Research Article

Optimal Harvest Control in a Singular Prey-Predator Fishery Model with Maturation Delay and Gestation Delay

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This paper presents a singular prey-predator fishery model, where maturation delay for prey and gestation delay for predator are considered. Fishing efforts are introduced to harvest prey and predator population, which are developed as control instruments to investigate optimal utilization of fishery resource. By analyzing associated characteristic equation, local stability analysis is studied due to combined variations of double time delays. Furthermore, Pontryagin's maximum principle is utilized to characterize optimal harvest control, and the optimality system is numerically solved based on an iterative method.

1. Introduction

According to statistics from the Food and Agriculture Organization of United Nations [1], approximate fifty-three percent of fish stock under observation has experienced overexploitation or depletion, which reiterate the fact that fishery needs to be managed with an effective and carefully defined objective to prevent overexploitation and replenish depleted stock [1]. In order to ensure sustainable fishery, it requires precision in stock assessment and reliability in fishery modelling [2, 3]. In recent decades, there are growing research interests in formulating bioeconomic mathematical models to obtain more comprehensive indications of feedback effect between exploitation activity and fishery resources [4–13].

In [5], authors establish a dynamic model where both prey and predator fishery resource population are exploitable, which is as follows:

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) - \frac{\beta x(t)\,y(t)}{\alpha + x(t)} - E(t)\,x(t),$$

$$\dot{y}(t) = y(t)\left[\frac{m\beta x(t)}{\alpha + x(t)} - d\right] - E(t)\,y(t), \tag{1}$$

where $x(t)$ and $y(t)$ represent population density of prey and predator population with time t, respectively. r is the intrinsic growth rate of prey, K stands for maximum environmental capacity, β denotes predation rate of predator, m represents

biomass transmission rate from predation activity, and d is death rate of predator.

It is well known that commercial harvesting fluctuates with dynamical variation of economic interest [14, 15]. Although the harvested biological models have received great attention from both mathematical and theoretical biologists, little work has been done on dynamic effect of economic interest on harvested prey-predator fishery model [14, 15]. In 1954, Gordon proposes common-property resource economic theory [14], which investigates dynamic effect of commercial harvesting on ecosystem from an economic perspective [14, 15]. In [14], an algebraic equation is proposed to investigate economic interest of commercial harvesting:

$$\begin{aligned}
&\text{Net Economic Revenue} \\
&\quad = \text{Total Revenue (TR)} - \text{Total Cost (TC)}.
\end{aligned} \tag{2}$$

In order to show dynamical variation of unit price of harvested population due to change of amount of harvested population, $a_1/(b_1 + E(t)x(t))$ is assumed to be unit price of commercially harvested prey, and $a_2/(b_2 + E(t)y(t))$ is assumed to be unit price of commercially harvested predator. Hence, TR in (2) takes the following form TR $= a_1 E(t)x(t)/(b_1 + E(t)x(t)) + a_2 E(t)y(t)/(b_2 + E(t)y(t))$, where a_1, a_2, b_1, b_2 are all positive constants. Let c represent cost of

commercial harvesting. Hence, TC in (2) takes the following form TC $= cE(t)$.

Recently, many theoreticians and experimentalists have investigated complex dynamics of prey-predator fishery system [16, 17]; it reveals that time delay may cause the loss of stability and other complicated dynamical behavior such as the periodic structure and bifurcation phenomenon [15, 18, 19].

Keeping these aspects in view, we will extend work in [5] by incorporating maturation delay for prey and gestation delay for predator into system (1). In this paper, prey is considered to be delayed by maturation delay $\tau_1 \geq 0$ due to population crowding [15, 16], and predator is assumed to be delayed by gestation delay $\tau_2 \geq 0$ [16]. Based on (1) and (2), a singular prey-predator fishery model is established as follows:

$$\dot{x}(t) = rx(t)\left(1 - \frac{x(t - \tau_1)}{K}\right) - \frac{\beta x(t) y(t)}{\alpha + x(t)} - E(t)$$

$$\cdot x(t),$$

$$\dot{y}(t) = y(t)\left[\frac{m\beta x(t - \tau_2)}{\alpha + x(t - \tau_2)} - d\right] - E(t) y(t),$$

$$0 = E(t)$$

$$\cdot \left[x(t)\frac{a_1}{b_1 + E(t) x(t)} + y(t)\frac{a_2}{b_2 + E(t) y(t)} - c\right]$$

$$- v,$$

where c represents cost of commercial harvesting, v is assumed to be economic interest of commercial harvesting, $E(t)$ represents commercial harvesting amount with respect to time t, and interpretations for other parameters and state variables share the same interpretations introduced in system (1) and (2). Furthermore, system (3) is investigated with initial conditions

$$x(\theta) \geq 0,$$

$$y(0) \geq 0,$$

$$E(0) \geq 0,$$

$$\theta \in [-\tau, 0], \quad \tau = \max\{\tau_1, \tau_2\}.$$

System (3) is rewritten as follows:

$$\Xi \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} rx(t)\left(1 - \frac{x(t - \tau_1)}{K}\right) - \frac{\beta x(t) y(t)}{\alpha + x(t)} - E(t) x(t) \\ y(t)\left[\frac{m\beta x(t - \tau_2)}{\alpha + x(t - \tau_2)} - d\right] - E(t) y(t) \\ E(t)\left[x(t)\frac{a_1}{b_1 + E(t) x(t)} + y(t)\frac{a_2}{b_2 + E(t) y(t)} - c\right] - v \end{bmatrix},$$

where the third equation in system (3) does not contain any differentiated variables; hence the third row in leading matrix $\Xi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has a zero row.

The rest of the sections of this paper are organized as follows. Positivity of any solution of system (3) and uniform persistence of system (3) are investigated in the second section. Local stability analysis of system (3) around interior equilibrium is discussed in the third section. The optimal system is derived and solved numerically based on an iterative method with Runge-Kutta fourth order scheme. Optimal harvest control problems associated with maximizing total discounted net revenues from the fishery and minimizing harvest cost are discussed in the fourth section. Numerical simulations are carried out to support theoretical analysis in the fifth section. Finally, this paper ends with a conclusion.

