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SCATTERING BY PENETRABLE ACOUSTIC TARGETS

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An acoustic target of constant density ρ_t and variable index of refraction is imbedded in a surrounding acoustic fluid of constant density ρ_a. A time harmonic wave propagating in the surrounding fluid is incident on the target. We consider two limiting cases of the target where the parameter $\varepsilon \equiv \rho_a/\rho_t \to 0$ (the nearly rigid target) or $\varepsilon \to \infty$ (the nearly soft target). When the frequency of the incident wave is bounded away from the 'in-vacuo' resonant frequencies of the target, the resulting scattered field is essentially the field scattered by the rigid target for $\varepsilon = 0$ or the soft target if $\varepsilon \to \infty$. However, when the frequency of the incident wave is near a resonant frequency, the target oscillates and its interaction with the surrounding fluid produces peaks in the scattered field amplitude. In this paper we obtain asymptotic expansions of the solutions of the scattering problems for the nearly rigid and the nearly soft targets as $\varepsilon \to 0$ or $\varepsilon \to \infty$, respectively, that are uniformly valid in the incident frequency. The method of matched asymptotic expansions is used in the analysis. The outer and inner expansions correspond to the incident frequencies being far or near to the resonant frequencies, respectively. We have applied the method only to simple resonant frequencies, but it can be extended to multiple resonant frequencies. The method is applied to the incidence of a plane wave on a nearly rigid sphere of constant index of refraction. The far field expressions for the scattered fields, including the total scattering cross-sections, that are obtained from the asymptotic method and from the partial wave expansion of the solution are in close agreement for sufficiently small values of ε.

1. Introduction

The target is an acoustic fluid of constant density ρ_t and variable index of refraction $n_t(\boldsymbol{x})$, where $\boldsymbol{x}$ is the coordinate vector with components (x, y, z). It is embedded in an acoustic fluid with constant density ρ_a and constant index of refraction n_a. We assume that the density ratio $\varepsilon \equiv \rho_a/\rho_t$ is either small or large so that the target is either nearly rigid or nearly soft, respectively.

A time harmonic wave propagating in the surrounding fluid is incident on the target. If the frequency of the incident wave is bounded away from the 'in-vacuo' or the resonant frequencies of the target, then the scattered field is essentially either the rigidly scattered or the softly scattered field depending on whether the target is either nearly rigid or nearly soft. We refer to the fields scattered by either the perfectly rigid or the perfectly soft targets as the background fields. However, if the incident frequency is close to one of the resonant frequencies of the target, then the oscillations of the target will considerably alter the scattered fields. The resonant frequencies for the nearly rigid (soft) target are the eigenfunctions of the target with soft (rigid) boundary conditions.

In this paper we employ the method of matched asymptotic expansions [1] to obtain asymptotic expansions of the solutions of the scattering problems as $\varepsilon \to 0$, that are uniformly valid in the frequency of the incident wave. In the analysis we assume that the background fields are 'known'. That is, for targets with simple geometries the solutions are obtained analytically by e.g. partial wave (eigenfunctions) expansions. More generally, we assume that the background fields are known accurately from numerical

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solutions of the perfectly rigid or perfectly soft scattering problems. Then our method shows how to construct asymptotically the solutions of the penetrable target scattering problem when ε is small, or large. Of course, approximate solutions of the penetrable target problem can also be obtained by numerical computation. However, then the resulting algebraic system is ill-conditioned for incident frequencies near the resonant frequencies of the target for small ε. This results in computational difficulties and inaccuracies. Our method analyzes the near resonance structure of the solution resulting in a specific formula for the solution, thus obviating near-resonant numerical difficulties and reducing the amount of computation as we discuss briefly in Section 8.

The scattering problems are formulated in Section 2. The outer expansion of the method of matched asymptotic expansions, which is obtained in Section 3 for the nearly rigid target, is valid if the frequency of the incident wave is bounded away from all of the resonant frequencies of the target. This expansion becomes unbounded as the frequency approaches a resonant frequency. The inner expansions, for the nearly rigid target which are obtained in Section 4, are valid near the resonant frequencies. Finally, the composite expansion of the method of matched asymptotic expansions, which is obtained in Section 5 for the nearly rigid target by appropriately combining the inner and outer expansions, yields the desired expansion that is uniform in the incident frequency. The method is applied in Section 7 to the scattering of a plane wave by a nearly rigid spherical target with a constant index of refraction. The present asymptotic representations are compared with the partial wave expansion for this scattering problem, and the results are summarized in Figs. 2–5. Finally, the results of the asymptotic analysis for the nearly soft target $\varepsilon \to \infty$ are summarized in Section 9.

We have previously employed this method to analyze the scattering of acoustic waves from baffled membranes [2]. The idea of solving scattering problems approximately for baffled flexible surfaces using the small density ratio between the fluid and the target was first employed by Leppington [3], but he used a different method of analysis.

The methods of geometrical acoustics and the geometrical theory of diffraction [4, 5] are frequently used to obtain asymptotic expansions for the fields scattered from penetrable targets, such as we are considering in this paper. These expansions are valid for *high* frequency incident waves and for *arbitrary* density ratios. The expansions that are obtained in this paper are for *small* density ratios but they are *uniform* in the incident frequency. A variety of iterative methods, such as Born's method [6], and numerical methods [7] are employed to solve the scattering problems for low frequency incident waves and arbitrary density ratios. The method of matched asymptotic expansions has also been used [8] to solve low frequency scattering problems. There the expansions are valid for small values of the incident frequency and they are uniformly valid in the spatial variables.

When $n_t \equiv$ constant and the shape of the scatterer is simple, e.g. a cyclinder or a sphere, then a representation of the scattered field can be obtained by a partial wave, or eigenfunction expansion. The linear approximation method of nuclear resonance theory has been applied to each term of this partial wave expansion for related problems of scattering by elastic targets to study the resonant interaction of the incident field with the target for incident frequencies near the simple resonant frequencies of the target; see [9] for an extensive review of this work. The resulting resonance approximation, which does not require small density ratios, is equivalent to our inner expansion. However, by numerical evaluations of the partial wave series for special problems it was previously observed [9] that for small density ratios the scattered field was essentially given by the background fields if the incident frequencies are bounded away from the resonant frequencies of the target. Near the resonant frequencies the amplitude of the scattered field is sharply peaked as suggested by the nuclear resonance approximation. The asymptotic analysis in this paper clearly reveals this structure of the scattered field and its dependence on the incident frequency for arbitrary geometries of the targets and for arbitrary variations in $n_t(\boldsymbol{x})$. Thus, explicit partial wave representations of the scattered field are not required to determine its features. Furthermore, an asymptotic approximation is obtained for the scattered field that is uniformly valid in the incident frequency, which is not obtained by the nuclear resonance theory. Finally, the asymptotic analysis suggests more efficient procedures for the numerical solution of the scattering problem, as we have already mentioned.

