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ESTIMATION OF PARTIAL FREQUENCY RESPONSE FUNCTIONS OF A PARAMETRICALLY EXCITED FLEXIBLE MULTIBODY SYSTEM

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ABSTRACT

Flexible multibody system simulation allows for fast and adequate investigation of the dynamics of mechanical systems. But in case of a system response with large deformations the time response does not uncover the causes, i. e. the resonances of the system. The identification of the systems eigenfrequencies gives more insight in resonance phenomena, but in case of periodic time-variant systems the often used snap-shot-eigenfrequencies do not reveal the real system dynamics, which has to be described by more than only one frequency response function. Based on the formulation of a flexible multibody system and the theory of ordinary linear periodic differential equations, partial frequency response functions, describing the real characteristics of a periodic system, are calculated and compared to the snapshot-frequency response functions.

NOMENCLATURE

$\mathbf{A}$ orientation matrix (sec. 2), system matrix (sec. 3)
E youngs modulus
$\mathbf{H}$ HOOKE matrix (sec. 2), frf matrix (sec. 3-5)
$\mathbf{J}$ JACOBIan matrix
$\mathbf{L}$ differential operator
$\mathbf{M}, \mathbf{D}, \mathbf{K}$ mass, damping, stiffness matrix
$\mathbf{N}$ matrix of interpolation functions
$\mathbf{R}$ vector of material coordinates
$\mathbf{a}$ acceleration vector
$\mathbf{f}$ force vector
$\mathbf{g}$ gravitational field intensity vector
$\mathbf{h}$ external force vector
$\mathbf{k}$ vector of gyroscopic terms
$\mathbf{p}$ vector of modal coordinates
$\mathbf{q}$ vector of nodal coordinates
$\mathbf{r}$ position vector
$\mathbf{u}$ displacement vector
$\mathbf{v}$ translational velocity vector
$\mathbf{x}$ state space vector
$\mathbf{y}$ diplacement variable
Ψ modal matrix
Θ matrix of eigenvectors
$\boldsymbol{\epsilon}, \boldsymbol{\sigma}$ stress, strain
$\boldsymbol{\lambda}$ diagonal matrix of eigenfrequencies
μ squared ratio of lowest weak and stiff eigenfrequency
ν POISSON's ratio
ρ density
$\boldsymbol{\omega}$ angular velocity vector

1 INTRODUCTION

A glance at various fields of technology where Mechanical Engineering plays a central role such as aerospace (air- and spacecraft), transportation (road and rail vehicles), automation (robotics and machine tools) reveals very similar present tendencies: In order to save weight and energy drastic reduction of material is a major request in all fields, lightweight constructions have highest priority in all major design groups. As a consequence, the elastic deformations and structural vibrations become of focal interest, even in situations where formerly a rigid body modelling was more than appropriate.

Two major options are available to analyse flexible mechanical systems: The finite element method (FEM) and the multibody system (MBS) approach [1]. A simulation based on finite element models is – despite of the labour for setting up the model data – straightforward, the corresponding codes are well developed, and include linear and nonlinear theory of elasticity. But there is a disadvantage: a dynamic analysis with FEM-codes is very time consuming. In many applications one is confronted with system models, in which the deformations of the flexible bodies are small but superimposed on a large reference- or "rigid-body-motion". In multibody system simulation one exploits this fact to reduce the computational burden for such applications by linearizing the equations of motion assuming small deflections. Using relative variables to represent the reference motion and applying $O(N)$-formalisms [2], MBS-codes provide an efficient alternative for system analysis via simulation [3]. Following such arguments, flexible multibody formalisms have been used frequently [4, 5].

Geometric stiffening is a well known effect in dynamics and its importance for multibody system simulation has been pointed out clearly, in [6]. The modelling of flexible bodies in the MBS-program SIMPACK [7] has been described in various places [8, 9] including the problem of how to compute geometric stiffening terms for beams and arbitrary structures. Recent investigations [10] demonstrate of how to increase computational efficiency by neglecting terms not required to represent the body deformations with the accuracy guaranteed by a linear approximation.

For technical applications the knowledge of the dynamic behaviour in terms of resonances or instabilities is important to ensure proper operation conditions of the above mentioned systems. The extraction of this information from the MBS simulation is not straight forward in case of parametrically excited systems, e.g. flexible rotor systems, and needs consideration in more detail [11].

2 BASIC EQUATIONS FOR AN ELASTIC MULTIBODY SYSTEM

The starting point for the development of the $O(N)$-formalism in SIMPACK is a set of equations of motion of the flexible bodies. The flexible body data required to generate the MBS-equations are those, which are needed to compute the elements of the mass matrix, the gyroscopic terms and the stiffness terms. To compute such data preprocessors have been developed. After reading the flexible body data and the remainder of the system data, SIMPACK can simulate the MBS-motion. The coordinate systems shown in Fig. 1 will be used to describe the body's motion. They are

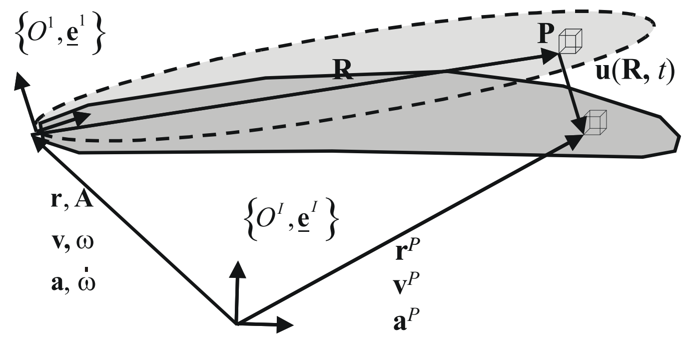

Figure 1. REPRESENTATION OF MOTION.

the inertial frame $\{O^I, \underline{\mathbf{e}}^I\}$ and the body reference frame $\{O^1, \underline{\mathbf{e}}^1\}$ located at one end of the body in the reference configuration. In general all vectors will be resolved in the reference frame $\{O^1, \underline{\mathbf{e}}^1\}$, e.g. $r = \underline{\mathbf{e}}^{1T}\mathbf{r}$. The motion of a representative material point P with respect to inertial space is described as:

$$\mathbf{r}^P(\mathbf{R}, t) = \mathbf{r}(t) + \mathbf{R} + \mathbf{u}(\mathbf{R}, t). \quad (1)$$

In this representation the body motion is given in terms of the reference or rigid body motion as described by the position vector $\mathbf{r}(t)$ and the orientation matrix $\mathbf{A}(t)$ and in terms of small displacements $\mathbf{u}$, i.e.

