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Control of 2D behaviors by partial interconnection

Paula Rocha and Diego Napp Avelli

Dedicated to U.Oberst on the occasion of his 70th birthday

***Abstract*— In this paper we study the stability of two dimensional (2D) behaviors with two types of variables: the variables that we are interested to control (the *to-be-controlled variables*) and the variables on which we are allowed to enforce restrictions (the *control variables*). We derive conditions for the stabilization of the to-be-controlled variables by regular partial interconnection, i.e., by imposing non-redundant additional restrictions to the control variables.**

I. INTRODUCTION

In this contribution we deal with 2D behaviors whose system variables are divided into two sets: the variables that we are interested to control (called *to-be-controlled variables*) and the variables on which we are allowed to enforce restrictions (called *control variables*). In this context, control is viewed as the ability to impose adequate additional restrictions to the control variables in order to obtain a desired overall functioning pattern of the to-be-controlled variables. Hence, in this situation although we can not act directly upon the to-be-controlled variables, we can nevertheless influence their dynamics by imposing restrictions on the control variables. This situation is known as *partial* control or *partial* interconnection [21], [1], [10], [16], [13]. The situation in which the set of to-be-controlled variables coincides with the set of control variables is referred to as *full* control or *full* interconnection.

In particular we are interested in imposing restrictions to the control variables by means of *regular interconnections*. In such interconnection, the restrictions imposed on the behavior by the controller are independent of the restrictions already present. This type of interconnection is closely related to the notion of feedback control in the classical state-space systems, see [12], [26].

The problem of stabilization is well understood for 1D behaviors in both contexts of full and partial control, see for instance [21], [1], [9]. However, stabilization of 2D and nD behaviors has only been studied in the context of full control, see for instance [8], [14], [24], [17], with different underling notions of stability. Here we adopt a notion of stability defined with respect to a stability cone as considered in [8], [14] and investigate the problem of stabilization of 2D behaviors by regular partial interconnection.

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The paper is organized as follows: In Section 2 we present all the necessary definitions and results on the theory of 2D behaviors. In Section 3 we study the problem of stabilization of 2D behaviors by means of regular partial interconnections and the main results of the paper are presented.

II. PRELIMINARIES

In this section we introduce the fundamental material on 2D behaviors. To this end, we divide the section in four parts. In the first subsection we introduce 2D discrete behaviors and their basic properties. The second part is devoted to introduce the notion of stability considered. The last two subsections are concerned with the theory of behaviors with two different types of variables.

A. 2D (kernel) behaviors

Throughout the paper $\mathbb{R}[\underline{s},\underline{s}^{-1}] := \mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}]$ denotes the ring of Laurent polynomials, in the indeterminates s_1 and s_2, with coefficients in $\mathbb{R}$. We consider 2D behaviors $\mathfrak{B}$ defined over $\mathbb{Z}^2$ that can be described by a set of linear partial difference equations, i.e.,

$$\mathfrak{B} = \ker R(\underline{\sigma},\underline{\sigma}^{-1}) := \{z \in \mathcal{U}^q \mid R(\underline{\sigma},\underline{\sigma}^{-1})w \equiv 0\} \subset \mathcal{U}^q,$$

where $\mathcal{U}$ is the trajectory universe, here taken to be $(\mathbb{R})^{\mathbb{Z}^2}$, $\underline{\sigma} = (\sigma_1, \sigma_2)$, $\underline{\sigma}^{-1} = (\sigma_1^{-1}, \sigma_2^{-1})$, the σ_i's are the elementary 2D shift operators (defined by $\sigma_i w(\underline{k}) = w(\underline{k} + e_i)$, for $\underline{k} \in \mathbb{Z}^2$, where e_i is the ith element of the canonical basis of $\mathbb{R}^2$) and $R(\underline{s},\underline{s}^{-1})$ is a 2D Laurent-polynomial (or in short, L-polynomial) matrix known as *representation* of $\mathfrak{B}$. Throughout this paper these behaviors are simply referred to as *behaviors*. If no confusion arises, given an L-polynomial matrix $A(\underline{\sigma},\underline{\sigma}^{-1})$, we sometimes write A instead of $A(\underline{\sigma},\underline{\sigma}^{-1})$ and $A(\underline{s},\underline{s}^{-1})$.

