Finite Element and Finite Difference Methods in Engineering

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By means of a simple example, a stretched string under transverse load, finite element and finite difference methods which are so widely used in engineering are illustrated. The finite element method is shown to be an essentially modified Raleigh-Ritz procedure. The finite difference technique is applied directly to the string differential equation; an energy related approach is also discussed. The manner in which a combination finite element/finite difference solution can be effected for the same physical problem is treated. Application of both the finite element and finite difference methods to more complex problems as well as selected programs and depositories are mentioned.

Introduction

For many years, including to a fair extent at the present time, engineering analysts and designers have utilized handbooks and design charts as a prime mechanism in making decisions. With the advent of easily available computer assistance, analytical and numerical methods are being utilized with increased frequency in engineering decision making. Through necessity, some fields such as aerospace have been in the conspicuous forefront with respect to the early and necessary usage of rather sophisticated analytical methods in an engineering design mode. For example, the Boeing 747 aircraft could never have been designed in such an efficient and rapid manner relative to its earlier predecessor, the 707, without extensive and imaginative use of numerical and analytical techniques. It is interesting to note, however, that traditionally less analytic intensive engineering fields such as automotive and civil engineering, are also utilizing many relatively sophisticated computerized techniques both for analysis and manufacturing.

In this paper, prime attention is focused on the two most widely utilized numerical tools in engineering: finite element and finite difference methods. While finite difference techniques have been used for almost a century, the finite element method has been a serious tool for but a decade. The growth and breadth of finite element applications is indeed almost virulent. The recent maturity of the field is perhaps best evidenced by the rash of books published on the subject during the last three years [1–7].[1] While the prime initial technical area of application in finite elements was in structural mechanics, current fields of application include fluids, heat transfer, and electronics.

A prime purpose of this paper is to describe the essence of the finite element and finite difference methods via relatively simple examples. In addition, a description by example will also be presented of how these two powerful techniques could be combined.

In the next four sections, the string under constant tension is used as the general prototype of the finite difference or finite element problem. Firstly, an exact and a Raleigh-Ritz type solution is obtained; subsequently, a modification of the Raleigh-Ritz method is presented to demonstrate the close link with the finite element method; solutions to the same stretched string problem are then obtained by a direct differential equation and an energy approach; finally, the sequence is completed by illustrating for the same problem how the finite element method can be combined with the finite difference technique. A general discussion follows describing how finite element methodology with various elements is used as a solution technique. The more recent generalization of finite difference techniques with variable mesh arrangements is subsequently treated. Finally, a description of representative computer programs which might be of interest from a design viewpoint are enumerated along with possible sources of obtaining the same. The paper closes with some conclusions and discussion.

String Under Tension—Exact and Raleigh-Ritz Solutions

The prototype example selected, which will be utilized to demonstrate much of the theory discussed is that of a stretched string under constant tension T; the tension is presumed large enough that it does not change significantly due to the application of a transverse loading P, Fig. 1. The transverse load is supported by the constant tension T in the string which under-

[1]Numbers in brackets designate References at end of paper.

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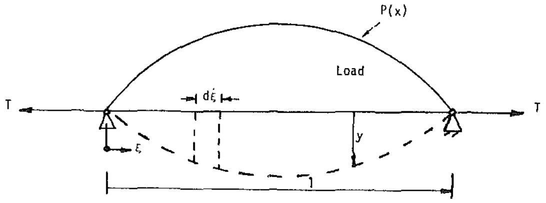

Fig. 1 Initially stretched string with transverse load

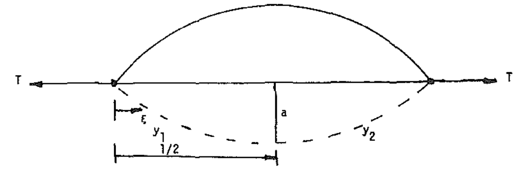

Fig. 2 Modified Raleigh-Ritz with two functionals: y_1, y_2

goes a deformation pattern shown by the dashed line. Enforcement of equilibrium for an elemental section $d\xi$ of the string results in a simple differential equation shown in equation (1). This example is essentially the one-dimensional simple analogue of the well-known two-dimensional membrane problem used in elasticity theory for torsion [8].