$$\mathbf{u} = \bar{\mathbf{u}}(\mathbf{R}^u, \mathbf{y}(\mathbf{R}^y, t)), \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}^u \\ \mathbf{R}^y \end{bmatrix}. \quad (2)$$

The displacements are expressed in terms of deformation variables $\mathbf{y}$ [12], e.g. for beams the motion of the axis. This representation of body motion is suitable for incorporating the assumption of small deformations. The coordinates of the absolute velocity and acceleration of mass element $\mathrm{d}m$ at point P with respect to reference frame are derived from Eqn. (1)

$$\mathbf{v}^P = \mathbf{v} + \tilde{\boldsymbol{\omega}}(\mathbf{R} + \mathbf{u}) + \dot{\mathbf{u}}, \quad \mathbf{v} = \dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}}\mathbf{r}, \quad (3)$$

$$\mathbf{a}^P = \mathbf{a} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}(\mathbf{R} + \mathbf{u}) + \dot{\tilde{\boldsymbol{\omega}}}(\mathbf{R} + \mathbf{u}) + 2\tilde{\boldsymbol{\omega}}\dot{\mathbf{u}} + \ddot{\mathbf{u}}, \quad (4)$$

$$\mathbf{a} = \dot{\mathbf{v}} + \tilde{\boldsymbol{\omega}}\mathbf{v},$$

where $\mathbf{v}$ and $\mathbf{a}$ are the reference point velocity and acceleration, $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are the angular velocity and acceleration of the reference frame respectively. The tilde operator represents the cross product of vectors. Differentiation of displacements $\mathbf{u}$ with respect to time yields

$$\dot{\mathbf{u}} = \mathbf{J}(\mathbf{y})\dot{\mathbf{y}}, \tag{5}$$

with the JACOBIan

$$\mathbf{J} = [J_{\alpha i}] = \left[\frac{\partial u_\alpha}{\partial y_i}\right] \tag{6}$$

containing the derivatives of $\mathbf{u}$ w.r.t. $\mathbf{y}$. Deformations are measured by the symmetric GREEN-LAGRANGE strain tensor. A common matrix form is

$$\boldsymbol{\epsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{23}, 2\varepsilon_{31}]^T = \left[\mathbf{L}^0 + \frac{1}{2}\mathbf{L}^1(\mathbf{u})\right]\mathbf{u}, \tag{7}$$

where operator matrix $\mathbf{L}^0$, which is well known from the linear theory of elasticity [13], contains partial derivatives with respect to the material coordinates $\partial_\alpha = \partial()/\partial R_\alpha$. Products of ∂_α and $u_{\alpha,\beta}$, so-called bilinear terms, are collected in $\mathbf{L}^1$. They result from the nonlinear terms of the strain-displacement relation, Eqn. (7). Stresses are represented here by the symmetric second PIOLA-KIRCHHOFF stress tensor. For a linear material law, the stresses are related to the strains by

$$\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^T = \mathbf{H}\boldsymbol{\epsilon}, \tag{8}$$

with the 6×6 HOOKEan matrix $\mathbf{H}$. Its elements are given in terms of two independent elasticity constants YOUNG's modulus E and POISSON's ratio ν of the isotropic material.

2.1 Approximation Methods for Elastic Deformations

A RITZ-approximation of the deformation variables $\mathbf{y}$

$$\mathbf{y}(\mathbf{R}^y, t) = \mathbf{N}(\mathbf{R}^y)\mathbf{q}(t) \tag{9}$$

is used with the n^q unknown nodal coordinates $\mathbf{q}(t)$ and finite elements as interpolation functions $\mathbf{N}$. Introducing Eqn. (9) into Hamilton's principle and using the fundamental theorem of variational calculus one finds the equations of motion

$$\mathbf{M}(\mathbf{q})\begin{bmatrix}\mathbf{a}\\ \dot{\boldsymbol{\omega}}\\ \ddot{\mathbf{q}}\end{bmatrix} + \mathbf{k}(\boldsymbol{\omega}, \mathbf{q}, \dot{\mathbf{q}}) + \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{K}(\mathbf{q})\mathbf{q}\end{bmatrix} - \mathbf{h}(\mathbf{r}, \mathbf{A}, \mathbf{q}, \dots) = \mathbf{0}. \tag{10}$$

Matrix $\mathbf{M}$ contains inertia terms, $\mathbf{k}$ gyroscopic terms, $\mathbf{K}$ stiffness terms and $\mathbf{h}$ the applied forces.

The number of coordinates is reduced by modal transformation. The eigenmodes are computed by FE-analysis of the flexible body in an inertial reference frame i. e.

$$\mathbf{a} \equiv \dot{\boldsymbol{\omega}} \equiv \mathbf{v} \equiv \boldsymbol{\omega} \equiv \mathbf{0}. \tag{11}$$

Then Eqn. (10) is linearized w.r.t. $\mathbf{q}$ for the undeformed configuration $\mathbf{q}^0 \equiv \mathbf{0}$ to obtain the linear homogenous equations of motion in terms of the node coordinates

$$\mathbf{M}^{e0}\ddot{\mathbf{q}} + \mathbf{K}^0\mathbf{q} = \mathbf{0}, \tag{12}$$

with the positive definite mass matrix

$$\mathbf{M}^{e0} = \iiint_V \mathbf{N}^T\mathbf{J}^{0T}\mathbf{J}^0\mathbf{N}\rho \mathrm{d}V, \quad \text{where } \mathbf{J}^0 = \mathbf{J}(\mathbf{q}^0), \tag{13}$$

and the positive semi definite stiffness matrix

$$\mathbf{K}^0 = \iiint_V \mathbf{N}^T\mathbf{J}^{0T}\mathbf{L}^{0T}\mathbf{H}\mathbf{L}^0\mathbf{J}^0\mathbf{N}\mathrm{d}V, \quad \text{where } \mathbf{L}^0 = \mathbf{L}(\mathbf{q}^0). \tag{14}$$

Solving Eqn. (12) one obtains n^q eigenfrequencies $\boldsymbol{\lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_{n^q})$ and the associated real eigenvectors $[\Theta_j]_i$. Now the node coordinates $\mathbf{q}$ are approximated by

$$\mathbf{q} = \Theta\mathbf{p}, \tag{15}$$

with n^p properly chosen modes

$$\Theta = [\Theta_{ij}], \quad i = 1, \dots, n^q, \quad j = 1, \dots, n^p, \tag{16}$$

and with the associated modal coordinates $\mathbf{p}$. To summarize: the deformation variables $\mathbf{y}$ are represented in view of Eqn. (9) and (15) as

$$\mathbf{y}(\mathbf{R}^y, t) = \Psi(\mathbf{R}^y)\mathbf{p}(t), \tag{17}$$

with

$$\Psi := \mathbf{N}(\mathbf{R}^y)\Theta. \tag{18}$$

Now the eigenvectors are normalized such that the modal mass matrix is the identity matrix

$$\bar{\mathbf{M}}^{e0} = \Theta^T \mathbf{M}^{e0} \Theta = \mathbf{1}. \tag{19}$$