2. Positivity and Uniform Persistence

Theorem 1. *Any solutions of system (3) with initial conditions (4) are positive.*

Proof. Due to lemma in [20] and Theorem A.4 in [21], any solution of system (3) with initial conditions (4) exists uniquely and each component of solution remains within interval $[0, X_0]$ for some $X_0 > 0$. Standard and simple arguments show that any solution of system (3) always exists and stays positive. □

Theorem 2. *If $v > 0$, τ_1 and τ_2 are bounded, and the following inequalities (6) hold, and then system (3) with initial conditions (4) is uniformly persistent:*

$$m\beta Ke^{r\tau_1} > \alpha d,$$

$$a_1 b_2 \beta Ke^{r\tau_1} + a_2 b_1 r (\alpha + Ke^{r\tau_1}) > cb_1 b_2 \beta,$$

$$c\alpha b_1 > a_1 Ke^{r\tau_1} (m\beta Ke^{r\tau_1} - \alpha d),$$

$$cb_2 \alpha \beta > a_2 r (\alpha + Ke^{r\tau_1}) (m\beta Ke^{r\tau_1} - \alpha d).$$

Proof. By using Taylor series expansion [2], for $x(t)$ and $\tau_1 > 0$, $\tau_2 > 0$ we have

$$x(t - \tau_1) = x(t) - \tau_1 \frac{d}{dt}(x(t) - \tau_1 \dot{x}(t) + \cdots),$$

$$x(t - \tau_2) = x(t) - \tau_2 \frac{d}{dt}(x(t) - \tau_2 \dot{x}(t) + \cdots).$$

Consequently, it can be derived that

$$x(t - \tau_1) \leq x(t),$$

$$x(t - \tau_2) \leq x(t).$$

By using (8) and the first equation of system (3), it gives that

$$x(t) \leq x(t - \tau_1) e^{r\tau_1},$$

$$x(t) \leq rx(t)\left(1 - \frac{x(t)}{Ke^{r\tau_1}}\right),$$

which derives that

$$\limsup_{t \to \infty} x(t) \leq Ke^{r\tau_1} := Q_1.$$

If τ_1 is bounded, then it is easy to show that $Q_1 > 0$ and is bounded. According to (10), it can be derived that there exists $T_1 > 0$ such that $x(t) \leq Q_1$ holds for $t > T_1 + \tau_1$.

Based on the first equation of system (3) and practical interpretations of prey survival, it can be derived that there exists $T_2 > T_1 + \tau_1$ such that $y(t) \leq r[\alpha + x(t)]/\beta$, and

$$\limsup_{t \to \infty} y(t) \leq \frac{r[\alpha + x(t)]}{\beta} \leq \frac{r(\alpha + Ke^{r\tau_1})}{\beta} := Q_2. \quad (11)$$

Similarly, based on the second equation of system (3) and practical interpretations of predator survival, it is shown that there exists $T_3 > T_2$ such that

$$E(t) \leq \frac{m\beta x(t - \tau_2)}{\alpha} - d. \quad (12)$$

By using (8), if τ_2 is bounded then it follows that $E(t) \leq (m\beta x(t)/\alpha) - d$, and

$$\limsup_{t \to \infty} E(t) \leq \frac{m\beta Ke^{r\tau_1} - \alpha d}{\alpha} := Q_3. \quad (13)$$

If $m\beta Ke^{r\tau_1} > \alpha d$, then it can be derived that $Q_3 > 0$ and is bounded. According to (13), it is shown that there exists $T_4 > T_3$ such that $E(t) \leq Q_3$ holds for $t > T_4$.

By virtue of the third equation of system (3), when $v > 0$, it can be derived that there exists $T_5 > T_4$ such that

$$\frac{v}{E(t)} = x(t) \frac{a_1}{b_1 + E(t)x(t)} + y(t) \frac{a_2}{b_2 + E(t)y(t)} \\ - c \leq \frac{a_1 x(t)}{b_1} + \frac{a_2 y(t)}{b_2} - c, \quad (14)$$

which implies that

$$\liminf_{t \to \infty} E(t) \\ \geq \frac{vb_1 b_2 \beta}{a_1 b_2 \beta Ke^{r\tau_1} + a_2 b_1 r(\alpha + Ke^{r\tau_1}) - cb_1 b_2 \beta} := W_3. \quad (15)$$

If $a_1 b_2 \beta Ke^{r\tau_1} + a_2 b_1 r(\alpha + Ke^{r\tau_1}) > cb_1 b_2 \beta$, then it is easy to show that $W_3 > 0$ and is bounded, and then there exists $T_6 > T_5$ such that $E(t) \geq W_3$ holds for $t > T_6$.

By virtue of the third equation of system (3), when $v > 0$, it can be derived that there exists $T_7 > T_6$ such that

$$E(t)x(t) > \frac{b_1}{a_1}\left[c - \frac{a_2}{b_2}E(t)y(t)\right], \\ E(t)y(t) > \frac{b_2}{a_2}\left[c - \frac{a_1}{b_1}E(t)x(t)\right]. \quad (16)$$

According to (10), (11), and (13), it can be obtained that

$$\liminf_{t \to \infty} x(t) \\ \geq \frac{cb_2 \alpha\beta - a_2 r(\alpha + Ke^{r\tau_1})(m\beta Ke^{r\tau_1} - \alpha d)}{\alpha\beta} := W_1, \quad (17)$$

$$\liminf_{t \to \infty} y(t) \geq \frac{b_2[\alpha cb_1 - a_1 Ke^{r\tau_1}(m\beta Ke^{r\tau_1} - \alpha d)]}{a_2 b_1(m\beta Ke^{r\tau_1} - \alpha d)} \\ := W_2.$$

If the following two inequalities hold

$$c\alpha b_1 > a_1 Ke^{r\tau_1}(m\beta Ke^{r\tau_1} - \alpha d), \\ cb_2 \alpha\beta > a_2 r(\alpha + Ke^{r\tau_1})(m\beta Ke^{r\tau_1} - \alpha d), \quad (18)$$

then $W_1 > 0$ and $W_2 > 0$ and is bounded, and there exists $T_8 > T_7$ such that $x(t) \geq W_1$ and $y(t) \geq W_2$ hold for $t > T_8$.