2. Formulation

The target, or scatterer, is an acoustic fluid that occupies the three dimensional region V, with boundary B. In V the constant fluid density is ρ_t and the index of refraction is $n_t(\boldsymbol{x})$, where the dimensionless space coordinates $\boldsymbol{x} = (x, y, z)$ are obtained from the dimensional coordinates by dividing them by a characteristic length, L of the target, such as its maximum 'diameter'. The surrounding acoustic fluid, has constant density ρ_a, constant index of refraction n_a, and a constant sound speed C_a. We assume that the incident and the resulting scattered fields are proportional to $e^{-i\omega t}$, where ω is the specified circular frequency of the incident field. This factor is omitted in the remainder of the paper.

We denote the acoustic velocity potentials in V and exterior to V by $\Phi^{(t)}$ and $\Phi^{(a)}$, respectively. They satisfy the Helmholtz equations

$$\Delta\Phi^{(t)} + k^2\mu^2(\boldsymbol{x})\Phi^{(t)} = 0 \quad \text{for } \boldsymbol{x} \in V, \tag{2.1a}$$

$$\Delta\Phi^{(a)} + k^2\Phi^{(a)} = 0 \quad \text{for } \boldsymbol{x} \in \bar{V}. \tag{2.1b}$$

In (2.1) Δ is the Laplacian, $\bar{V}$ denotes the exterior of V, $\mu \equiv n_t/n_a$, and k is the dimensionless wave number defined by

$$k \equiv \omega L/C_a. \tag{2.2}$$

The boundary conditions corresponding to (2.1) are that, $\Phi^{(a)}$ satisfies an appropriate condition as $r \equiv |\boldsymbol{x}| \to \infty$, and on the interface B

$$\Phi^{(t)} = \varepsilon\Phi^{(a)}, \qquad \Phi_n^{(t)} = \Phi_n^{(a)} \quad \text{for } \boldsymbol{x} \in B. \tag{2.3a,b}$$

Here, the subscript n denotes the normal derivative taken along the outward unit normal $\boldsymbol{n}$ to V on B and ε is defined by

$$\varepsilon = \rho_a/\rho_t. \tag{2.4}$$

The conditions (2.3a, b) imply that the acoustic pressures and velocities normal to B are continuous across B. For the limiting cases $\varepsilon \to 0$ and $\varepsilon \to \infty$ the limit problems correspond respectively to either an acoustically rigid or an acoustically soft target, as we demonstrate. Thus, for ε small (large) we refer to the target as nearly rigid (nearly soft).

A time harmonic source with velocity potential $\Phi^{(I)}(\boldsymbol{x}, k)$ propagates in $\bar{V}$ and is incident on V. Then we express the total acoustic fields in V and $\bar{V}$ by

$$\Phi^{(t)} = \phi^{(t)} \quad \text{for } \boldsymbol{x} \in V, \qquad \Phi^{(a)} = \Phi_b + \phi^{(a)} \quad \text{for } \boldsymbol{x} \in \text{V}, \tag{2.5a}$$

where the background field Φ_b is defined by

$$\Phi_b \equiv \Phi^{(I)} + \Phi^{(R)}, \tag{2.5b}$$

and $\phi^{(a)}$ and $\phi^{(t)}$ are the scattered acoustic potentials. In addition, $\Phi^{(R)}$ is the acoustic potential in $\bar{V}$ that results from the scattering of the incident field from either a rigid or soft target depending on whether we consider $\varepsilon \to 0$ or $\varepsilon \to \infty$. This is, $\Phi^{(R)}$ satisfies (2.1b), the radiation condition as $r \equiv |x| \to \infty$, and either

$$\Phi_n^{(R)} = -\Phi_n^{(I)} \quad \text{for } x \in B, \text{ and a nearly rigid target} \tag{2.6a}$$

or

$$\Phi^{(R)} = -\Phi^{(I)} \quad \text{for } x \in B, \text{ and a nearly soft target.} \tag{2.6b}$$

We assume that Φ_b is either known explicitly, such as from a partial wave expansion, or it is known approximately from an accurate numerical evaluation, such as by the T matrix method [7]. We can equivalently assume that the Green's function for (2.1b), which satisfies the radiation condition as $r \to \infty$, is known for both the Neumann and Dirichlet problems. Denoting them by $G(x, \xi; k)$ and $\hat{G}(x, \xi; k)$ respectively, we have $G_n = \hat{G} \equiv 0$ for $x \in B$. In addition, the reflected fields in the surrounding fluid are given for the rigid and soft targets in terms of the incident field, respectively by,

$$\Phi^{(R)}(x; k) = -\iint_B G(x, \xi; k)\Phi_\nu^{(I)}(\xi)\, d\Omega(\xi), \qquad \Phi^{(R)}(x; k) = \iint_B \hat{G}_\nu(x, \xi; k)\Phi^{(I)}(\xi)\, d\Omega(\xi) \tag{2.7}$$

where $d\Omega(\xi)$ is a differential element of surface area on B and the subscripts ν denote normal derivatives in the ξ variable.

By inserting (2.5) into (2.1) and (2.3), and using (2.6) we conclude that the scattered potentials must satisfy

$$\Delta\phi^{(t)} + k^2\mu^2\phi^{(t)} = 0 \quad \text{for } x \in V, \tag{2.8}$$

$$\Delta\phi^{(a)} + k^2\phi^{(a)} = 0 \quad \text{for } x \in \bar{V}, \tag{2.9}$$

and for $x \in B$:

$$\phi_n^{(a)} = \phi_n^{(t)}, \quad \phi^{(t)} = \varepsilon[\Phi_b + \phi^{(a)}] \qquad \text{for a nearly rigid target} \tag{2.10}$$

$$\phi_n^{(t)} = \phi_n^{(a)} + \frac{\partial}{\partial n}\Phi_b, \quad \phi^{(a)} = \varepsilon^{-1}\phi^{(t)} \quad \text{for a nearly soft target.} \tag{2.11}$$