Instead of characterizing $\mathfrak{B}$ by means of a representation matrix R, it is also possible to characterize it by means of its *orthogonal module* $\mathrm{Mod}(\mathfrak{B})$, which consists of all the 2D L-polynomial rows $r(\underline{s},\underline{s}^{-1}) \in \mathbb{R}^{1\times q}[\underline{s},\underline{s}^{-1}]$ such that $\mathfrak{B} \subset \ker\, r(\underline{\sigma},\underline{\sigma}^{-1})$, and can be shown to coincide with the $\mathbb{R}[\underline{s},\underline{s}^{-1}]$-module $\mathrm{RM}(R)$ generated by the rows of R, i.e., $\mathrm{Mod}(\mathfrak{B}) = \mathrm{RM}(R(\underline{s},\underline{s}^{-1}))$, see [22] for details.

For a full column rank L-polynomial matrix $R \in \mathbb{R}^{p\times q}[\underline{s},\underline{s}^{-1}]$ define its Laurent variety (or zeros) as

$$\mathcal{V}(R) = \{(\lambda_1,\lambda_2) \in \mathbb{C}^2 \mid \mathrm{rank}(R(\lambda_1,\lambda_2)) < \mathrm{rank}(R)\},$$

where the first rank is taken over $\mathbb{C}$ and the second one over $\mathbb{R}[\underline{s},\underline{s}^{-1}]$. Note that $\mathcal{V}(R)$ is equal to the set of common zeros of the $q \times q$ minors of R.

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It is worth pointing out that although the representation of a behavior $\mathfrak{B}$ is highly non-unique, any two different representations of $\mathfrak{B}$ share the same Laurent variety, i.e., if $\mathfrak{B} = \ker R_1 = \ker R_2$, then $\mathcal{V}(R_1) = \mathcal{V}(R_2)$.

Definition 1: A full column rank L-polynomial matrix $R \in \mathbb{R}^{p\times q}[\underline{s}, \underline{s}^{-1}]$ is said to be *right minor prime* (rMP) if $\mathcal{V}(R)$ is finite and *right zero prime* (rZP) if $\mathcal{V}(R)$ is empty. A full row rank L-polynomial matrix $R \in \mathbb{R}^{p\times q}[\underline{s}, \underline{s}^{-1}]$ is said to be left minor/zero prime (ℓMP/ℓZP) if R^T is right minor/zero prime, respectively. An L-polynomial matrix L is called a *minimal left annihilator* (MLA) of R if it has full row rank, $LR = 0$, and for any other L-polynomial matrix S such that $SR = 0$ we have that $S = AL$ for some L-polynomial matrix A. We define minimal right annihilators in a similar way, with the obvious adaptations.

Note that L is an MLA of R if it has full row rank and $\mathrm{im}R = \ker L$. It can be shown that an MLA always exists, it is ℓMP and it is uniquely determined modulo a unimodular matrix.

We next review the notions of controllability and autonomy in the context of the behavioral approach.

Definition 2: A behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^2}$ is said to be *controllable* if for all $z_1,\ z_2 \in \mathfrak{B}$ there exists $\delta > 0$ such that for all subsets $U_1,\ U_2 \subset \mathbb{Z}^2$ with $d(U_1, U_2) > \delta$, there exists a $z \in \mathfrak{B}$ such that $z|_{U_1} = z_1|_{U_1}$ and $z|_{U_2} = z_2|_{U_2}$.

In the above definition, $d(\cdot,\cdot)$ denotes the Euclidean metric on $\mathbb{Z}^n$ and $z|_U$, for some $U \subset \mathbb{Z}^n$, denotes the trajectory z restricted to the domain U.