Based on the above analysis, it can be concluded that system (3) with initial conditions (4) is uniformly persistent. □

3. Local Stability Analysis

Firstly, interior equilibrium P^* of system (3) in the case of $v > 0$ is computed as follows: $P^*(x^*, y^*, E^*) = (x^*, (\alpha + x^*)[Kd + r(K - x^*)]/K\beta - mx^*, m\beta x^*/(\alpha + x^*) - d)$, and x^* satisfy the following equation:

$$A_0 x^{*6} + A_1 x^{*5} + A_2 x^{*4} + A_3 x^{*3} + A_4 x^{*2} + A_5 x^* \\ + A_6 = 0, \quad (19)$$

where A_k $(k = 0, 1, \ldots, 6)$ are defined as follows:

$$A_0 = r(d - m\beta)^2(a_1 + a_2 - v + d - m\beta),$$

$$A_1 = (m\beta - d)^2[3\alpha dr \\ + (Kd + Kr - \alpha r - mK\beta)(m\beta - d + v - a_1 - a_2) \\ - cb_1 r],$$

$$A_2 = K\alpha(d + r)(m\beta - d)^3 - 3\alpha d(m\beta - d)^2(Kd \\ + Kr - \alpha r - mK\beta) - 3\alpha^2 d^2 r(m\beta - d) + (m\beta \\ - d)^2[cb_1(Kd + Kr - \alpha r - mK\beta) + K\beta cb_2 \\ - \alpha K(d + r)(a_1 + a_2 - v)] + \alpha(m\beta - d)(m\beta \\ - 3d)[(a_1 + a_2 - v)(Kd + Kr - \alpha r - mK\beta) \\ - cb_1 r],$$

$$A_3 = \alpha K(d + r)(cb_1 - 3\alpha d)(m\beta - d)^2 + r\alpha^3 d^3 \\ + Kcb_1 b_2 \beta(m\beta - d) + vb_1 b_2 K\beta + \alpha(m\beta - d)(Kd \\ + Kr - \alpha r - mK\beta)[3\alpha d^2 + cb_1(m\beta - 3d)] + (v \\ - a_1)b_2 K\beta(m\beta - d) + rd\alpha^2(3d - 2m\beta) \\ \cdot [(a_1 + a_2 - v)(Kd + Kr - \alpha r - mK\beta) - cb_1 r] \\ + \alpha(m\beta - d)(m\beta - 3d)[K\beta cb_2 \\ - \alpha K(d + r)(a_1 + a_2 - v)],$$

$$A_4 = \alpha^3 d^3 \left[3K (d + r) (m\beta - d) \right.$$

$$- d (Kd + Kr - \alpha r - mK\beta) \right] + b_2 K\beta\alpha^2 (m\beta - d)$$

$$\cdot (v - a_1) + cb_1\alpha k (d + r) (m\beta - d) (m\beta - 3d)$$

$$\cdot \left[K\beta cb_2 + cb_1 (Kr + Kd - a\alpha r - mK\beta) \right] + 2 (a_1$$

$$- v) b_2 K\alpha\beta (m\beta - d) + 2 (a_2 - v) b_1 d\alpha^2 K (d + r$$

$$- m\beta) + b_2 K\alpha\beta (v - a_1) (\alpha - 2) (m\beta - d),$$

$$A_5 = cb_1\alpha^3 dK (d + r) (3d - 2m\beta) + b_2 Kd\alpha^3 \beta (a_1$$

$$- v) + \alpha^3 d^3 \left[K\beta cb_2 - \alpha K (d + r) (a_1 + a_2 - v) \right.$$

$$+ cb_1 (Kr + Kd - \alpha r - mK\beta) \right] + b_1 (v - a_2)$$

$$\cdot \left[\alpha^3 K (d + r) (m\beta - d) - m\beta K - 2\alpha^2 dK (d + r) \right],$$

$$A_6 = cb_1\alpha^4 Kd^2 (d + r) + (a_2 - v) b_1\alpha^3 K (d + r)$$

$$+ K\beta b_1 b_2\alpha^3 (v - cd).$$

$$(20)$$

Furthermore, it follows from the practical interpretations of interior equilibrium that P^* exists provided that $y^* > 0$ and $E^* > 0$, which derives that

$$rx^{*2} + (\alpha r + Km\beta - Kd - Kr) x^* - \alpha K (d + r) < 0,$$

$$(m\beta - d) x^* - \alpha d > 0.$$

$$(21)$$

It follows from Routh-Hurwitz criterion [21], simple sufficient conditions for existence of P^* of system (3) are as follows:

$$0 < v < \min \left\{ a_1 + a_2 + d \right.$$

$$- m\beta, \frac{(1 - a_2) b_2\beta - d\alpha (d + r)}{b_2\beta - d - r} \left. \right\}.$$

$$(22)$$

Based on Jacobian matrix of system (3) evaluated around P^* and Ξ defined in (5), characteristic equation of system (3) around P^* is as follows:

$$\lambda^2 + B_1\lambda + B_2 + (B_3\lambda + B_4) e^{-\lambda\tau_1} + B_5 e^{-\lambda\tau_2} = 0, \quad (23)$$

where B_i, $i = 1, 2, \ldots, 5$, are defined as follows:

$$B_1 = -\frac{\beta x^* y^*}{(\alpha + x^*)^2} - \frac{a_2 b_2 E^* y^* (b_1 + E^* x^*)^2 + a_1 b_1 E^* x^* (b_2 + E^* y^*)^2}{a_1 b_1 x^* (b_2 + E^* y^*)^2 + a_2 b_2 y^* (b_1 + E^* x^*)^2 - c (b_1 + E^* x^*)^2 (b_2 + E^* y^*)^2},$$

$$B_2 = \frac{a_2 b_2\beta x^* y^{*2} E^* (b_1 + E^* x^*)^2 + a_1 b_1\beta x^* y^* E^* (\alpha + x^*) (b_2 + E^* y^*)^2}{(\alpha + x^*)^2 \left[a_1 b_1 x^* (b_2 + E^* y^*)^2 + a_2 b_2 y^* (b_1 + E^* x^*)^2 - c (b_1 + E^* x^*)^2 (b_2 + E^* y^*)^2 \right]},$$

$$B_3 = \frac{rx^*}{K}, \quad (24)$$

$$B_4 = -\frac{ra_2 b_2 x^* y^* E^* (b_1 + E^* x^*)^2}{K \left[a_1 b_1 x^* (b_2 + E^* y^*)^2 + a_2 b_2 y^* (b_1 + E^* x^*)^2 - c (b_1 + E^* x^*)^2 (b_2 + E^* y^*)^2 \right]},$$

$$B_5 = \frac{m\alpha\beta x^* y^*}{(\alpha + x^*)^2} \left[\frac{\beta}{\alpha + x^*} - \frac{a_2 b_2 E^* (b_1 + E^* x^*)}{a_1 b_1 x^* (b_2 + E^* y^*)^2 + a_2 b_2 y^* (b_1 + E^* x^*)^2 - c (b_1 + E^* x^*)^2 (b_2 + E^* y^*)^2} \right].$$

3.1. Case I: $\tau_1 > 0$, $\tau_2 = 0$.

When $\tau_1 > 0$ and $\tau_2 = 0$, (23) can be rewritten as follows:

$$\lambda^2 + B_1\lambda + B_2 + B_5 + (B_3\lambda + B_4) e^{-\lambda\tau_1} = 0. \quad (25)$$

By substituting $\lambda = i\omega_1$ ($\omega_1 > 0$) into (25) and separating real and imaginary parts, it gives that

$$\omega_1^2 - (B_2 + B_5) = B_4 \cos (\omega_1\tau_1) + B_3\omega_1 \sin (\omega_1\tau_1),$$

$$B_1\omega_1 = -B_3\omega_1 \cos (\omega_1\tau_1) + B_4 \sin (\omega_1\tau_1),$$

$$(26)$$

which derives that

$$\omega_1^4 + \left[B_1^2 - 2 (B_2 + B_5) - B_3^2 \right] \omega_1^2 + (B_2 + B_5)^2 - B_4^2$$

$$= 0.$$

$$(27)$$

According to Routh-Hurwitz criterion [21], a simple sufficient condition $(B_2 + B_5)^2 - B_4^2 < 0$ guarantees existence of positive root ω_1^* of (27). Hence, (25) has a pair of purely imaginary roots of the form $\pm i\omega_1^*$.

By eliminating $\sin(\omega_1\tau_1)$ from (26), it can be calculated that τ_{1k}^* corresponding to ω_1^* is as follows:

$$\tau_{1k}^* = \frac{1}{\omega_1^*} \arccos \left\{ \frac{(B_4 - B_1 B_3) \omega_1^{*2} - (B_2 + B_5) B_4}{B_4^2 + B_3^2\omega_1^{*2}} \right\}$$

$$+ \frac{2k\pi}{\omega_1^*},$$

$$(28)$$

where $k = 0, 1, \ldots$.

It follows from Butler's lemma [22] that system (3) is locally stable around P^* when $0 < \tau_1 < \tau_{10}^*, \tau_2 = 0$.

Based on the above analysis, local stability analysis of system (3) around P^* is concluded in Theorem 3.

Theorem 3. *If $(B_2 + B_5)^2 - B_4^2 < 0$, then system (3) is locally stable around P^* when $0 < \tau_1 < \tau_{10}^*, \tau_2 = 0$, and τ_{10}^* is defined in (28).*

3.2. Case II: $\tau_1 = 0, \tau_2 > 0$. When $\tau_1 = 0, \tau_2 > 0$, by using similar arguments in Section 3.1, local stability analysis of system (3) around P^* can be concluded in the following Theorem 4.

Theorem 4. *If $(B_2 + B_4)^2 - B_5^2 < 0$, then system (3) is locally stable around P^* when $\tau_1 = 0, 0 < \tau_2 < \tau_{20}^*$, and τ_{20}^* is defined as follows:*

$$\tau_{2k}^* = \frac{1}{\omega_2^*} \arccos \left\{ \frac{\omega_2^{*2} - B_2 - B_4}{B_5} \right\} + \frac{2k\pi}{\omega_2^*}, \quad (29)$$

where $k = 0, 1, \ldots$ and ω_2^* is a positive root of the following equation:

$$\omega_2^4 + \left[(B_1 + B_3)^2 - 2 (B_2 + B_4) \right] \omega_2^2 + (B_2 + B_4)^2 - B_5^2 \\ = 0. \quad (30)$$

3.3. Case III: $\tau_1 > 0, \tau_2 \in (0, \tau_{20}^*), \tau_1 \neq \tau_2$. In this subsection, τ_1 is considered to be bifurcation parameter and τ_2 is regarded as fixed value $\tilde{\tau}_2 \in (0, \tau_{20}^*)$, where τ_{20}^* is determined in Theorem 4.