We shall now reformulate the scattering problems (2.8)–(2.11) as a 'boundary value problem' for the target potentials. The 'boundary conditions' are integral equations for $\phi^{(t)}$ on B. To do this we first express the exterior potential $\phi^{(a)}$ in terms of the interior potential $\phi^{(t)}$. By employing the exterior Green's functions G and $\hat{G}$ and the first equation in (2.10) and the last equation (2.11) we then obtain from the integral representation of $\phi^{(a)}$ that

$$\phi^{(a)}(x) = \iint_B G(x, \xi; k)\phi_\nu^{(t)}\, d\Omega(\xi) \quad \text{for } x \in \bar{V}, \text{ for a nearly rigid target,} \tag{2.12}$$

$$\phi^{(a)}(x) = -\frac{1}{\varepsilon}\iint_B \hat{G}_\nu(x, \xi; k)\phi^{(t)}\, d\Omega(\xi) \quad \text{for } x \in \bar{V}, \text{ for a nearly soft target.} \tag{2.13}$$

Then we insert (2.12)–(2.13) into the second equation (2.10) and the first equation in (2.11), respectively, to obtain

$$\phi^{(t)} = \Phi_b + \varepsilon \iint_B G\phi_\nu^{(t)}\, d\Omega(\xi) \quad \text{for } x \in B, \text{ for a nearly rigid target,} \tag{2.14}$$

$$\phi_n^{(t)} = \frac{\partial}{\partial n}\Phi_b - \frac{1}{\varepsilon}\frac{\partial}{\partial n}\iint_B \hat{G}_\nu \phi^{(t)}\, d\Omega(\xi) \quad \text{for } x \in B, \text{ for a nearly soft target.} \tag{2.15}$$

Consequently, (2.14) and (2.15) are integral equations for $\phi^{(t)}$ on the interface B. Thus, the nearly rigid and nearly soft scattering problems are reformulated as, solving (2.8) in V subject to either (2.14) or (2.15) on B. When $\phi^{(t)}$ is determined, we use either (2.12) or (2.13) to find $\phi^{(a)}$.

In Sections 3–5 we determine uniform asymptotic expansions as $\varepsilon \to 0$ of the nearly rigid scattering problem. Similar asymptotic expansions as $\varepsilon \to 0$ of the nearly soft scattering problem are presented in Section 8.

3. The nearly rigid scattering problem: The outer expansion

We seek an asymptotic expansion as $\varepsilon \to 0$ of the solution to the nearly rigid scattering problem in the form

$$\phi^{(t)}(x; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j \phi_j^{(t)}(x; k). \tag{3.1}$$

The coefficients $\phi_j^{(t)}$, $j = 0, 1, \ldots$, are determined by inserting (3.1) into (2.8) and (2.14), and equating coefficients of the same powers of ε. Thus, we find that $\phi_j^{(t)}$ satisfy the Helmholtz equation (2.8) and that for $x \in B$ they satisfy the conditions

$$\phi_j^{(t)} = \Phi_b \delta_{j1} + \iint_B G(x, \xi; k) \frac{\partial \phi_{j-1}^{(t)}}{\partial \nu}\, d\Omega, \quad j = 0, 1, 2, \ldots, \tag{3.2}$$

where $\phi_{-1}^{(t)} \equiv 0$, and δ_{jk} is the Kronecker delta function.

To analyze these problems, we first denote the eigenvalues and eigenfunctions of

$$\Delta\psi + k^2\mu^2(x)\psi = 0 \quad \text{for } x \in V, \qquad \psi = 0, \quad x \in B \tag{3.3}$$

by k_i and $\psi_i(x)$, $i = 1, 2, 3, \ldots$. They correspond to the resonant frequencies and normal modes of the target immersed in a vacuum, i.e. the boundary conditions in (3.3) correspond to a soft scatterer. We assume that these eigenfunctions form a complete set which are normalized by the conditions

$$\iiint_V \mu^2(x)\psi_j(x)\psi_i(x)\, dx = \delta_{ij}, \quad i, j = 1, 2, 3, \ldots. \tag{3.4}$$

Since $\phi_0^{(t)}$ satisfies (2.8) and since $\phi_0^{(t)} = 0$ on B, from (3.2) with $j = 0$, we conclude that $\phi_0^{(t)} \equiv 0$ in V if $k \neq k_i$, $i = 1, 2, 3, \ldots$. It then follows from (2.12) that $\phi_0^{(a)} \equiv 0$ for $x \in \bar{V}$. Furthermore, it follows from (3.2) with $j = 1$ that $\phi_1^{(t)}$, which is a solution of (2.8) for $x \in V$, satisfies

$$\phi_1^{(t)} = \Phi_b \quad \text{for } x \in B. \tag{3.5}$$

We solve (2.8) and (3.5) when $k \neq k_i$, $i = 1, 2, \ldots$, by employing the Green's function $g(x, \xi; k)$ of (2.8) that vanishes for $x \in B$. The solution is given by

$$\phi_1^{(t)}(x; k) = \iint_B g_\nu(x, \xi; k)\Phi_b(\xi; k)\, d\Omega(\xi). \tag{3.6}$$

The eigenfunction representation of $g(x, \xi; k)$ is

$$g(x, \xi; k) = \sum_{m=1}^{\infty} \frac{\psi_m(\xi)\psi_m(x)}{k^2 - k_m^2}. \tag{3.7}$$

This series converges in the L_2 norm to g. By inserting (3.7) into (3.6) we obtain the normal mode expansion of the interior potential as

$$\phi_1^{(t)}(x, k) = \sum_{m=1}^{\infty} \frac{\beta_m(k)\psi_m(x)}{k^2 - k_m^2}, \quad x \in V \tag{3.8}$$

where the coefficients β_m are defined by

$$\beta_m(k) \equiv \iint_B \Phi_b(\xi; k) \frac{\partial \psi_m}{\partial \nu}(\xi)\, d\Omega. \tag{3.9}$$

Again, the series (3.8) converges in the L_2 norm to $\phi_1^{(t)}$ for $x \in V$.

It follows from (3.8) that $\phi_1^{(t)}$ and hence from (2.12) that $\phi^{(a)}$ are both singular as $k \to k_j$, $j = 1, 2, 3, \ldots$. Consequently, the expansion (3.1), which we write as

$$\phi^{(t)} = \varepsilon \phi_1^{(t)} + O(\varepsilon^2), \tag{3.10}$$

and which we call the outer expansion, is invalid near the resonant frequencies $k_1, k_2, \ldots$ of the target. The corresponding expression for $\phi^{(a)}$

$$\phi^{(a)}(x; k) = \varepsilon \sum_{m=1}^{\infty} \frac{\beta_m(k)}{k^2 - k_m^2} \iint_B G(x, \xi; k) \frac{\partial \psi_m(\xi)}{\partial \nu}\, d\Omega(\xi) + O(\varepsilon^2) \tag{3.11}$$

follows from (2.13) and (3.10). Thus, for all k bounded away from the resonant frequencies the scattered fields are $O(\varepsilon)$. Hence, the acoustic potential is $O(\varepsilon)$ inside the target and, to lowest order, it is obtained from the background field outside the target.