The modal stiffness matrix then is a diagonal matrix whose nonzero elements are squares of the eigenfrequencies λ_i, i.e.

$$\bar{\mathbf{K}}^0 = \Theta^T \mathbf{K}^0 \Theta = \mathrm{diag}(\lambda_1^2, \dots, \lambda_{n^p}^2). \tag{20}$$

We arrange the elements of $\bar{\mathbf{K}}^0$ in ascending order. The modal-coordinates are partitioned in non stiff "weak" modes (index w) and in stiff modes (index s)

$$\mathbf{p} = \left[\mathbf{p}^{wT}, \mathbf{p}^{sT}\right]^T. \tag{21}$$

For linearization the equations of motion, Eqn. (10), are rewritten in dimensionless form. This allows to compare the order of magnitudes. A time scale is $1/\lambda_1^w$ with the lowest eigenfrequency λ_1^w and a distance scale is the characteristic length l. In the equations of motion one can identify the small parameter

$$\mu = \left(\frac{\lambda_1^w}{\lambda_1^s}\right)^2 \ll 1, \tag{22}$$

which is the squared ratio of the lowest eigenfrequency in non stiff direction λ_1^w and the lowest eigenfrequency in stiff direction λ_1^s. The stiffness matrix is now arranged as

$$\hat{\mathbf{K}} = \begin{bmatrix} \left\lceil \hat{\lambda}_i^w \right\rfloor^2 & \mathbf{O} \\ \mathbf{O} & \frac{1}{\mu}\left\lceil \lambda_i^{*s} \right\rfloor^2 \end{bmatrix} + \hat{\mathbf{K}}^N(\mathbf{p}, 1/\mu), \tag{23}$$

with the small parameter μ in the constant term. This is important during linearization while looking at the order of magnitude of the terms. Now we assume that the deformation coordinates $\mathbf{p}$ remain small: $p_k \ll 1$. Using a perturbation technique, one expands the deformation coordinates into a series with the small parameter μ

$$\mathbf{p} = \mathbf{p}_0 + \mu\mathbf{p}_1 + \mu^2\mathbf{p}_2 + \mu^3\mathbf{p}_3 + \dots, \quad \mathbf{p}_0 = \mathbf{0}. \tag{24}$$

By introducing Eqn. (24) in the dimensionless nonlinear equations of motion and expanding nonlinear functions into Taylor-series in terms of μ results in the so called reduced equations of motion. Comparison of terms of order $O(\mu^0)$ leads to an algebraic equation for the deformation coordinates in stiff directions (written here with dimensions)

$$\mathbf{p}^s = (\boldsymbol{\lambda}^s)^{-1}\left(\mathbf{h}^{s0} - \mathbf{M}^{ts}\mathbf{a} - \mathbf{M}^{rs}\dot{\boldsymbol{\omega}} - \mathbf{k}^{s0}(\boldsymbol{\omega})\right). \tag{25}$$

Herein the matrices contain constant volume integrals

$$\mathbf{M}^{ts} = \mathbf{C}1^s, \quad \mathbf{k}^s = \left[\boldsymbol{\omega}^T \mathbf{C}4_k^s \boldsymbol{\omega}\right], \tag{26}$$

$$\mathbf{M}^{rs} = \mathbf{C}2^s, \quad \mathbf{h}^s = \sum_{i=1}^{n^f} \left(\Psi^{sT}\mathbf{J}^{0T}\right)\Big|_{\mathbf{R}=\mathbf{R}_i} \mathbf{f}_i^e + \mathbf{C}1^{sT}\mathbf{g},$$

which are defined in (31). The rigid body motion and the deformation in non stiff directions results in an ordinary differential equation including terms of order $O(\mu^0)$ and $O(\mu^1)$

$$\begin{bmatrix} \mathbf{M}^{tt} & \mathbf{M}^{tr} & \mathbf{M}^{tw} \\ & \mathbf{M}^{rr} & \mathbf{M}^{rw} \\ \text{sym.} & & \mathbf{M}^{ww} \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \dot{\boldsymbol{\omega}} \\ \ddot{\mathbf{p}}^w \end{bmatrix} + \begin{bmatrix} \mathbf{k}^t \\ \mathbf{k}^r \\ \mathbf{k}^w \end{bmatrix} +$$

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\lambda}^w \mathbf{p}^w + \mathbf{K}^\sigma(\mathbf{a}, \dot{\boldsymbol{\omega}}, \boldsymbol{\omega}, \mathbf{h}^s)\mathbf{p}^w \end{bmatrix} - \begin{bmatrix} \mathbf{h}^t \\ \mathbf{h}^r \\ \mathbf{h}^w \end{bmatrix} = \mathbf{0}. \tag{27}$$

Equation (27) contains the so called geometric stiffness terms $\mathbf{K}^\sigma$ and the following submatrices

$$\begin{aligned} \mathbf{M}^{tt} &= m\mathbf{1}, \quad \mathbf{M}^{rr} = \mathbf{I} + \mathbf{C}4_k p_k - \mathbf{C}4_k^T p_k, \\ \mathbf{M}^{tr} &= \widetilde{\mathbf{C}0}, \quad \mathbf{M}^{rw} = \mathbf{C}2 + \mathbf{C}5_k p_k + \mathbf{C}8_k p_k, \\ \mathbf{M}^{tw} &= \mathbf{C}1 + \mathbf{C}7_k p_k \quad \mathbf{M}^{ww} = \mathbf{C}3, \end{aligned} \tag{28}$$