Let $\lambda = i\sigma_1 (\sigma_1 > 0)$ be the root of (23), and two transcendental equations are obtained as follows:

$$\sigma_1^2 - (B_2 + B_5 \cos (\sigma_1 \tilde{\tau}_2)) \\ = B_4 \cos (\sigma_1 \tau_1) + B_3 \sigma_1 \sin (\sigma_1 \tau_1), \\ B_1 \sigma_1 - B_5 \sin (\sigma_1 \tilde{\tau}_2) \\ = B_4 \sin (\sigma_1 \tau_1) - B_3 \sigma_1 \cos (\sigma_1 \tau_1), \quad (31)$$

where $B_k, k = 1, 2, \ldots, 5$, are defined in (23).

By eliminating $\tilde{\tau}_2$ from (31), it can be derived that

$$\cos (\sigma_1 \tau_1) = \frac{f_{10} (\sigma_1, \tilde{\tau}_2)}{f_{12} (\sigma_1, \tilde{\tau}_2)}, \\ \sin (\sigma_1 \tau_1) = \frac{f_{11} (\sigma_1, \tilde{\tau}_2)}{f_{12} (\sigma_1, \tilde{\tau}_2)}, \quad (32)$$

where $f_{1k}(\sigma_1, \tilde{\tau}_2), k = 0, 1, 2$, are defined as follows:

$$f_{10} (\sigma_1, \tilde{\tau}_2) = (B_4 - B_1 B_3) \sigma_1^2 - B_2 B_4 \\ - B_4 B_5 \cos (\sigma_1 \tilde{\tau}_2) \\ + B_3 B_5 \sigma_1 \sin (\sigma_1 \tilde{\tau}_2), \\ f_{11} (\sigma_1, \tilde{\tau}_2) = B_3 \sigma_1^3 + (B_1 B_4 - B_2 B_3) \sigma_1$$

$$- B_3 B_5 \sigma_1 \cos (\sigma_1 \tilde{\tau}_2) \\ - B_4 B_5 \sin (\sigma_1 \tilde{\tau}_2), \\ f_{12} (\sigma_1, \tilde{\tau}_2) = B_4^2 + B_3^2 \sigma_1^2. \quad (33)$$

Based on $f_{1k}(\sigma_1, \tilde{\tau}_2), k = 0, 1, 2$, defined in (32), we define

$$F_1 (\sigma_1, \tilde{\tau}_2) = f_{10}^2 (\sigma_1, \tilde{\tau}_2) + f_{11}^2 (\sigma_1, \tilde{\tau}_2) - f_{12}^2 (\sigma_1, \tilde{\tau}_2) \\ = 0. \quad (34)$$

It is difficult to discuss root properties of transcendental equation (34) due to its complicated form. Without investigating the properties of roots of (34) in detail, it follows from simple computations that (23) has a pair of purely imaginary roots when (34) has at least two roots $0 < \sigma_{10} < \sigma_{11} < \cdots$.

By denoting $\sigma_{1w} = \max\{\sigma_{1k}, k = 0, 1, \ldots\}$ and τ_1 which is considered to be bifurcation parameter, we can obtain the corresponding critical value τ_{1w}:

$$\tau_{1k} = \frac{\psi_{1w} + 2k\pi}{\sigma_{1w}}, \\ \tau_{1w} = \min \{\tau_{1k}, k = 0, 1, \ldots\}, \quad (35)$$

where $k = 0, 1, \ldots, \psi_{1c} \in [0, 2\pi)$ satisfies the following equations:

$$\cos \psi_{1w} = \frac{f_{10} (\sigma_{1w}, \tilde{\tau}_2)}{f_{12} (\sigma_{1w}, \tilde{\tau}_2)}, \\ \sin \psi_{1w} = \frac{f_{11} (\sigma_{1w}, \tilde{\tau}_2)}{f_{12} (\sigma_{1w}, \tilde{\tau}_2)}. \quad (36)$$

Based on the above analysis, local stability analysis around P^* due to dynamical variation of bifurcation parameter τ_1 can be summarized as follows.

Theorem 5. *If (34) has at least two roots $0 < \sigma_{10} < \sigma_{11} < \cdots$, and there exists τ_{1w} defined in (35), then system (3) is locally stable around P^* when $0 < \tau_1 < \tau_{1w}, \tau_2 = \tilde{\tau}_2$.*

3.4. Case IV: $\tau_2 > 0, \tau_1 \in (0, \tau_{10}^*), \tau_1 \neq \tau_2$. In this subsection, τ_2 is considered to bifurcation parameter and τ_1 is regarded as fixed value $\tilde{\tau}_1 \in (0, \tau_{10}^*)$, where τ_{10}^* is determined in Theorem 3. By using similar arguments in Theorem 5 of this paper, local stability analysis around P^* can be concluded in Theorem 6.

Firstly, we define following equation:

$$F_2 (\sigma_2, \tilde{\tau}_1) = f_{20}^2 (\sigma_2, \tilde{\tau}_1) + f_{21}^2 (\sigma_2, \tilde{\tau}_1) - f_{22}^2 (\sigma_2, \tilde{\tau}_1) \\ = 0, \quad (37)$$

where $f_{2k}(\sigma_2, \tilde{\tau}_2), k = 0, 1, 2$, are defined as follows:

$$f_{20} (\sigma_2, \tilde{\tau}_1) = \sigma_2^2 - B_2 - B_4 \cos (\sigma_2 \tilde{\tau}_1) \\ - B_3 \sigma_2 \sin (\sigma_2 \tilde{\tau}_1),$$

$$f_{21}(\sigma_2, \tilde{\tau}_1) = B_1\sigma_2 + B_3\sigma_2 \cos(\sigma_2\tilde{\tau}_1)$$

$$- B_4 \sin(\sigma_2\tilde{\tau}_1),$$

$$f_{22}(\sigma_2, \tilde{\tau}_1) = B_5.$$

(38)