4. The nearly rigid scattering problem: The inner expansion

We obtain an asymptotic expansion as $\varepsilon \to 0$ of the nearly rigid scattering problem, that is valid for k near a resonant frequency k_m, by first defining the stretched frequency parameter α by

$$k = k_m(1 + \varepsilon\alpha). \tag{4.1}$$

Then we seek an asymptotic expansion for $\phi^{(t)}$, which we call the inner expansion, in the form

$$\phi^{(t)} = \sum_{j=0} U_j(x; k)\varepsilon^j. \tag{4.2}$$

An inner expansion for $\phi^{(a)}$ is determined by inserting (4.2) into (2.12).

We find in the usual way by substituting (4.1)–(4.2) into (2.8) and (2.14) that the coefficients U_0 and U_1 must satisfy the boundary value problems

$$\Delta U_0 + k_m^2 \mu^2 U_0 = 0 \quad \text{for } x \in V, \qquad U_0 \equiv 0 \quad \text{for } x \in B, \tag{4.3}$$

$$\Delta U_1 + k_m^2 \mu^2 U_1 = -2k_m^2 \alpha \mu^2 U_0 \quad \text{for } \boldsymbol{x} \in V, \qquad U_1 = \Phi_b + \iint_B G \frac{\partial U_0}{\partial \nu} \mathrm{d}\Omega \quad \text{for } \boldsymbol{x} \in B. \tag{4.4}$$

The coefficients U_j, $j = 2, 3, 4, \ldots$ satisfy similar inhomogeneous problems.

In the following analysis we assume that k_m is a simple eigenvalue of (3.3). A discussion of multiple eigenvalues for a related scattering problem of the baffled membrane is given in [2]. Thus, the solution of (4.3) is

$$U_0 = A_m \psi_m(\boldsymbol{x}). \tag{4.5}$$

The constant amplitude A_m is to be determined. Since the homogeneous problem corresponding to (4.4) has a non-trivial solution, the inhomogeneous terms in (4.4) must satisfy an appropriate solvability condition. To obtain this condition, we multiply both sides of the differential equation for U_1 in (4.4) by ψ_m and then integrate the result over V. This gives

$$\iiint_V \psi_m [\Delta U_1 + k_m^2 \mu^2 U_1] \, \mathrm{d}\boldsymbol{x} = -2k_m^2 \alpha A_m, \tag{4.6}$$

where we have used (4.5) and the normalization condition (3.4). Then by applying the divergence theorem to the left-hand side of (4.6), using the differential equation (3.3) for ψ_m, and the boundary condition of (4.4) for U_1, (4.6) is reduced to a linear algebraic equation for A_m. The solution of this equation is

$$A_m = \frac{\beta_m(k_m)}{2k_m^2 \alpha + a_m} \tag{4.7}$$

where the constant β_m is defined by (3.9) and a_m is defined by the double surface integral

$$a_m \equiv -\iint_B \iint_B G(\boldsymbol{x}, \boldsymbol{\xi}; k_m) \frac{\partial \psi_m(\boldsymbol{x})}{\partial \nu} \frac{\partial \psi_m(\boldsymbol{\xi})}{\partial \nu} \mathrm{d}\Omega(\boldsymbol{\xi}) \, \mathrm{d}a(\boldsymbol{x}) \tag{4.8}$$

Here $\mathrm{d}a$ is a differential element of area.

If the ψ_m and/or $G(\boldsymbol{x}, \boldsymbol{\xi}; k_m)$ are known only numerically, then the integrals in (4.8) must be evaluated numerically. Thus, the inner expansion, which is valid for k near k_m, is given by

$$\phi^{(t)} = A_m \psi_m(\boldsymbol{x}) + \mathrm{O}(\varepsilon) \tag{4.9}$$

where A_m is given in (4.7). The corresponding expression for $\phi^{(a)}$

$$\phi^{(a)} = A_m \iint_B G(\boldsymbol{x}, \boldsymbol{\xi}; k_m) \frac{\partial \psi_m(\boldsymbol{\xi})}{\partial \nu} \mathrm{d}\Omega(\boldsymbol{\xi}) + \mathrm{O}(\varepsilon) \tag{4.10}$$

follows from (2.13) and (4.9). The inner expansion is $\mathrm{O}(1)$ as $\varepsilon \to 0$, and thus it is of the same order as the background field, see (2.5a). This demonstrates that the target's main influence on the scattered field occurs when k is near k_m.

5. The nearly rigid scattering problem: The uniform expansion

According to the method of matched asymptotic expansions, see [1, pp. 11–13], there is an 'overlap' interval for k near k_m, ($\alpha \to \infty$), in which the outer and inner expansions are both valid asymptotic expansions of $\phi^{(t)}$. This leads to the matching conditions [1] of the method of matched asymptotic expansions. They are conditions on the coefficients of the outer and inner expansions, which we can show are identically satisfied. This is to be expected as there are no undetermined constants or functions in the outer or inner expansions. This occurs in other applications of the method of matched asymptotic expansions where the singularity occurs as a parameter approaches a critical value; e.g. see [10]. We omit all details of the present analysis.

The composite expansion of the method of matched asymptotic expansions gives the required uniform asymptotic expansion of the scattered fields for k in an interval about k_m. It is given by the sum of the inner and outer expansions minus the outer expansion in terms of the 'inner' variable α, as $\alpha \to \infty$. It can be shown that to lowest order the composite expansion is given by

$$\phi^{(t)} = \varepsilon \sum_{j=1}^{\infty} \left[\frac{\beta_j(k)}{k^2 - k_j^2} - \frac{\varepsilon \beta_m(k_m) a_m \delta_{mj}}{d_m(d_m + \varepsilon a_m)} \right] \psi_j(x) \quad \text{for } x \in V \tag{5.1}$$

where $d_m \equiv 2k_m(k - k_m)$. Inserting (5.1) into (2.13) we find that the corresponding uniform expansion of $\phi^{(a)}$ for k in an interval about k_m is given by

$$\phi^{(a)} = \varepsilon \sum_{j=1}^{\infty} \left[\frac{\beta_j(k)}{k^2 - k_j^2} - \frac{\varepsilon \beta_m(k_m) a_m \delta_{mj}}{d_m(d_m + \varepsilon a_m)} \right] \phi_j^{(a)}(x; k) \quad \text{for } x \in \bar{V}, \tag{5.2}$$

where the functions $\phi_j^{(a)}$ are solutions of (2.9) which satisfy the radiation condition and the boundary conditions $\partial \phi_j^{(a)}/\partial \nu = \partial \psi_j/\partial \nu$ for $x \in B$, $j = 1, 2, 3, \ldots$. They are given by

$$\phi_j^{(a)}(x; k) = \iint_B G(x, \xi; k) \psi_\nu(\xi) \, d\Omega(\xi). \tag{5.3}$$