$$\begin{aligned} \mathbf{k}^t &= \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{C}0 + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{C}1\mathbf{p} + 2\tilde{\boldsymbol{\omega}}\mathbf{C}1\dot{\mathbf{p}}, \\ \mathbf{k}^r &= \tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega} - \tilde{\boldsymbol{\omega}}\mathbf{C}4_k\boldsymbol{\omega}p_k + \tilde{\boldsymbol{\omega}}\mathbf{C}4_k^T\boldsymbol{\omega}p_k + 2\mathbf{C}4_k\dot{p}_k\boldsymbol{\omega}, \\ \mathbf{k}^w &= \left[\boldsymbol{\omega}^T\mathbf{C}4_k\boldsymbol{\omega}\right] + \left[\boldsymbol{\omega}^T\mathbf{C}9_{lk}\boldsymbol{\omega}p_l\right] + \\ &\quad \left[\boldsymbol{\omega}^T\mathbf{C}6_{lk}\boldsymbol{\omega}p_l\right] + 2\mathbf{C}5_k^T\boldsymbol{\omega}\dot{p}_l, \end{aligned} \tag{29}$$

$$
\begin{aligned}
\mathbf{h}^t &= \sum_{i=1}^{n^f} \mathbf{f}_i^e + m\mathbf{g}, \\
\mathbf{h}^r &= \sum_{i=1}^{n^f} \tilde{\mathbf{R}}_i \mathbf{f}_i^e + \frac{\widetilde{\mathbf{C}0}}{m}\mathbf{g} + \\
&\qquad \sum_{i=1}^{n^f} \left(\mathbf{J}^0[\Psi_\alpha]_k\right)^\sim \Big|_{\mathbf{R}=\mathbf{R}_i} \mathbf{f}_i^e p_k + \widetilde{\mathbf{C}1}_k \mathbf{g} p_k, \qquad (30) \\
\mathbf{h}^w &= \sum_{i=1}^{n^f} \left(\mathbf{J}^0\Psi\right)^T \Big|_{\mathbf{R}=\mathbf{R}_i} \mathbf{f}_i^e + \mathbf{C}1^T\mathbf{g} + \\
&\qquad \sum_{i=1}^{n^f}\sum_{k=1}^{n^p} \Psi^T \frac{\partial \mathbf{J}(\mathbf{p})}{\partial p_k}\Big|_{\mathbf{p}=\mathbf{0}}\Big|_{\mathbf{R}=\mathbf{R}_i} \mathbf{f}_i^e p_k + \mathbf{C}3\mathbf{g}p_k,
\end{aligned}
$$

with the constant volume integrals

$$
\begin{aligned}
m\mathbf{1} &= \iiint_V \mathbf{1}\rho \mathrm{d}V, \quad \mathbf{C}1 = \iiint_V \mathbf{J}^0\Psi\rho\mathrm{d}V, \\
\mathbf{C}0 &= \iiint_V \mathbf{R}\rho\mathrm{d}V, \quad \mathbf{C}2 = \iiint_V \tilde{\mathbf{R}}\mathbf{J}^0\Psi\rho\mathrm{d}V, \\
\mathbf{I} &= \iiint_V \tilde{\mathbf{R}}\tilde{\mathbf{R}}\rho\mathrm{d}V, \quad \mathbf{C}3 = \iiint_V \Psi^T\mathbf{J}^{0T}\mathbf{J}^0\Psi\rho\mathrm{d}V, \\
\mathbf{C}4_k &= \iiint_V \tilde{\mathbf{R}}\left(\mathbf{J}^0[\Psi_\alpha]_k\right)^\sim \rho\mathrm{d}V, \\
\mathbf{C}5_k &= \iiint_V \left(\mathbf{J}^0[\Psi_\alpha]_k\right)^\sim \mathbf{J}^0\Psi\rho\mathrm{d}V, \\
\mathbf{C}6_{lk} &= \iiint_V \left(\mathbf{J}^0[\Psi_\alpha]_l\right)^\sim \left(\mathbf{J}^0[\psi_a]_k\right)^\sim \rho\mathrm{d}V, \qquad (31) \\
\boldsymbol{\lambda} &= \iiint_V \Psi^T\mathbf{J}^{0T}\mathbf{L}^{0T}\mathbf{H}\mathbf{L}^0\mathbf{J}^0\Psi\rho\mathrm{d}V, \\
\mathbf{C}7_k &= \iiint_V \frac{\partial\mathbf{J}(\mathbf{p})}{\partial p_k}\Big|_{\mathbf{p}=\mathbf{0}} \Psi\rho\mathrm{d}V, \\
\mathbf{C}8_k &= \iiint_V \tilde{\mathbf{R}}\frac{\partial\mathbf{J}(\mathbf{p})}{\partial p_k}\Big|_{\mathbf{p}=\mathbf{0}} \Psi\rho\mathrm{d}V, \\
\mathbf{C}9_{lk} &= \iiint_V \left(\frac{\partial\mathbf{J}(\mathbf{p})}{\partial p_l}\Big|_{\mathbf{p}=\mathbf{0}} [\Psi_\alpha]_k\right)^\sim \tilde{\mathbf{R}}^T\rho\mathrm{d}V.
\end{aligned}
$$

Where $\mathbf{f}^e$ are the external forces acting on the body surface and $\mathbf{g}$ is the vector of the gravitational field intensity. The density of the flexible body is ρ. In the reduced equations of motion we have to solve the time independent volume integrals, Eqn. (31). This is done with the preprocessor FEMBS.

3 DYNAMICS OF A PERIODICAL-PARAMETRICALLY EXCITED SYSTEM

3.1 Equation of Motion of a Flexible Rotorsystem

Mechanical systems which are described by a mathematical model with time-variant coefficient matrices are called parametrically excited systems. Structures consisting of an elastic rotor which is attached to an elastic base structure, e.g. windturbines or helicopters, are examples for this kind of systems. Such a structure can be represented by a multibody system with flexible bodies. The system