Theorem 6. *If (37) has at least two roots* $0 < \sigma_{20} < \sigma_{21} < \cdots$, *there exists a critical delay* τ_{2w}, *and then system (3) is locally stable around* P^* *when* $0 < \tau_2 < \tau_{2w}, \tau_1 = \tilde{\tau}_1$, *where* $\tau_{2w} = \min\{(\psi_{2w} + 2k\pi)/\sigma_{2w}, k = 0, 1, \ldots\}, \sigma_{2w} = \max\{\sigma_{2k}, k = 0, 1, \ldots\}$ *and* $\psi_{2w} \in [0, 2\pi)$ *satisfies the following equations:*

$$\cos\psi_{2w} = \frac{f_{20}(\sigma_{2w}, \tilde{\tau}_1)}{f_{22}(\sigma_{2w}, \tilde{\tau}_1)},$$

$$\sin\psi_{2w} = \frac{f_{21}(\sigma_{2w}, \tilde{\tau}_1)}{f_{22}(\sigma_{2w}, \tilde{\tau}_1)}.$$

(39)

4. Optimal Control Problem

In this section, τ_1 and τ_2 are considered to be a fixed value $\hat{\tau}_1 \in (0, \tau_{1w})$ and $\hat{\tau}_2 \in (0, \tau_{20}^*)$, respectively; $\hat{\tau}_1 \neq \hat{\tau}_2$, τ_{1w}, and τ_{20}^* have been determined in Theorems 5 and 4, respectively. It follows from Theorem 5 that system (3) with $\hat{\tau}_1$ and $\hat{\tau}_2$ is locally stable around P^*, and we will investigate optimal control $E(t)$ with maximizing total discounted net revenues and minimizing commercial harvesting cost during $[t_0, t_f]$. The optimal harvest control problem is formulated as follows:

$$J(E) = \int_{t_0}^{t_f} e^{-\delta t} \left[E(t) x(t) \frac{a_1}{b_1 + E(t) x(t)} \right.$$

$$\left. + E(t) y(t) \frac{a_2}{b_2 + E(t) y(t)} - cE(t) \right] dt,$$

(40)

where δ is instantaneous annual discount rate [7], and optimal control problem (40) is subject to the first and second delayed differential equations of system (3) with $\hat{\tau}_1$ and $\hat{\tau}_2$. Furthermore, convexity of $J(E)$ with respect to $E(t)$, linearity of the delayed differential equations with $E(t)$, and range values compactness of state variables can be uniformly investigated when discussing existence of optimal harvest control $E(t)$. By assuming E_δ as an optimal harvest control with corresponding optimal state variables x_δ and y_δ of system (3) with $\hat{\tau}_1$ and $\hat{\tau}_2$, and E_δ satisfies the following condition:

$$J(E_\delta) = \max\{J(E(t)) \mid E(t) \in U\},$$

(41)

where U represents control set that is defined as follows:

$$U = \{E : [t_0, t_f]$$

$$\longrightarrow [0, E_{\max}] \mid E \text{ is Lebesgue measurable.}\}.$$

(42)

The Hamiltonian function $H(t)$ associated with system (3) with $\hat{\tau}_1$ and $\hat{\tau}_2$ is

$$H(t) = e^{-\delta t} \left[E(t) x(t) \frac{a_1}{b_1 + E(t) x(t)} \right.$$

$$\left. + E(t) y(t) \frac{a_2}{b_2 + E(t) y(t)} - cE(t) \right] + \lambda_1(t)$$

$$\cdot \left[rx(t) \left(1 - \frac{x_{\hat{\tau}_1}}{K} \right) - \frac{\beta x(t) y(t)}{\alpha + x(t)} - E(t) x(t) \right]$$

$$+ \lambda_2(t) \left[y(t) \left(\frac{m\beta x_{\hat{\tau}_2}}{\alpha + x_{\hat{\tau}_2}} - d \right) - E(t) y(t) \right],$$

(43)

where $x_{\hat{\tau}_1} := x(t - \hat{\tau}_1)$, $x_{\hat{\tau}_2} := x(t - \hat{\tau}_2)$, and $\lambda_1(t), \lambda_2(t)$ are adjoint functions.

Theorem 7. *There exists an optimal harvest* E_δ *which is determined by (44); furthermore, there exist adjoint functions* $\lambda_1(t)$, $\lambda_2(t)$ *satisfying system (47) with* $\lambda_k(t_f) = 0, k = 1, 2$.

$$\frac{a_1 b_1 x_\delta}{(b_1 + E_\delta x_\delta)^2} + \frac{a_2 b_2 y_\delta}{(b_2 + E_\delta y_\delta)^2}$$

$$= c + [x_\delta \lambda_1(t) + y_\delta \lambda_2(t)] e^{\delta t}.$$

(44)

Proof. By differentiating $H(t)$, the adjoint system can be obtained as follows:

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}(t) - \chi_{1[t_0, t_f - \hat{\tau}_1]}(t) \frac{\partial H}{\partial x_{\hat{\tau}_1}}(t + \hat{\tau}_1)$$

$$- \chi_{2[t_0, t_f - \hat{\tau}_2]}(t) \frac{\partial H}{\partial x_{\hat{\tau}_2}}(t + \hat{\tau}_2),$$

(45)

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y}(t),$$

where

$$\chi_{1[t_0, t_f - \hat{\tau}_1]}(t) = \begin{cases} 1, & t \in [0, t_f - \hat{\tau}_1], \\ 0, & \text{otherwise,} \end{cases}$$

$$\chi_{2[t_0, t_f - \hat{\tau}_2]}(t) = \begin{cases} 1, & t \in [0, t_f - \hat{\tau}_2], \\ 0, & \text{otherwise.} \end{cases}$$

(46)