In the far field where $r \to \infty$, the Green's function g is given by the spherical wave

$$G(x, \xi; k) = D(\hat{r}, \xi; k) \frac{e^{ikr}}{r} [1 + O(1/r)] \tag{5.4}$$

where $\hat{r} = x/r$ is the unit vector in the observation direction. The directivity factor D of the Green's function, which depends on the shape of the target and on k, is assumed to be known either analytically or numerically because we assume that the solution of the rigid scattering problem is known similarly. It is related to the directivity factor A_R of the rigid target by

$$A_R(\hat{r}; k) = -\iint_B D(\hat{r}, \xi; k) \Phi_\nu^{(1)}(\xi) \, d\Omega(\xi). \tag{5.5}$$

We next insert (5.4) into (5.3) and substitute this result into (5.2) to obtain the far field uniform representation

$$\phi^{(a)} \sim A(\hat{r}; k) \frac{e^{ikr}}{r} [1 + O(1/r)] \quad \text{for } r \to \infty. \tag{5.6}$$

The directivity factor in (5.6) is given by

$$A(\hat{\boldsymbol{r}}; k) \equiv \varepsilon \sum_{j=1}^{\infty} \left\{ \frac{\beta_j(k)}{k^2 - k_j^2} - \frac{\varepsilon a_m \beta_m(k_m)\delta_{mj}}{d_m(d_m + \varepsilon a_m)} \right\} F_j(\hat{\boldsymbol{r}}; k) \tag{5.7}$$

where the F_j are defined by

$$F_j(\hat{\boldsymbol{r}}; k) \equiv \iint_B D(\hat{\boldsymbol{r}}, \boldsymbol{\xi}; k) \frac{\partial}{\partial \nu} \psi_j(\boldsymbol{\xi})\, \mathrm{d}\Omega, \quad j = 1, 2, 3, \ldots . \tag{5.8}$$

The functions F_j are the 'modal' directivity factors, that is, they are the directivity factors of the solutions $\phi_j^{(a)}(\boldsymbol{x}; k)$.

The composite expansions (5.1) and (5.2) are uniformly valid for k in an O(1) interval about the simple eigenvalue k_m. They are also valid for k bounded away from the other eigenvalues k_j, $j \neq m$. If each eigenvalue is simple, then we obtain an expansion that is uniformly valid for all k by summing the terms depending on m in (5.1)–(5.2). This gives

$$\phi^{(t)} = \varepsilon \sum_{j=1}^{\infty} Z_j(k, \varepsilon, \alpha)\psi_j + \cdots \qquad \text{for } \boldsymbol{x} \in V \tag{5.9a}$$

$$\phi^{(a)} = \varepsilon \sum_{j=1}^{\infty} Z_j(k, \varepsilon, \alpha)\phi_j^{(a)}(\boldsymbol{x}; k) + \cdots \quad \text{for } \boldsymbol{x} \in \bar{V} \tag{5.9b}$$

where the coefficients Z_j are defined by

$$Z_j \equiv \frac{\beta_j(k)}{k^2 - k_j^2} - \frac{\varepsilon \beta_j(k_j) a_j}{d_j(d_j + \varepsilon a_j)}, \quad j = 1, 2, \ldots . \tag{5.9c}$$

Similarly, in the far field we obtain

$$\phi^{(a)} = A_c(\hat{\boldsymbol{r}}; k) \frac{\mathrm{e}^{ikr}}{r} \tag{5.10a}$$

where the uniform directivity factor A_c is given by

$$A_c(\hat{\boldsymbol{r}}; k) \sim \varepsilon \sum_{j=1}^{\infty} Z_j(k, \varepsilon, \alpha) F_j(\hat{\boldsymbol{r}}; k) + \cdots . \tag{5.10b}$$

The total scattered potential is then obtained by adding (5.10a) to the far field expansion of $\Phi^{(R)}$. We find as $r \to \infty$ that

$$\Phi^{(R)} + \phi^{(a)} = A_{CT} \frac{\mathrm{e}^{ikr}}{r} + \cdots, \qquad A_{CT} \equiv A_R + A_c \tag{5.11}$$

where the directivity factor A_R for the rigid target is defined by (5.5).

6. Interpretation and discussion of results

The inner and outer expansions can be recovered from the uniform expansions (5.1), (5.2), (5.6) and (5.7) by taking appropriate limits in these equations. Thus, if $k - k_m = O(1)$ as $\varepsilon \to 0$, then the second term of (5.1) is $O(\varepsilon^2)$. Consequently, the first term dominates and (5.1) reduces to the outer expansion (3.10).

Similarly, the far field potential given by (5.6), (5.7) and (5.8) is reduced in the outer limit to

$$\phi^{(a)}=\left[\varepsilon \sum_{j=1}^{\infty} \frac{\beta_j(k)}{k^2-k_j^2} F_j(\hat{\boldsymbol{r}}; k)\right]\frac{e^{ikr}}{r}+\cdots \tag{6.1}$$

which is the far field limit of (3.11). Thus, $\phi^{(t)}$ and $\phi^{(a)}$ are $O(\varepsilon)$ when the incident frequency is bounded away from all of the acoustic target's resonant frequencies. Then the acoustic target behaves essentially like a rigid scatterer.

However, when the incident frequency approaches a resonant frequency, i.e. when $k=k_m(1+\varepsilon\alpha)$ for $\alpha=O(1)$ as $\varepsilon\to 0$, the first term in (5.1) is O(1). It combines with the second term to yield the inner result (4.9). Similarly, the far field expression (5.6)–(5.8) for the scattered potential is reduced, for k near k_m, to

$$\phi^{(a)}=A_m F_m(\hat{\boldsymbol{r}}; k_m)\frac{e^{ik_m r}}{r}+\cdots \tag{6.2}$$

which is the far field limit of (4.10). This is $O(1/\varepsilon)$ larger than the outer expansion (6.1) and it is of the same order as $\Phi^{(I)}$ and $\Phi^{(R)}$. Thus, the scattered potential $\phi^{(a)}$ contributes to the lowest order approximation only when k is near k_j, $j=1, 2, \ldots$.