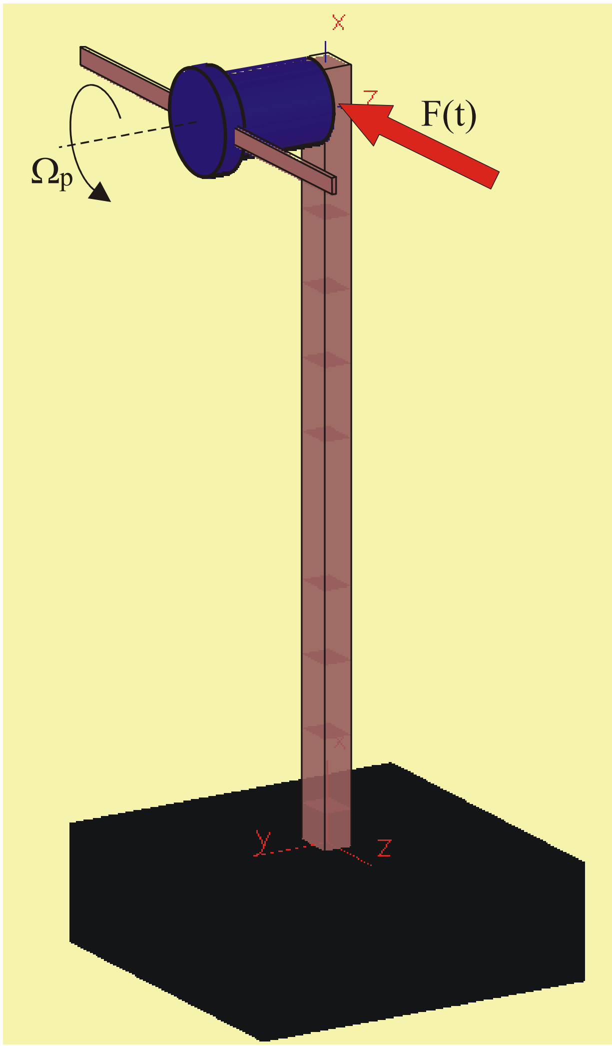

Figure 2. MBS MODEL OF A FLEXIBLE ROTORSYSTEM.

shown in Fig. 2 consists of a cantilever, a cylindrical rigid hub with a revolution joint and two elastic blades. The cantilever (index C) with a fixed base can be expressed in an inertial frame of reference using nodal degrees of freedom according to Eqn. (12)

$$\mathbf{M}_C^{e0}\ddot{\mathbf{q}}_C + \mathbf{K}_C^0\mathbf{q}_C = \mathbf{h}_C, \tag{32}$$

with the vector of forces and moments $\mathbf{h}_C = [\mathbf{h}_{C\backslash K}^T, -\mathbf{h}_K^T]^T$ where the forces and moments of the coupling point $\mathbf{h}_K$ (index K) are denoted separately from the loads $\mathbf{h}_{C\backslash K}$ applied to the other degrees of freedom of the cantilever without the coupling point (index $C\backslash K$). A modal representation $\mathbf{q}_C = \Theta_C\mathbf{p}_C$, see Eqn. (15), with the separated degrees of freedom of the coupling point $\mathbf{q}_K$

$$\mathbf{q}_C = \begin{bmatrix}\mathbf{q}_{C\backslash K}\\ \mathbf{q}_K\end{bmatrix} = \underbrace{\begin{bmatrix}\Theta_{C\backslash K}\\ \Theta_K\end{bmatrix}}_{\Theta_C}\mathbf{p}_C, \tag{33}$$

is given by

$$\mathbf{M}_C\ddot{\mathbf{p}}_C + \mathbf{K}_C\mathbf{p}_C = \Theta_C^T\mathbf{h}_C. \tag{34}$$

The equations of motion of the rotor (index R) with two elastic blades including the interia and gyroscopic terms of the rotating rigid hub have the form of Eqn. (27)

$$\begin{bmatrix}\mathbf{M}^{KK} & \mathbf{M}^{KR}\\ \text{sym.} & \mathbf{M}^{RR}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}_K\\ \ddot{\mathbf{p}}_R\end{bmatrix} + \begin{bmatrix}\mathbf{k}_K\\ \mathbf{k}_R\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \boldsymbol{\lambda}_R^w\mathbf{p}_R + \mathbf{K}^\sigma(\mathbf{a}_K, \dot{\boldsymbol{\omega}}_K, \boldsymbol{\omega}_K, \mathbf{h}^s)\mathbf{p}_R\end{bmatrix} - \begin{bmatrix}\mathbf{h}_K\\ \mathbf{h}_R\end{bmatrix} = \mathbf{0}. \tag{35}$$

Applying the relations of kinematic compatibility and the equilibrium of reaction forces and moments at the coupling point to Eqn. (34) and (35) yields the equations of motion for the whole structure.

Operation conditions with the rotor rotating at a constant angular frequency $\Omega_p = 2\pi/T_p$ yield T_p-periodic time-variant system matrices. In case of small elastic deformations, which are expressed in moving frames of reference attached to each flexible body, the equations of motion may be linearized w.r.t these small deformations even if there are large rigid body motions. Then a linear periodic time-variant system of differential equations of order n^p is obtained in terms of modal coordinates $\mathbf{p} = [\mathbf{p}_C^T, \mathbf{p}_R^T]^T$

$$\mathbf{M}(t)\ddot{\mathbf{p}}(t) + \mathbf{D}(t)\dot{\mathbf{p}}(t) + \mathbf{K}(t)\mathbf{p}(t) = \Theta^T\mathbf{h}(t), \tag{36}$$

with $\mathbf{M}(t+T_p) = \mathbf{M}(t)$ the symmetric mass matrix, with $\mathbf{D}(t+T_p) = \mathbf{D}(t)$ as an abbreviation for the sum of damping and gyroscopic matrices, with $\mathbf{K}(t+T_p) = \mathbf{K}(t)$ denoting the sum of stiffness and centrifugal matrices and with the transfomation matrix Θ between modal and nodal degrees of freedom of the flexible rotor system

$$\underbrace{\begin{bmatrix}\mathbf{q}_C\\ \mathbf{q}_R\end{bmatrix}}_{\mathbf{q}(t)} = \underbrace{\begin{bmatrix}\Theta_C & \mathbf{O}\\ \mathbf{O} & \Theta_R\end{bmatrix}}_{\Theta}\underbrace{\begin{bmatrix}\mathbf{p}_C\\ \mathbf{p}_R\end{bmatrix}}_{\mathbf{p}(t)}. \tag{37}$$

3.2 Solution of Linear Periodic Time Variant Systems

Equation (36) can be rewritten in state space

$$\dot{\mathbf{x}}(t) - \mathbf{A}(t)\mathbf{x}(t) = \mathbf{A}_1^{-1}(t)\bar{\mathbf{h}}(t), \tag{38}$$

with the state vector $\mathbf{x}(t) = [\mathbf{p}^T(t), \dot{\mathbf{p}}^T(t)]^T$, the excitation vector $\bar{\mathbf{h}}(t) = [(\Theta^T\mathbf{h}(t))^T, \mathbf{0}^T]^T$ and the T_p-periodic system matrices

$$\mathbf{A}(t) = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}(t)\mathbf{K}(t) & -\mathbf{M}^{-1}(t)\mathbf{D}(t)\end{bmatrix}, \tag{39}$$

$$\mathbf{A}_1^{-1}(t) = \begin{bmatrix}\mathbf{0} & \mathbf{M}^{-1}(t)\\ \mathbf{M}^{-1}(t) & -\mathbf{M}^{-1}(t)\mathbf{D}(t)\mathbf{M}^{-1}(t)\end{bmatrix}. \tag{40}$$