$$\frac{d^2y}{d\xi^2} = -\frac{P}{T} \tag{1}$$

For definiteness, let us select a load function $P = P_0 \sin \pi\xi$. Inserting this load function into equation (1) and integrating twice with respect to ξ satisfying the obvious boundary conditions that the deflection y vanishes at 0 and 1, we find the exact solution of equation (2).

$$\text{exact solution} \quad y = \frac{PoL^2}{T\pi^2} \sin \pi\xi \tag{2}$$

Next, let us review the so-called Raleigh-Ritz procedure which has been widely used as an approximate technique to solve problems in mechanics. With this procedure, one assumes that the displacement function y is equal to the product of an undetermined parameter a_1 and a shape parameter $f(\xi)$, equation (3a). The shape parameter is selected so that it does conform qualitatively to what one would expect intuitively, and most importantly satisfies the geometric boundary conditions of no displacement at either end, equation (3b). The amplitude parameter a_1 is calculated by determining the total potential energy in the system and minimizing this quantity with respect to a_1.

$$\text{Raleigh-Ritz procedure} \quad y = a_1 f(\xi) \tag{3a}$$

$$\text{B.C.'s satisfied} \quad f(0) = f(1) = 0 \tag{3b}$$

The total potential energy of the string is defined with respect to its original straight undeformed configuration. It consists of two terms expressed mathematically by equation (4); the physical interpretation of the first term is the strain energy associated with the stretching of the elastic string, and the second is related to the potential energy of the external load (the work it can do in going from the deformed position back to the original configuration).

$$\text{potential energy} \quad V = \frac{T}{2}\int_0^1 y'^2 d\xi - \int_0^1 P y d\xi \tag{4}$$

To illustrate this process, let us select for the deflection function of (3a) a parabola as illustrated by equation (5). The boundary conditions are clearly satisfied and a, the amplitude parameter, represents the maximum deformation at the center ($\xi = {}^1/_2$).

$$y = a[4\xi(1 - \xi)] \tag{5}$$

If we substitute equation (5) into the total potential energy expression of equation (4) and minimize with respect to the amplitude parameter a ($\partial V/\partial a = 0$) we find that $a = 0.955$ ($P_0/T\pi^2$). When compared with the exact solution of equation (2), this approximate solution is only in error by 4 ${}^1/_2$ percent.

As originally conceived, the Raleigh-Ritz approach could be applied with a series of terms of the type shown in equation (3a), namely a series of products of amplitude parameters by functionals each of which satisfy the boundary conditions. The functionals should differ from each other and the amplitude parameters also have different designations (say, $a_1, a_2, \ldots a_n$). By substituting all these expressions in the energy equation (4) and minimizing, in turn, with respect to each amplitude parameter, a series of simultaneous equations will be obtained for the n parameters which could be solved.

The difficulty in applying the Raleigh-Ritz method in this traditional form is that for complex realistic situations, it is not readily feasible to select single functionals which satisfy the boundary conditions.

Modified Raleigh-Ritz Approach (Finite Element Method)

A simple but powerful modification of the Raleigh-Ritz procedure can be effected which overcomes the previously mentioned difficulty of selecting functions which simultaneously satisfy all boundary conditions. This procedure is illustrated by Fig. 2 which pertains to the same stretched string problem discussed in the previous section. For this case, however, two functionals, y_1 and y_2, are selected such that the boundary conditions associated with each are satisfied and they meet at a common point (and slope) in the center. As before, the amplitude parameter a remains open and obtainable by an energy minimization process.