The coefficient of the outgoing spherical wave in (6.2) is the product of the amplitude A_m and the modal directivity factor F_m. Furthermore, $|A_m F_m|^2$ is the differential cross-section of the scattered acoustic potential for k near k_m. The function $|F_m(\hat{\boldsymbol{r}}; k_m)|^2$ is the differential cross-section of the scattered acoustic modal potential $\phi_m^{(a)}(\boldsymbol{x}; k_m)$. The amplitude A_m contains information about the coupling between the acoustic medium and the target, which we now describe.

As in [2], we can show that the four fold integral (4.8) which defines a_m is given by

$$a_m=R+iI \tag{6.3}$$

where R and I are defined by

$$R\equiv \iiint_{\bar{V}} \{|\nabla\phi_m^{(a)}(\boldsymbol{x}; k_m)|^2-k_m^2|\phi_m^{(a)}(\boldsymbol{x}; k_m)|^2\}\,dx\,dy\,dz,$$

$$I\equiv k_m\int_0^{2\pi}\int_0^{\pi}|F_m(\hat{\boldsymbol{r}}; k_m)|^2 \sin\phi\, d\phi\, d\theta. \tag{6.4}$$

Here I/k_m is the total cross-section of $\phi_m^{(a)}(\boldsymbol{x}; k_m)$ and R is twice the corresponding dimensionless Lagrangian. The total acoustic scattering cross-section for k near k_m can therefore be written as

$$\sigma_T=\frac{|A_m|^2}{k_m} I \tag{6.5}$$

where

$$|A_m|^2=\frac{|\beta_m(k_m)|^2}{(2k_m^2\alpha+R)^2+I^2}. \tag{6.6}$$

The result (6.6) is sketched as a function of α in Fig. 1. Since the maximum occurs at $\alpha=-R/2k_m^2$, the largest response does not occur at $k=k_m$ (or equivalently, $\alpha=0$). Thus, R gives the detuning of the target due to the surrounding fluid. Furthermore, it follows from (6.6) that I is the bandwidth of $|A_m|$. To evaluate $|A_m|^2$ it is assumed that k_m and ψ_m are known explicitly or by numerical computation. Then the integrals that define β_m, R and I must be determined similarly.

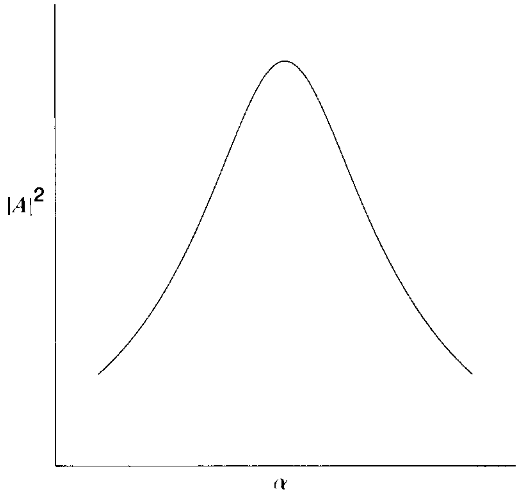

Fig. 1. A graph of $|A|^2$ for a simple eigenvalue.

If $\Phi^1 \equiv 0$, then $\beta_m(k_m) = 0$ and the solvability condition (4.6) gives

$$(2k_m^2\alpha + a_m)A_m = 0. \tag{6.7}$$

A nonzero solution of this equation requires $\alpha = -a_m/2k_m^2$. From (4.1), (6.3) and (6.4) it follows that the complex eigenfrequency of the composite medium consisting of the target and the surrounding fluid is then given asymptotically by,

$$k \sim \left(k_m - \frac{\varepsilon R}{2k_m}\right) - \mathrm{i}\left(\frac{\varepsilon I}{2k_m}\right) + \mathrm{O}(\varepsilon^2). \tag{6.8}$$

The negative imaginary part in (6.8) corresponds to damping because the assumed time dependence is $\mathrm{e}^{-\mathrm{i}\omega t}$. Since the decay rate is proportional to I, this parameter measures the ability of the target, 'vibrating near frequency k_m', to radiate acoustic energy into the region $\bar{V}$.

7. An illustrative example: A nearly rigid sphere

We consider a unit sphere, whose index of refraction $\mu \equiv$ constant, that is insonified by the incident acoustic potential

$$\Phi^{(\mathrm{I})} = \mathrm{e}^{\mathrm{i}kz}. \tag{7.1}$$

A partial wave representation of the solution of the total scattering problem (2.1)–(2.4) is [6],

$$\Phi^{(\mathrm{a})} = \sum_{l=0}^{\infty} D_l\left[j_l(kr) - \frac{j_l'(k)}{h_l'(k)}h_l(kr)\right]P_l(\cos\phi) + \sum_{l=0}^{\infty} e_l h_l(kr)P_l(\cos\phi) \quad \text{for } r > 1 \tag{7.2a}$$

$$\Phi^{(\mathrm{t})} \equiv \phi^{(\mathrm{t})} = \sum_{l=0}^{\infty} f_l j_l(k\mu r)P_l(\cos\phi) \qquad \text{for } r < 1. \tag{7.2b}$$

Here, j_l and h_l are spherical Bessel and Hankel functions of the first kind, respectively, P_l is the Legendre polynomial and ϕ is the polar angle measured from the positive z-axis. The primes denote differentiation with respect to the argument of the function. The constants in (7.2) are given by

$$D_l \equiv (i)^l(2l+1), \quad e_l \equiv \frac{i\varepsilon\mu D_l j'(k\mu)}{k^2 h_l'(k)\Delta},$$

$$\Delta \equiv h_l'(k) j_l(k\mu) - \varepsilon\mu j_l'(k\mu) h_l(k), \quad f_l \equiv \frac{i\varepsilon D_l}{k^2\Delta}. \tag{7.3}$$

The first sum in (7.2a) is the background potential $= \Phi^{(I)} + \Phi^{(R)}$ in the surrounding fluid and the second sum is $\phi^{(a)}$. The numbers

$$k_l^s \equiv z_l^s/\mu, \quad l = 0, 1, 2, \ldots, s = 1, 2, 3, \ldots, \tag{7.4}$$

where z_l^s are the zeros of $j_l(z)$, are the eigenvalues of (3.3) for the rigid sphere. They are all simple. The corresponding eigenfunctions, are

$$\psi_l^s \equiv \frac{1}{\mu}\sqrt{\frac{2l+1}{2\pi}}\, j_l(z_l^s r) P_l(\cos\phi)/j_l'(z_l^s) \tag{7.5}$$

where we have used the customary double index notation for the eigenvalues and eigenfunctions.