By using the optimal conditions, (45) can be rewritten as follows:

$$\frac{d\lambda_1}{dt}$$

$$= -e^{-\delta t} \frac{a_1 b_1 E(t)}{(b_1 + E(t) x(t))^2}$$

$$+ \lambda_1(t) \left[r\left(\frac{x(t)}{K} - 1 \right) + \frac{\alpha\beta y(t)}{(\alpha + x(t))^2} + E(t) \right]$$

$$+ \chi_{1[t_0, t_f - \hat{\tau}_1]}(t) \frac{rx(t)\lambda_2(t + \hat{\tau}_1)}{K}$$

$$- \chi_{2[t_0, t_f - \hat{\tau}_2]}(t) \frac{m\alpha\beta y(t)}{(\alpha + x(t))^2} \lambda_2(t + \hat{\tau}_2),$$

$$\frac{d\lambda_2}{dt}$$

$$= -e^{-\delta t} \frac{a_2 b_2 E(t)}{(b_2 + E(t)y(t))^2} + \frac{\beta x(t)\lambda_1(t)}{\alpha + x(t)}$$

$$+ \lambda_2(t) \left(E(t) + d - \frac{m\beta x(t)}{\alpha + x(t)} \right). \tag{47}$$

In optimal control problem (40), there is not terminal cost and final state is free. Consequently, transversality conditions (bounded conditions) for adjoint functions are as follows: $\lambda_k(t_f) = 0$, $k = 1, 2$.

It is easy to obtain characterization of optimal harvest control E_δ, which derives that $\partial H / \partial E|_{E=E_\delta} = 0$. Consequently, it follows from simple computations that there exists an optimal harvest E_δ determined by

$$\frac{a_1 b_1 x_\delta}{(b_1 + E_\delta x_\delta)^2} + \frac{a_2 b_2 y_\delta}{(b_2 + E_\delta y_\delta)^2}$$

$$= c + [x_\delta \lambda_1(t) + y_\delta \lambda_2(t)] e^{\delta t}, \tag{48}$$

where $\lambda_1(t)$, $\lambda_2(t)$ satisfies system (47) with $\lambda_k(t_f) = 0$, $k = 1, 2$. □

Remark 8. It follows from similar arguments in Theorem 7 that symmetric optimal control problem on cases in Section 3.4 can be also obtained, which are omitted in this paper. τ_1 and τ_2 can be set to be a fixed value $\hat{\tau}_1 \in (0, \tau_{10}^*)$ and $\hat{\tau}_2 \in (0, \tau_{2w})$, respectively; $\hat{\tau}_1 \neq \hat{\tau}_2$, τ_{10}^*, and τ_{2w} are determined in Theorems 3 and 6, respectively.

In the following part, the optimal system is numerically solved based on an iterative method with Runge-Kutta fourth order scheme. Firstly, by assuming that there exist a step size $\rho > 0$ and integers i, j, n with $\tau_1 = i\rho, \tau_2 = j\rho$, and $t_f - t_0 = n\rho$. Secondly, we initialize optimal harvest control with a given initial value $E_0 = 0.09$. By using forward difference approximation, we solve the first and second differential equation of system (3) with initial conditions $x_0 = 0.38$, $y_0 = 0.43$. Thirdly, by using backward difference approximation, we solve the adjoint functions (47) with transversality conditions. Finally, the optimal harvest control is updated by values of the state and adjoint variables, and the updated optimal harvest control is replaced with the initial value given in the second step. This process will be repeated until successive iterates of harvest control values become sufficiently close.

By utilizing combinations of forward and backward difference approximations, it can be derived that for $k = 0, \ldots, n-1$

$$x_{k+1} = x_k + \rho \left[rx_k \left(1 - \frac{x_{k-i}}{K} \right) - \frac{\beta x_k y_k}{\alpha + x_k} - E_k x_k \right],$$

$$y_{k+1} = y_k + \rho \left[y_k \left(\frac{m\beta x_{k-j}}{\alpha + x_{k-j}} - d \right) - E_k y_k \right],$$

$$\lambda_1^{n-k-1} = \lambda_1^{n-k} - \rho e^{-\delta(n-k)} \frac{a_1 b_1 E_{k+1}}{(b_1 + E_{k+1} x_{k+1})^2}$$

$$+ \rho \lambda_1^{n-k} \left[r \left(\frac{x_{k+1}}{K} - 1 \right) + \frac{\alpha\beta y_{k+1}}{(\alpha + x_{k+1})^2} + E_{k+1} \right]$$

$$+ \rho \left[\chi_{1[t_0, t_f - \hat{\tau}_1]} \lambda_1^{n-k+i} \frac{rx_{k-j}}{K} \right.$$

$$\left. - \chi_{2[t_0, t_f - \hat{\tau}_2]} \lambda_2^{n-k+j} \frac{m\alpha\beta y_{k+1}}{(\alpha + x_{k+1})^2} \right],$$

$$\lambda_2^{n-k-1} = \lambda_2^{n-k} - \rho \left[e^{-\delta(n-k)} \frac{a_2 b_2 E_{k+1}}{(b_2 + E_{k+1} y_{k+1})^2} \right.$$

$$\left. - \frac{\beta x_{k+1} \lambda_1^{n-k}}{\alpha + x_{k+1}} - \lambda_2^{n-k} \left(E_{k+1} + d - \frac{m\beta x_{k+1}}{\alpha + x_{k+1}} \right) \right]. \tag{49}$$