To illustrate, let us select y_1 as a parabola and y_2 as a segment of a sine curve as described mathematically by equations (6)

$$y_1(\xi) = a[4\xi(1 - \xi)] \qquad 0 \le \xi \le {}^1/_2 \tag{6a}$$

$$y_2(\xi) = a(\sin \pi\xi) \qquad {}^1/_2 \le \xi \le 1 \tag{6b}$$

For the same sinusoidal loading described in the previous section, the functionals given by equations (6) are substituted into the energy expression of equation (4) (which must be partitioned into two regions commensurate with those described in equation (6)). We may then minimize the total potential energy with respect to the open amplitude parameter a and solve thereby for this value: $a = 0.976$ ($P_0/T\pi^2$). The error resulting relative to the exact solution of equation (2) is 2.4 percent. That this error is less than that of the previous solution associated with the complete parabolic functional of equation (5) is not surprising because in the second half of the current solution a sinusoidal function was assumed which is the shape of the exact solution of equation (2).

The methodology described by Fig. 2 is capable of extension in a convenient and methodical manner as exemplified by Fig. 3. Obviously, the $n + 1$ functionals (y_1 through y_{n+1}) could be used to describe the deflection profile in a piecewise fashion. The end functionals should satisfy appropriate boundary conditions; in addition, continuity between adjacent functionals should also be a requirement. For this generalized case, we would minimize the total potential energy (obtained by substituting in an appropriate piecewise fashion the n functionals into equation (4)) with respect to, in turn, a_1 through a_n. The resulting set of equations are readily solvable for all the open amplitude parameters $a_1, a_2, \ldots a_n$.

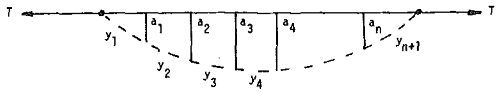

Fig. 3 Modified Raleigh-Ritz with $n+1$ functionals

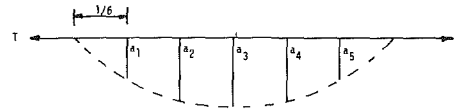

Fig. 5 String response described by finite number of parameters a_1 to a_5

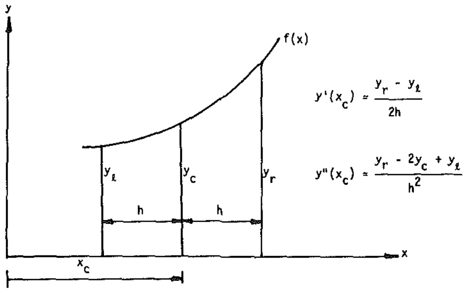

Fig. 4 Illustration of finite difference approximations to derivatives

The previous discussion illustrated by Figs. 2 and 3 constitutes the essence of the finite element procedure. The problem becomes largely a bookkeeping one when applied to one physical situation or another. The finite element method has been interestingly characterized as a type of patchquilt Raleigh-Ritz method and it is hoped that this illustration has amply portrayed that notion.

Finite Difference Technique—Direct Differential Equation and Energy Methods

The essence of the finite difference approach is to replace derivatives by algebraic quantities; subsequently, when we work with either the differential equations or an energy approach, we logically solve an evolving system of algebraic equations in an orderly manner.

An illustration of how we approximate the derivatives of a functional by algebraic quantities is shown in Fig. 4. In the limit, when the quantity h (the spacing between adjacent points) becomes vanishingly small, the finite difference approximations between the first and second derivatives shown adjacent to the curve are indeed rigorously the exact derivative of the function at the point x_c. The reader unfamiliar with finite difference approximations is encouraged to select specific functionals for $f(x)$ and prove to himself that the derivative approximations for some small but finite values of h do indeed work quite well.

Let us again turn to the stretched string of Fig. 1 and the associated differential equation of equation (1) to illustrate the first form of a finite difference solution.

