Psi \leq \left(\sum_{i=1}^{n} h_i^2\left|\omega_1\left(f^{(k,l)},\hat{x}_i,\hat{h}_i\right)\right|^p dxdx\right)^{\frac{1}{p}}$$

$$\leq C(P)\,\tau_1\left(f^{(k,l)};\left(\frac{1}{\sqrt{m-k}},\frac{1}{\sqrt{n-l}}\right)\right)_p \qquad \blacksquare$$

Theorem 3.2

Let $h_1, h_2\ v_1, v_2, r, p$ *be integers* $r \geq v_1 \geq h_1 \geq 0, p \geq v_2 \geq h_2 \geq 0$ *and let* $f \in L_p[-1,1]^2$. *let* $\alpha_{ij}(x,y)$, $i = h_1, h_1+1, \dots v_1, j = h_2, h_2+1, \dots\ v_2$ *be real- valued functions defined and bounded in* $L_p[-1,1]^2$ *and assume* $\alpha_{h_1h_2}$ *is either* $\geq \alpha > 0$ *or* $\leq \beta < 0$. *Throughout* $L_p[-1,1]^2$. *Consider the operator*

$$L = \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{ij}(x,y)\partial^{i+j}/\partial x^i \partial y^j$$

And assume that throughout $L_p[-1,1]^2$, $\quad L(f) \geq 0 \qquad$ (3.2.1)

Then for integers m, n *with* $m > r, n > p$ *there exists a polynomial* $Q_{m,n}(x,y)$ *of degree (m, n) such that* $L\left(Q_{m,n}(x,y)\right) \geq 0$ *throughout* $L_p[-1,1]^2$ *and*

$$\left\|f^{(k,l)} - Q_{m,n}^{(k,l)}\right\|_p \leq \frac{P_{m,n}(L,f)}{(h_1-k)!(h_2-l)!} + M_{m,n}^{(k,l)}(f) \qquad (3.2.2)$$

all $(0,0) \leq (k,l) \leq (h_1, h_2)$. Furthermore we get

$$\left\| f^{(k,l)} - (B_{m,n}f)^{(k,l)} \right\|_p \leq M_{m,n}^{(k,l)}(f) \qquad (3.2.3)$$

$for\ all\ (h_1+1, h_2+1) \leq (k,l) \leq (r,p).\ also$ (3.2.3) $is\ \ true\ whenever$

$0 \leq k \leq h_1\ and\ \ h_2+1 \leq l \leq p\ \ or\ \ h_1+1 \leq k \leq r\ \ and\ \ 0 \leq l \leq h_2.$ *Here*

$$M_{m,n}^{(k,l)} \equiv M_{m,n}^{(k,l)}(f) \equiv C(k,l,p).\,\omega_1\left(f^{(k,l)};\left(\frac{1}{\sqrt{m-k}},\frac{1}{\sqrt{n-l}}\right)\right)_p$$

$$+\max\{\frac{k(k-1)}{m},\frac{l(l-1)}{n}\}\ .\ \left\|f^{(k,l)}\right\|_p$$

$$P_{m,n} \equiv P_{m,n}(L,f) \equiv \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} L_{ij}\ .\ M_{m,n}^{(i,j)}$$

Where C is a positive constant depending on k , l *and* p, $0 < p < 1$.

$$L_{ij} \equiv \left(\int_{-1}^{1}\int_{-1}^{1}\left|\alpha^{-1}{}_{h_1h_2}.\,\alpha_{ij}(x,y)\right|^p dxdy\right)^{\frac{1}{p}} < \infty$$

Proof: let m,n be integers such that $m > r$, $n > p$

Case (i): Assume that throughout $[-1,1]^2$, $\alpha_{h_1h_2} \geq \alpha > 0$

From Theorem 1.3.1 we have

$$\left\|(f + P_{m,n}.\frac{x^{h_1}}{h_1!}.\frac{y^{h_2}}{h_2!})^{(k,l)} - (Q_{m,n}(x,y))^{(k,l)}\right\|_p \leq M_{m,n}^{(k,l)} \qquad (3.2.4)$$

all $0 \leq k \leq r$, $0 \leq l \leq p$, where

$$Q_{m,n}(x,y) \equiv B_{m,n}(f;x,y) + P_{m,n}.\frac{x^{h_1}}{h_1!}.\frac{y^{h_2}}{h_2!}$$

when $(0,0) \leq (k,l) \leq (h_1,h_2)$, (3.2.4) becomes

$$\left\|f^{(k,l)}(x,y) + P_{m,n}.\frac{x^{h_1-k}}{(h_1-k)!}.\frac{y^{h_2-l}}{(h_2-l)!} - Q_{m,n}^{(k,l)}(x,y)\right\|_p \leq M_{m,n}^{(k,l)}$$

Hence by the triangle inequality property of $\|.\|_p$ and $(x,y) \in [-1,1]^2$

We have the validity (3.2.2). Furthermore, if $(x,y) \in [-1,1]^2$, then

$$\alpha^{-1}{}_{h_1h_2}(x,y).\,L\left(Q_{m,n}(x,y)\right) \geq P_{m,n} - \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} L_{ij}\ .\ M_{m,n}^{(i,j)} = 0 \qquad [1]$$

The last is true inequality (3.2.4). Therefore $L\left(Q_{m,n}(x,y)\right) \geq 0$.

Case (ii): Assume that throughout $[-1,1]^2$, $\alpha_{h_1h_2} \leq \beta < 0$.

From Theorem 1.3.1 we have

$$\left\|(f - P_{m,n}.\frac{x^{h_1}}{h_1!}.\frac{y^{h_2}}{h_2!})^{(k,l)} - (Q_{m,n}(x,y))^{(k,l)}\right\|_p \leq M_{m,n}^{(k,l)} \qquad (3.2.5)$$

$all\ 0 \leq k \leq r, 0 \leq l \leq p,$ Where

$$Q_{m,n}(x,y) \equiv B_{m,n}(f;x,y) - P_{m,n}.\frac{x^{h_1}}{h_1!}.\frac{y^{h_2}}{h_2!}$$

When $(0,0) \leq (k,l) \leq (h_1,h_2)$, (3.2.5) becomes

$$\left\|f^{(k,l)}(x,y) - P_{m,n}.\frac{x^{h_1-k}}{(h_1-k)!}.\frac{y^{h_2-l}}{(h_2-l)!} - Q_{m,n}^{(k,l)}(x,y)\right\|_p \leq M_{m,n}^{(k,l)}$$

Hence by the triangle inequality property of $\|.\|_p$ and $(x,y) \in [-1,1]^2$

we have the validity (3.2.2). Furthermore, if $(x,y) \in [-1,1]^2$, then

$\alpha^{-1}{}_{h_1h_2}(x,y).L\left(Q_{m,n}(x,y)\right) \leq -P_{m,n} + \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} L_{ij}\,.\ M_{m,n}^{(i,j)} = 0$ [1]

The last is true inequality (3.2.5). Therefore again $L\left(Q_{m,n}(x,y)\right) \geq 0$.

In the cases of either

$(h_1+1, h_2+1) \leq (k,l) \leq (r,p)$ or $(0 \leq k \leq h_1, h_2+1 \leq l \leq p)$ *or*

$(h_1+1 \leq k \leq r, 0 \leq l \leq h_2)$,

We have

$$\left(f \pm P_{m,n}.\frac{x^{h_1}}{h_1!}.\frac{y^{h_2}}{h_2!}\right)^{(k,l)} = f^{(k,l)} . \qquad \blacksquare$$

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