3.2.1 Free Vibrations.

The solution of the periodic homogenous system of differential equations related to Eqn. (38) is found by an approximation method according to Hill [14, 15, 16]. The Floquet set up $\mathbf{x}(t) = \boldsymbol{\xi}(t)e^{\lambda t}$, with a T_p-periodic eigenvector $\boldsymbol{\xi}(t+T_p) = \boldsymbol{\xi}(t)$, and a Fourier-series expansion of the eigenvector $\boldsymbol{\xi}_F(t) = \sum\boldsymbol{\xi}_v e^{jv\Omega_p t}$ and the T_p-periodic system matrix $\mathbf{A}_F(t) = \sum\mathbf{A}_w e^{jw\Omega_p t}$ allows applying harmonic balance to the resulting equation

$$\left(\sum_{v=-\infty}^{\infty}(jv\Omega_p + \lambda)\,\boldsymbol{\xi}_v e^{jv\Omega_p t}\right) - \left(\sum_{w=-\infty}^{\infty}\mathbf{A}_w e^{jw\Omega_p t}\right)\left(\sum_{v=-\infty}^{\infty}\boldsymbol{\xi}_v e^{jv\Omega_p t}\right) = \mathbf{0}, \tag{41}$$

which yields an infinite set of coupled matrix equations. If the Fourier-series are truncated, the matrix equations can be rearranged in a finite hypermatrix scheme denoting a hyper-eigenvalueproblem from which the eigenvalues λ_k and eigenvectors $\boldsymbol{\xi}_{k,F}(t)$ are calculated. Because of a redundance in the hypermatrix scheme only $2n^p$ linearly independent fundamental solutions $\mathbf{x}_k(t)$ exist although the dimension of the hypermatrix is greater than $2n^p$.

3.2.2 Forced Vibrations.
Applying a coordinate transformation $\mathbf{x}(t) = \Xi(t)\bar{\mathbf{q}}(t)$ with the modal matrix $\Xi(t) = [\boldsymbol{\xi}_{1,F}(t), ..., \boldsymbol{\xi}_{2n^p,F}(t)]$ to Eqn. (38) yields a system of linear differential equations with constant coefficients according to the LJAPUNOV reducibility theorem [17], which is decoupled in case of $2n^p$ linearly independent eigenvectors,

$$\dot{\bar{q}}_k(t) - \lambda_{k,0}\bar{q}_k(t) = \frac{{}_L\boldsymbol{\xi}_{k,F}^T(t)\,\bar{\mathbf{h}}(t)}{a_k}, \quad k = 1, ..., 2n^p, \tag{42}$$

with the generalized modal masses a_k and the left hand eigenvectors ${}_L\boldsymbol{\xi}_{k,F}^T(t)$, which are calculated from ${}_L\,\Xi^T(t) = \lceil a_k \rfloor\,\Xi^{-1}(t)\mathbf{A}_1^{-1}(t)$. Solving Eqn. (42) in the frequency domain of the FOURIER-transformation finally yields the system response $\mathcal{X}(\Omega)$ due to forced vibrations in state space

$$\mathcal{X}(\Omega) = \sum_{k=1}^{2n^p} \sum_{v=-V}^{V} \sum_{u=-U}^{U} \bar{\mathbf{H}}_{k,vu}\overline{\mathcal{H}}(\Omega - (u+v)\Omega_p), \tag{43}$$

with the FOURIER-transforms $\mathcal{X}$ and $\overline{\mathcal{H}}$ of the response function $\mathbf{x}(t)$ and the excitation function $\bar{\mathbf{h}}(t)$, respectively, and with the frequency response matrices

$$\bar{\mathbf{H}}_{k,vu}(\Omega, \Omega_p) = \frac{\boldsymbol{\xi}_{k,v}\,{}_L\boldsymbol{\xi}_{k,u}^T}{a_k\left[j\left(\Omega - v\Omega_p\right) - \lambda_k\right]}. \tag{44}$$

If the solution according to the modal coordinates $\mathbf{p}(t)$ is extracted from Eqn. (43) and the modal transformation, Eqn. (37), is taken into account the system response in nodal coordinates is obtained

$$\mathcal{Q}_i(\Omega) = \sum_{k=1}^{2n^p} \sum_{v=-V}^{V} \underbrace{\sum_{u=-U}^{U} \frac{(\boldsymbol{\eta}_{k,v})_i({}_L\boldsymbol{\eta}_{k,u}^T)_j}{a_k[j\left(\Omega - v\Omega_p\right) - \lambda_k]}}_{H_{ij,k,vu}(\Omega,\Omega_p)} \mathcal{H}_j(\Omega - (u+v)\Omega_p), \tag{45}$$

with the response coordinate (i) and the excitation coordinate (j), with the frequency response functions $H_{ij,k,vu}(\Omega, \Omega_p)$, with the modeshape-vectors $\boldsymbol{\eta}_{k,F}(t) = \sum \boldsymbol{\eta}_{k,v} e^{jv\Omega_p t}$, $\boldsymbol{\eta}_{k,v} = [\Theta, \mathbf{O}]\boldsymbol{\xi}_{k,v}$ and the FOURIER-transform $\mathcal{H}$ of the excitation function $\mathbf{h}(t)$. As can be seen from Eqn. (45) a harmonic excitation with Ω_{exc} yields effective responses with several angular frequencies $\Omega_{res} = \Omega_{exc} + (u+v)\Omega_p$.

4 IDENTIFICATION OF THE FREQUENCY RESPONSE BEHAVIOUR

4.1 MBS program
The time-step integration of the nonlinear equations of motion allows to calculate the time response under any operating conditions. Applying the according forces to the system large deformations may be observed, but the time response does not uncover the reasons for inadmissible vibrations. The calculation of eigenfequencies and modeshapes gives more insight in the system behaviour. But in case of time-variant systems a linearisation at a fixed time is required to solve the eigenvalueproblem as it is usually done. This means to calculate the dynamic behaviour of the system in a fixed position but taking into account the effects of centrifugal, gyroscopic and maybe aerodynamic forces. These so called snap-shot eigenfrequencies cannot reveal the multiple resonance frequencies of the true modes $\boldsymbol{\eta}_{k,F}(t)$, because they are obtained without taking into account the extensive transfer behaviour of periodic time-variant systems.