The deformation domain could be characterized by the deformation functional y for the string as a five-parameter model, Fig. 5. If we could determine what a_1 through a_5 were, we would be able to determine to good approximation what is the actual deflectional function.

Firstly, from a symmetry viewpoint it is clear that $a_1 = a_5$, and $a_2 = a_4$. Hence, there are only three independent unknown quantities. By substituting for the finite difference equivalence of the derivative (see Fig. 4) into the differential equation of equation (1) at points associated with a_1, a_2, and a_3, three equations in algebraic form are obtained to solve the three quantities a_1 through a_3. The mesh spacing h associated with this exercise is one-sixth the total string length.

The resulting solution shows that the maximum amplitude parameter, a_3, is 2.3 percent more than the exact solution of equation (2).

In essence, what is conducted via this finite difference procedure is to replace a continuous differential equation which is valid at all points along the string by a finite number of approximate finite difference equilibrium equations at each of the five nodal points.

An alternative approach which has been used with some success in recent years is an energy finite difference method. One can visualize utilizing the finite difference mesh of Fig. 5 and substituting for the derivatives in accordance with the expressions of Fig. 4 into the energy expression of equation (4). By minimizing the total potential energy successively with respect to a_1, a_2, and a_3 (because of symmetry), one would obtain three equations in three unknowns, from which the values of a_1 through a_5 could be calculated.

Combination Finite Element/Finite Difference Solutions

One of the principle features of the finite element method is its ability to treat complex domains; on the other hand, the finite difference method, where applicable, is frequently very efficient with respect to computer time. The virtues of both techniques could be combined if an energy approach were utilized. A combination finite element/finite difference approach will be illustrated here by again selecting the stretched string problem of Fig. 1.

Again because of symmetry only three amplitude parameters a_1 through a_3 are to be determined, Fig. 6. As shown in Fig. 6, a finite element solution is applied to the first quarter of the string, and a finite difference approach is utilized for the second quarter (a symmetric situation exists in the right half of the string). As further illustrated in the figure, linear functionals are presumed in the finite element portion. In the finite difference portion, derivatives of the function are calculated at the nodal point by again using the finite difference approximations described in Fig. 4.

The total potential energy of equation (4) can be calculated for both the finite element and finite difference regimes as a function of the amplitude parameters a_1, a_2, and a_3. By again minimizing the total potential energy with respect to these three amplitude parameters, three equations arise which can be solved for a_1, a_2, and a_3. Accuracy of the solution is expected to be roughly comparable to the finite difference energy approach.

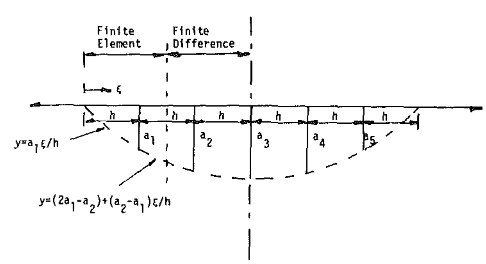

Fig. 6 Combination finite difference/finite element (FE) solution with linear functionals in FE part

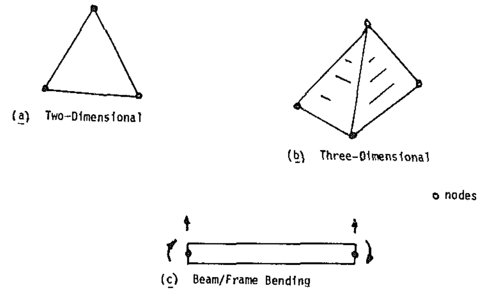

Fig. 7 Representative finite elements

If instead of three amplitude parameters there were n such parameters, then n minimization conditions are set up resulting in a determinate $n \times n$ system of equations. The illustration of Fig. 6 also demonstrates the most useful application of a combined finite difference/finite element approach in that the finite element technique is applied in the vicinity of the boundary where the likelihood of irregularities or complex shapes would cause difficulty for an exclusive finite difference approach.