We can show by taking the limit of (7.2) and (7.3) as $\varepsilon \to 0$, and for k bounded away from the eigenvalues k_l^s, that we recover the outer expansions (3.8), (3.9) and (3.12) for $\phi^{(a)}$ and $\phi^{(t)}$, as expected. Similarly, we can show that by taking the limit $\varepsilon \to 0$ in (7.2) where $k = k_l^s(1+\varepsilon\alpha)$ and α is O(1) that we recover the inner expansions (4.7)–(4.9) as expected. We omit all details of these calculations.

We now investigate the far field behavior of the partial wave expansion (7.2), which we refer to as the exact solution, and show numerically that it agrees closely, for sufficiently small ε, with the uniform asymptotic result given by (5.11). From (7.2a) we find as $r \to \infty$ that

$$\Phi_b = A_E \frac{e^{ikr}}{r} + \cdots, \qquad A_E(\phi; k) \equiv A_R + E \tag{7.6a}$$

where the directivity factors A_R and E are defined by

$$A_R \equiv \sum_{l=0}^{\infty} q_l D_l \frac{j_l'(k)}{h_l'(k)}, \qquad q_l \equiv \frac{(-i)^{l+1}}{k} P_l(\cos\phi) \tag{7.6b}$$

$$E \equiv \sum_{l=0}^{\infty} q_l e_l. \tag{7.6c}$$

The function A_R given by (7.6b) is the directivity factor of the rigid sphere. It agrees with the expression (5.5) for A_R specialized to the unit sphere. This can be seen by expressing the Green's function G for the rigid sphere in a modal series, expanding this result for $r \gg 1$, and by inserting the directivity factor D of G into (5.5).

The directivity factor F_j defined in (5.8) and used in the uniform result (5.10b) can be evaluated similarly for the sphere by substituting the directivity factor D for the sphere's Green's function into (5.8). Integrating the resulting expression yields,

$$F_j \equiv F_l^s = q_l \frac{k_l^s}{k h_l'(k)} \sqrt{\frac{2l+1}{2\pi}}. \tag{7.7}$$

The directivity factor A_{CT} for the total far field given by the uniform asymptotic expansion in (5.11) is now compared in Figs. 2 and 3 with the directivity factor A_E for the total exact far field given in (7.6a).

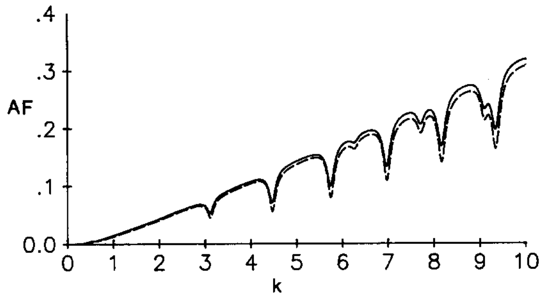

Fig. 2. Forward directivity factors AF for the nearly rigid sphere with $\varepsilon = 0.1$. The solid (dashed) curve is the exact (uniform asymptotic) result.

The solid line in Fig. 2 is the graph for $\varepsilon = 0.1$ of $|A_E(0; k)|$, i.e., in the forward scattered direction, and the dashed line is the graph $|A_{CT}(0; k)|$. The agreement is good. Presumably, by evaluating additional terms in the asymptotic expansions the agreement can be improved. The same functions are shown in Fig. 3 for $\varepsilon = 0.02$. Then it is impossible to graphically distinguish between the asymptotic and exact expressions. The sharp minima in Figs. 2 and 3 correspond to the inner regions which clearly become narrower as $\varepsilon \to 0$. The minima in Fig. 3 occur at the eigenfrequencies k_l^s in agreement with our theory.

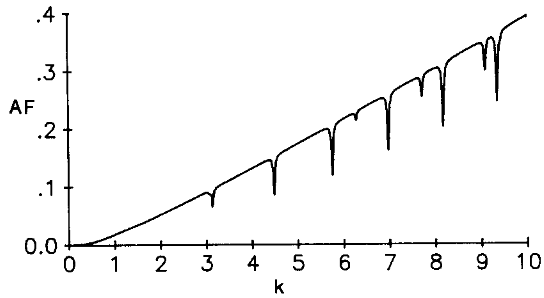

Fig. 3. Same as Fig. 2 but with $\varepsilon = 0.02$. The exact and asymptotic results are essentially indistinguishable.

The exact and uniform asymptotic total cross-sections σ_E and σ_{CT} respectively are graphed in Figs. 4 and 5 for $\varepsilon = 0.1$ and $\varepsilon = 0.02$, respectively. The solid line corresponds to σ_E while the oscillating dashed curve gives σ_{CT}. Note again that there is almost exact agreement for $\varepsilon = 0.02$ and good agreement when $\varepsilon = 0.1$. The sharp minima again occur at the eigenvalues k_l^s. The monotonic dashed curve in Fig. 4 is the cross-section for the rigid sphere. The small disagreement between the exact and the uniform asymptotic results, in the outer regions, is well within the $O(\varepsilon)$ error in the asymptotic approximation.

8. Remarks on numerical methods

As we discussed previously and observed for special geometries from partial wave representations by previous investigators [9], the nearly rigid (or soft) acoustic target responds like a rigid scatterer when the incident frequency is bounded away from the resonant frequencies k_m of the target. However, when $k \to k_m$ the resulting near-resonant oscillations of the target radically changes the scattered fields.

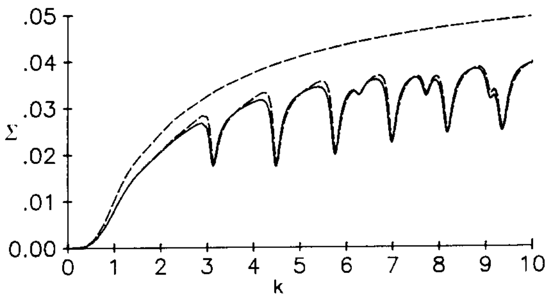

Fig. 4. Total cross-sections Σ for the nearly rigid sphere with $\varepsilon = 0.1$. The solid (dashed) curve is the exact (uniform asymptotic) result. The monotonically increasing dashed curve is the total cross-section for the rigid sphere ($\varepsilon = 0$).

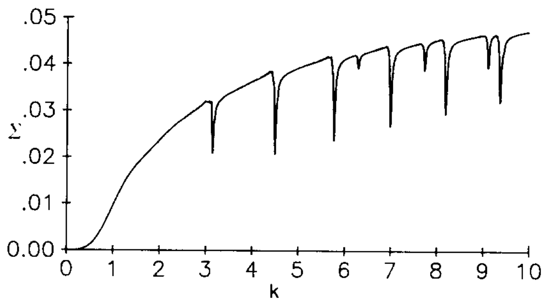

Fig. 5. Same as Fig. 4 but with $\varepsilon = 0.02$.