4.2 Partial Frequency Response Functions
The threefold sum in Eqn. (45) cannot be carried out independently of the excitation term. For a unique description of the systems transfer behaviour the terms on the right hand side have to be rearranged in groups with $u + v = z$

$$\mathcal{Q}_i(\Omega) = \sum_{z=-(V+U)}^{V+U} \underbrace{\sum_{k=1}^{2n^p} \sum_{v,u|v+u=z} H_{ij,k,vu}}_{H_{ij,z}(\Omega,\Omega_p)} \mathcal{H}_j(\Omega - z\Omega_p), \tag{46}$$

with the partial fequency response functions $H_{ij,z}(\Omega, \Omega_p)$. In case of a non-contact step-sine excitation the necessary signal amplitudes for accumulating the partial fequency response functions can be taken directly from the FOURIER-spectra of the time data

$$H_{ij,z}(\Omega + z\Omega_p, \Omega_p) = \frac{\mathcal{Q}_i(\Omega + z\Omega_p)}{\mathcal{H}_j(\Omega)}. \tag{47}$$

If there is a feedback due to a coupled excitation device or if a broadband excitation is applied, then Eqn. (47) cannot be used any longer because the system response with angular frequency Ω_{res} is caused by excitations not only with $\Omega_{exc} = \Omega_{res}$ but with $\Omega_{exc} = \Omega_{res} - (u+v)\Omega_p$, i. e. more than one partial frequency response function contributes to the system response. In this case a least squares estimation of the partial frequency response functions $H_{ij,z}(\Omega, \Omega_p)$ has to be carried out [11, 18].

5 APPLICATION TO A FLEXIBLE ROTOR SYSTEM

In order to demonstrate the differences which might occur if the effects of a periodical-parametrically excited system are neglected by a reduction to a time-invariant snapshot system, the flexible rotor system, Fig. 2, is investigated. The driving point function at the upper end of the cantilever is chosen as a representative of the system dynamics.

5.1 Non-rotating System

The non-rotating system is described by constant coefficient matrices and may be treated as usually. A finite element model of the whole system is used to determine the number of modes for the flexible bodies such that the dynamic behaviour of the non-rotating system is reliably modelled in a frequency range up to 400 Hz [11], Fig. 3. The notation "YY" indicates the direction along the axis of rotation while "ZZ" indicates the direction perpendicular to it. The according mode shapes are described in the table below.

Modes of Non-Rotating System				
	Frequency [Hz]		Description	
	FEM	MBS	Cantilever	Blades
1	24.2	24.2	1. bend. z	-
2	33.4	33.5	1. bend. y	-
3	100.8	101.8	1. torsion	1. bend. z in phase
4	113.4	113.4	-	1. bending z
5	181.3	185.7	2. bend. z	1. bending z
6	200.7	205.6	1. torsion	1. bend. z out of phase
7	332.4	338.4	2. bend. y / 1. torsion	1. bending z
8	437.1	492.6	3. bend. z	1. bend. y

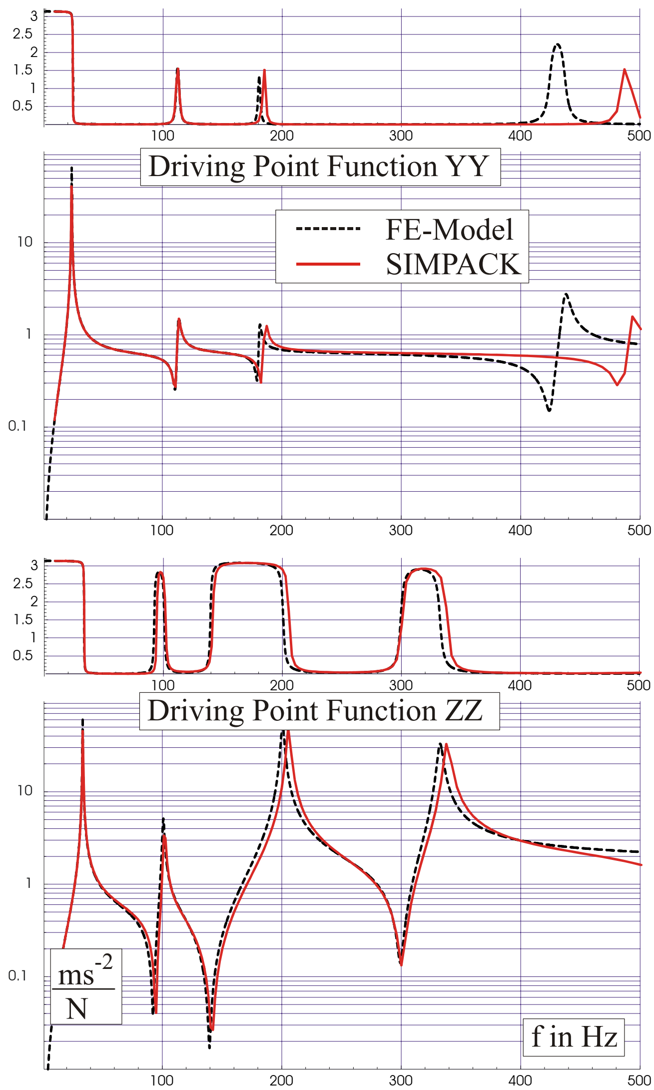

Figure 3. FREQUENCY RESPONSE FUNCTION OF THE NON-ROTATING SYSTEM AT THE END OF THE CANTILEVER.

5.2 Rotating System

The rotational speed of the rotor was set to $\Omega_p = 100\ \mathrm{s}^{-1} \approx 15.9$ Hz.

5.2.1 Snap-Shot-Frequency Response Function.

The snap-shot frequency response functions were calculated from the linear equations of motion of the MBS model for three different angular positions of the rotor, Fig. 4. As compared to the non-rotating system, Fig. 3, there are only slight differences in frequency and amplitude of the resonances for the horizontally oriented rotor. For the different angular positions of the rotor there are evident differences concerning the torsional modes of the cantilever (in the vincinity of 100 Hz and 200 Hz).