In the finite element regime of Fig. 6, a quadratic rather than a linear functional would probably produce a slightly improved result. In finite element parlance these functionals are known as shape functions. It is critical that no matter what the shape function selected, at a nodal point the functional must pass through the local amplitude parameter value. For example, in the domain from 0 to h in Fig. 6, the linear functional selected must pass through the points $y = 0$ and $y = a_1$.

Generalizations of the Finite Element and Finite Difference Method

While a relatively simple notion of the finite element has been introduced in relation to the string problem, it should now be apparent the same concept is applicable toward much more complex elements such as those illustrated in Fig. 7. Two- and three-dimensional continua, as well as bending-type problems, can be readily solved with this approach. A shape function is assumed for whatever the element happens to be; the deflections or rotations at the nodal point must again be associated with discrete amplitude parameters whose deformations are compatible with those of adjacent element in the field. The problem becomes largely a bookkeeping one rather than a conceptionally difficult one. Total potential energy is utlimately expressed in terms of all the amplitude parameters. Minimization with respect to each of these quantities produces a determinate set of equations.

The finite difference technique can be used in the one-dimensional example of the string treated in previous sections, as well as in two- or three-dimensional meshes which are normally uniformly mapped out in some domain pertinent to the physical problem. As mentioned earlier, one of the difficulties with finite difference techniques is the inability to treat complex domains with variable rather than regular finite difference meshes. This problem could be alleviated greatly. Unlike the illustration of Fig. 4, finite difference approximations to derivatives can be expressed in a much more general form should the surrounding points to a given central point be randomly selected. A number of solutions to linear and nonlinear problems have been obtained utilizing variable mesh finite difference techniques with a direct differential equation approach [9], as well as with an energy technique [10].

Useful Programs and Sources

Utilizing finite difference and finite element methods, general and special-purpose computer programs are available in structures, fluid mechanics, and heat transfer. Many programs have pre- and post-processors as well as graphics output routines attached to them, making them particularly convenient for the user. Representative examples in the structures field include SAP IV [11] and GIFTS [12]. With these codes the user can employ a variety of finite element types and assemble them for various load cases in a very efficient manner. These two computer programs have, in fact, been widely utilized nationally by the mechanical engineering community.

A number of depositories of computer information are available such as the National Information Service in Earthquake Engineering (NISEE), the NASA-sponsored COSMIC at the University of Georgia, and the Air Force-sponsored ASIAC located in California. Most computer software depositories are poorly funded relative to the mission they are chartered to perform. As a result, many users have difficulty in obtaining computer programs via this route. Regrettably, there is no large institutionalized computer software sharing scheme operating on a national basis which is adequately funded.

From symposia proceedings or other special events, compilations of software are occasionally obtained [13]. In addition special ASME publications relating to programs sponsored by, some Technical Divisions do have discussions or treatments on computer programs [14]. Another novel attempt to communicate information on computer software is the forthcoming Structural Mechanics Software Series [15]. These volumes will have survey information on selected segments of structural mechanics as well as documentation-type information on actual programs which can be accessed directly on national computer network systems.

Conclusions and Comments

Finite element and finite difference methods have been illustrated via a number of simplified examples. It has also been shown how a combined approach to both these techniques is possible. The application of finite elements with more advanced elements is discussed, and finite difference techniques with generalized grids are briefly treated. Information on computer programs, as well as sources of the same, are cited.

The analytical methodology outlined here represents the current rather advanced state of the art available in a relatively convenient form to engineers from many disciplines. Pre-processor and graphic capabilities, along with interactive techniques, enable many computer programs to be readily useable by the engineer and designer. In view of the serious growth of product liability litigation [16], it behooves engineers to avail themselves of, and become acquainted with, these new analysis and design techniques so as to effect the safest possible design of a given product.

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