The rapid change in the scattering characteristics near resonance suggests that numerical procedures, such as the T matrix method [7] or coupled integral equation techniques [11], are ill-conditioned for $k \to k_m$, $m = 1, 2, \ldots$. Then the accurate determination of the solution's structure for k near k_m would at least require repeated calculations for closely spaced values of k near k_m, resulting in an inefficient and costly computation.

Our analysis suggest the following numerical procedure for solving penetrable target scattering problems. Use a standard numerical method, see e.g. [7], to obtain $\Phi^{(R)}(x; k)$ for the values of k that are desired. Then determine k_m and ψ_m by solving the eigenvalue problem (3.3) numerically for the desired resonant values. Next, solve for $\phi_m^{(a)}(x; k_m)$, which is defined in the sentence following (5.8) for the desired values of k_m by the same numerical procedure used to determine $\Phi^{(R)}$. By using this result we express a_m given by (4.8) as

$$a_m = -\iint_B \frac{\partial \psi_m(x)}{\partial \nu} \phi_m^{(a)}(x; k_m)\, \mathrm{d}a(x) \tag{8.1}$$

and determine it by a numerical evaluation of the integral in (8.1). Similarly the coefficients β_m are determined by numerical evaluations of the integrals in (3.9). Finally, the amplitudes A_m given in (4.7) are determined.

Thus, the outer expansion of the total scattered field is given numerically by

$$\Phi^{(a)} = \Phi^{(R)} + \mathrm{O}(\varepsilon), \qquad \Phi^{(t)} = \mathrm{O}(\varepsilon) \tag{8.2}$$

A more accurate determination of the outer expansion can be obtained by numerically evaluating the quantities $\phi_1^{(a)}$ and $\phi_1^{(t)}$ given in Section 3. The inner expansion of the scattered field for k near k_m is given numerically by

$$\Phi^{(a)} = \Phi^{(R)} + A_m \phi^{(a)}(x; k_m) + \mathrm{O}(\varepsilon), \qquad \Phi^{(t)} = A_m \psi_m(x) + \mathrm{O}(\varepsilon), \tag{8.3}$$

see (4.10) and (4.9). Similarly, numerical evaluations of the uniform asymptotic approximation are obtained by using the formulas given in Section 5 and the quantities already evaluated numerically.

9. Nearly soft target

In this section we state the results for nearly soft targets described mathematically by (2.8), (2.15) and (2.13) in the limit as $\varepsilon \to \infty$. The outer expansions, analogous to (3.12) and (3.13) are

$$\phi^{(t)}(x; k) = \sum_{j=1}^{\infty} \frac{\gamma_j(k)}{k^2 - k_j^2} Q_j(x) + O(1/\varepsilon), \tag{9.1a}$$

$$\phi^{(a)}(x; k) = -\frac{1}{\varepsilon} \iint_B \phi^{(t)}(\xi; k) \hat{G}_\nu(x, \xi; k)\, d\Omega(\xi) + O(1/\varepsilon^2) \tag{9.1b}$$

In (9.1a) the coefficients γ_j are defined by

$$\gamma_j(k) = -\iint_B Q_j(\xi) \frac{\partial}{\partial \nu} [\Phi_b]\, d\Omega(\xi) \tag{9.2}$$

where $\Phi^{(R)}$ in (2.5b) is now the solution to (2.1b) which satisfies the Dirichlet condition (2.6b) and $Q_j(x)$ are the eigenfunctions of

$$\Delta Q + k^2 \mu^2(x) Q = 0 \quad \text{for } x \in V, \qquad Q_n = 0 \quad \text{for } x \in B \tag{9.3}$$

which are normalized by (3.4) with ψ_j replaced by Q_j. We observe that the boundary condition in (9.3) corresponds to a rigid target.

The inner expansions, which are analogous to those given by (4.9) and (4.10), are

$$\phi^{(t)}(x; k_m) = \varepsilon B_m Q_m(x) + O(1), \qquad \phi^{(a)}(x; k_m) = -B_m \iint_B \hat{G}_\nu(x, \xi; k_m) Q_m(\xi)\, d\Omega + O(1/\varepsilon), \tag{9.4}$$

where

$$B_m = \frac{\gamma_m(k_m)}{2\alpha k_m^2 + b_m}, \qquad \alpha \equiv \frac{\varepsilon(k - k_m)}{k_m} \tag{9.5}$$

and the constant b_m is defined by the double surface integral

$$b_m \equiv -\iint_B Q_m(x) \left\{ \frac{\partial}{\partial n} \iint_B \hat{G}_\nu(x, \xi; k_m) Q_m(\xi)\, d\Omega(\xi) \right\} da(x). \tag{9.6}$$

Similarly the composite expansions corresponding to the results given by (5.1)–(5.2) are

$$\phi^{(t)} = \sum_{j=1}^{\infty} \left[\frac{\gamma_j(k)}{k^2 - k_j^2} - \frac{\gamma_m(k_m) b_m \delta_{mj}}{d_m(\varepsilon d_m + b_m)} \right] Q_j(x) + \cdots \qquad \text{for } x \in V, \tag{9.7a}$$

$$\phi^{(a)} = \frac{1}{\varepsilon} \sum_{j=1}^{\infty} \left\{ \frac{\gamma_j(k)}{k^2 - k_j^2} - \frac{\gamma_m(k_m) b_m \delta_{mj}}{d_m(\varepsilon d_m + b_m)} \right\} J(x; k) Q_j + \cdots \qquad \text{for } x \in \bar{V} \tag{9.7b}$$

where the operator J is defined by

$$J(x; k)\psi \equiv -\iint_B \hat{G}_\nu(x, \xi; k) \psi(\xi)\, d\Omega(\xi). \tag{9.8}$$

Formulae analogous to (5.6)–(5.10) for far field quantities are derived in a straightforward manner and are omitted here.

The discussion and interpretation of the results for the nearly rigid target carry over completely with minor modifications except for one striking difference. Away from resonance the acoustic field within the target is O(1) for the nearly soft target whereas it is $O(\varepsilon)$ for the nearly rigid target. The scattered acoustic potentials for both problems vanish in their respective limits. Near resonance, the field within the body is very large ($O(1/\varepsilon)$ as $\varepsilon \to 0$) for the nearly soft body and it is O(1) for the nearly rigid body. The scattered acoustic potentials are both O(1) near resonance for both targets.

Acknowledgment

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