In order to show effectiveness of optimal harvest control designed in Theorem 7, parameters values of system (3) are partially from [5], $r = 2.5$, $K = 1.4$, $m = 0.6$, $\beta = 1$, $\alpha = 1.4$, $d = 0.09$, $a_1 = 1$, $a_2 = 0.5$, $b_1 = 1.2$, $b_2 = 1$, $c = 0.05$, $\delta = 0.01$, $t_0 = 0$, and $t_f = 120$ with appropriate units. By using given values of parameters and simple computations, interior equilibrium $P^*(0.2471, 3.3911, 0.7467)$ exists provided that $0 < v < 0.1491$. In the following part, $v = 0.075$ is utilized in the numerical simulations, which is arbitrarily selected within $(0, 0.1491)$ and is enough to merit theoretical analysis obtained in this paper. It follows from Theorem 3, Theorem 7, and simple computations that system (3) is locally stable around $(0.2471, 3.3911, 0.7467)$ when $0 < \tau_1 < \tau_{10}^* = 0.5632$ and $\tau_2 = 0$. By using similar arguments and Theorems 4 and 7, it is revealed that system (3) is locally stable around $(0.2471, 3.3911, 0.7467)$ when $0 < \tau_2 < \tau_{20}^* = 0.3894$ and $\tau_1 = 0$. As analyzed in Section 3.3, $\hat{\tau}_2 = 0.15$ is arbitrarily selected within $(0, \tau_{20}^*)$, that is, $(0, 0.3894)$, and it follows from Theorems 5 and 7 that system (3) is locally stable around $(0.2471, 3.3911, 0.7467)$ when $0 < \tau_1 < \tau_{1w} = 0.4126$ and $\hat{\tau}_2 = 0.15$. As analyzed in Section 3.4, $\hat{\tau}_1 = 0.28$ is arbitrarily selected within $(0, \tau_{10}^*)$, that is, $(0, 0.5632)$, and it follows from Theorems 6 and 7 that system (3) is locally stable around $(0.2471, 3.3911, 0.7467)$ when $\hat{\tau}_1 = 0.28$ and $0 < \tau_2 < \tau_{2w} = 0.2219$. Numerical simulations are made to investigate dynamic effects of double time delays on variation of optimal prey, predator biomass, and optimal harvest control, which can be found in Figures 1 and 2, respectively. The initial values of simulation work made in Figure 1 are as follows: $x(0) = 0.28$, $y(0) = 0.31$, $E(0) = 0.76$, and initial values of simulation work made in Figure 2 are as follows: $x(0) = 0.42$, $y(0) = 0.21$, $E(0) = 0.81$.
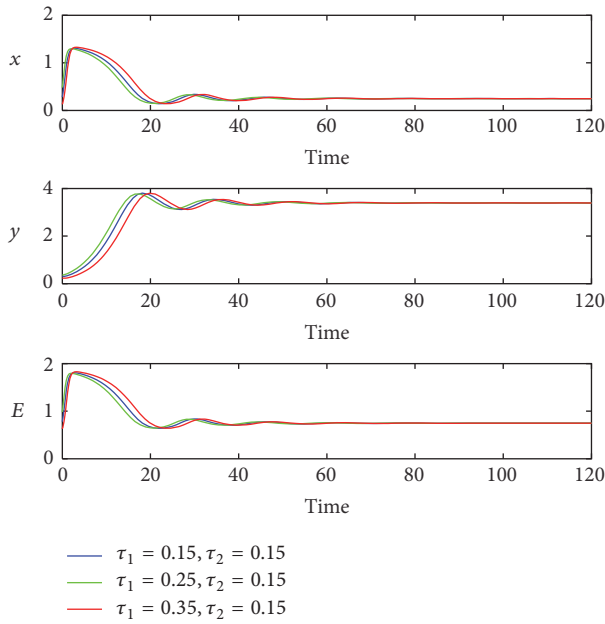
FIGURE 1: Stable dynamical variation of optimal prey, predator biomass and optimal harvest control with the increasing time under $\tau_1 = 0.15, 0.25, 0.35$ and $\tau_2 = 0.15$, respectively.
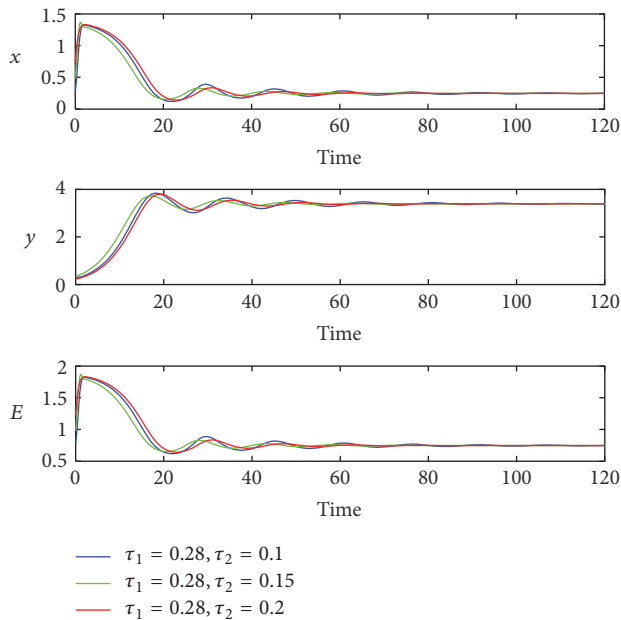

FIGURE 2: Stable dynamical variation of optimal prey, predator biomass and optimal harvest control with the increasing time under $\tau_1 = 0.28$ and $\tau_2 = 0.1, 0.15, 0.2$, respectively.

5. Conclusion

By incorporating maturation delay for prey population and gestation delay for predator population, we extend work done in [5]. In this paper, a singular prey-predator fishery model with double time delays is established. Fishing efforts are introduced to commercially harvest prey population and predator population, which are developed as control instruments to investigate optimal utilization of prey-predator fishery resource. Positivity of solutions and uniform persistence of system are discussed in Theorems 1 and 2. By analyzing associated characteristic transcendental equation, local stability around interior equilibrium is discussed due to combined variations of double time delays, which can be found in Theorem 3 to Theorem 6. Furthermore, an optimal harvest control is designed in Theorem 7. By using the proposed mathematical model and dynamical analysis, authors aim to obtain some results which are theoretically beneficial to discussing optimal harvesting strategies and sustainability mechanism of harvested prey-predator fishery system with double time delays. Furthermore, the theoretical results may be potentially constructive for administrative agencies to formulate regulatory policies to preserve economic optimality through ensuring global sustainability.

Competing Interests

All authors of this article declare that there is no conflict of interests regarding the publication of this article. We have no proprietary, financial, professional, or other personal interest of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, this article.

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