5.2.2 Partial Frequency Response Functions.

The partial frequency response functions, Eqn. (47), are calculated from the resulting time data of a multibody system simulation with step-sine excitation at the end of the cantilever perpendicular to the axis of rotation, Fig. 2. The step-sine excitation was applied in the range from $0 - 450$ Hz in steps of 0.6 Hz. The dominating partial frequency response functions are depicted in Fig. 5. Besides the "0"-partial frf the "$\pm 2\Omega_p$"-partial frfs contribute significantly to the sys-

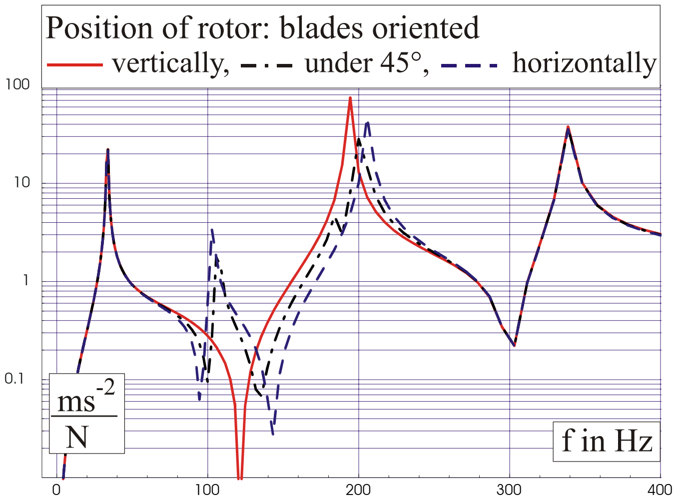

Figure 4. SNAP-SHOT FREQUENCY RESPONSE FUNCTIONS ZZ FOR DIFFERENT ANGULAR POSITIONS OF THE ROTOR.

tem response due to the change of the rotors mass geometry twice per revolution. In order to illustrate the use of the partial frequency response functions a harmonic excitation with a frequency of 200 Hz is plotted in Fig. 5 as an upward-arrow. The system responds with the excitation frequency and additional sidebands. In each partial frequency response function $H_{ZZ,z}$ the according harmonic response with a frequency of $[200 + z\Omega_p/(2\pi)]$ Hz is indicated by a downward-arrow.

5.2.3 Discussion of Results.

In order to compare the results of the snap-shot and the partial frequency response functions, the latter are added up, which yields the frequency response of the periodic system due to an impuls excitation. There are significant differences according to the torsional modes of the cantilever, Fig. 6. The snap-shot frequency response functions, even those for other angular positions, Fig. 4, are not able to detect the real resonances of the periodic system.

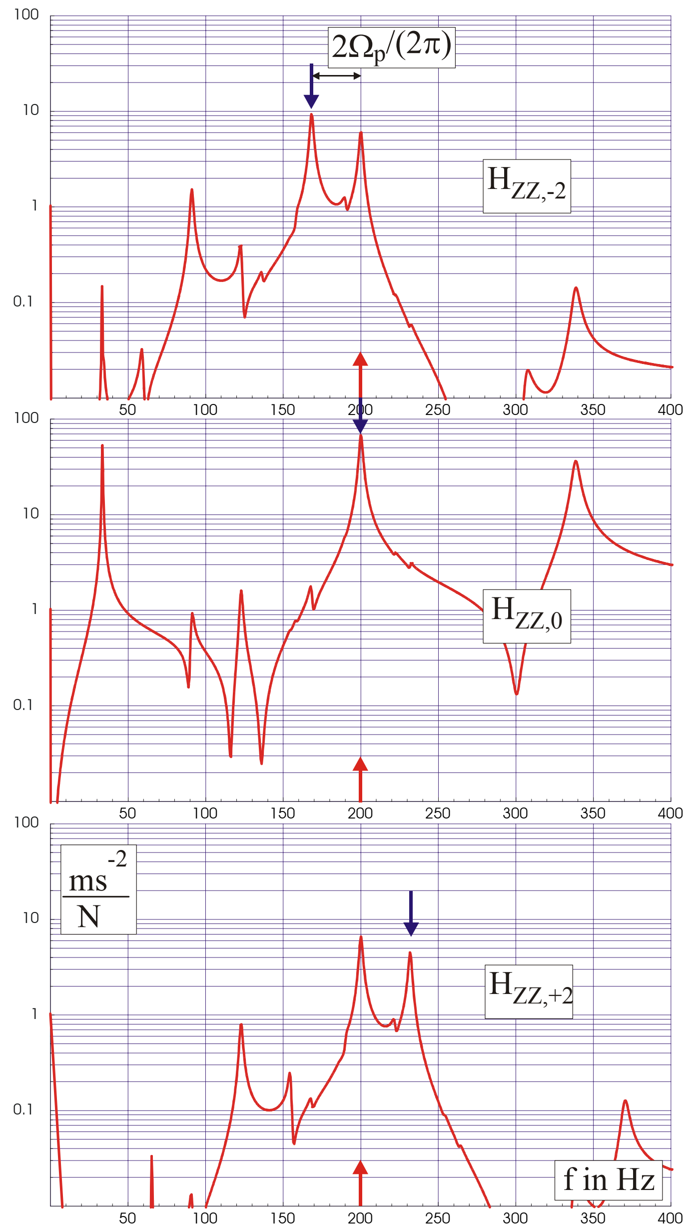

Figure 5. PARTIAL FREQUENCY RESPONSE FUNCTIONS ZZ WITH $\max|H_{ZZ,z}| > 1$.

6 CONCLUSION

The simulation of flexible multibody systems is an important numerical tool for dynamic studies of many structures. There is no difficulty in simulating parametrically excited systems even in a non-linear formulation by time-step integration. But if the focus is on the frequency response behaviour of such parametrically excited systems, one has to be aware of the consequences of neglecting the parametrical excitation. The consideration of snap-shot eigenfrequencies does not reveal all resonances of a parametrically excited system, which may lead to momentous misinterpretation of the system dynamics, as was shown for a periodic time-variant system.

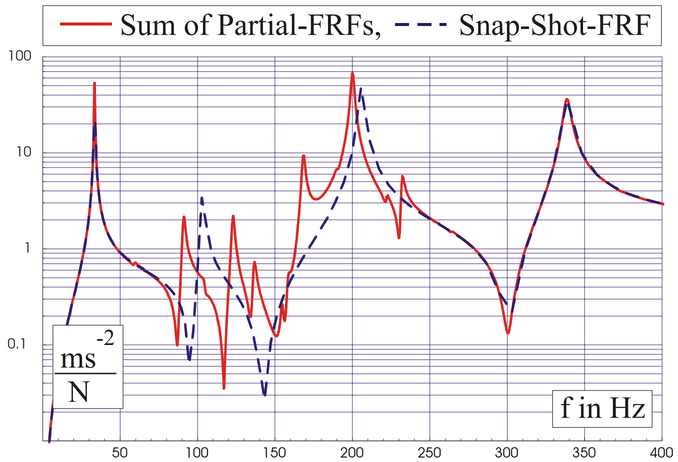

Figure 6. SNAP-SHOT FRF ZZ (BLADES HORIZONTALLY ORIENTED) VS. FREQUENCY RESPONSE ZZ OF THE PERIODIC SYSTEM.

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