In contrast with the one dimensional case, 2D behaviors admit a stronger notion of controllability called rectifiability. Whereas controllable behaviors are the ones that can be represented by a ℓMP L-polynomial matrix, or in other words by an MLA of some L-polynomial matrix, rectifiable behaviors correspond to ℓZP representation matrices.

On the other hand, we shall say that a behavior $\mathfrak{B} = \ker R$ is autonomous if R has full column rank. In the 1D case, all autonomous behaviors are finite dimensional vector spaces whereas in the 2D case this is no longer true. In fact, a 2D autonomous behavior $\mathfrak{B} = \ker R$ has finite dimension if and only if R is rMP.

Every 2D behavior $\mathfrak{B}$ can be decomposed into the sum $\mathfrak{B} = \mathfrak{B}^c + \mathfrak{B}^a$, where $\mathfrak{B}^c$ is the *controllable part* of $\mathfrak{B}$ (defined as the largest controllable sub-behavior of $\mathfrak{B}$) and $\mathfrak{B}^a$ is a (non-unique) autonomous sub-behavior. This sum can be chosen to be direct for 1D behaviors, but this is not always possible for multidimensional behaviors, see [25].

B. Stability

A discrete 1D behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ is said to be *stable* if all its trajectories tend to the origin as time goes to infinity. In the 2D case, we shall define stability with respect to a specified stability region, as in [14], by adapting the ideas in [8] to the discrete case. For this purpose we identify a *direction* in $\mathbb{Z}^2$ with an element $\underline{d} = (d_1, d_2) \in \mathbb{Z}^2$ whose components are coprime integers, and define a *stability cone* in $\mathbb{Z}^2$ as the set of all positive integer linear combinations of 2 linearly independent directions.

By a *half-line* associated with a direction $\underline{d} \in \mathbb{Z}^2$ we mean the set of all points of the form $\alpha\underline{d}$ where α is a nonnegative integer; clearly, the half-lines in a stability cone S are the ones associated with the directions $\underline{d} \in S$.

Given a stability cone $S \subset \mathbb{Z}^2$, a trajectory $z \in \mathcal{U}^q$ is said to be S*-stable* if it tends to zero along every half line in S. A behavior $\mathfrak{B}$ is S*-stable* if all its trajectories are S*-stable*. It turns out that stable behaviors on $\mathcal{U}^q$ (with respect a stability cone S) must be finite dimensional.

Lemma 3: ([14, Lemma 2]) Every 2D behavior $\mathfrak{B} \subset \mathcal{U}^q$ which is stable with respect to some stability cone S is a finite dimensional linear subspace of the trajectory universe, $\mathcal{U}^q = (\mathbb{R}^q)^{\mathbb{Z}^2}$.

In order to characterize stability, we introduce some preliminary notation. Given two elements $\underline{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\underline{k} = (k_1, k_2) \in \mathbb{Z}^2$, we define

$$\underline{\lambda}^{\underline{k}} := \lambda_1^{k_1}\lambda_2^{k_2}.$$

With this notation a 2D q-vector polynomial function $p(\underline{k})$ of $\underline{k}$ is such that

$$p(\underline{k}) = \sum_{\underline{i}\in I} \alpha_{\underline{i}}\underline{k}^{\underline{i}},$$

where $I \subset (\mathbb{Z}_+)^2$ is a finite bi-index set and $\alpha_{\underline{i}} \in \mathbb{R}^q$.

Definition 4: We say that z is *pure polynomial exponential* with *frequency* $\underline{\lambda}$ if $z(\underline{k}) = p(\underline{k})\underline{\lambda}^{\underline{k}}$, with $p(\underline{k})$ a q-vector polynomial function. If z is a linear combination of pure polynomial exponential we say that z is a *polynomial exponential*. The frequencies of a polynomial exponential $z = \sum \alpha_i z_i$, where z_i are pure polynomial exponential, are defined as all the frequencies of the pure polynomial exponential z_i. A frequency $\underline{\lambda} \in \mathbb{R}^2$ is said to be S-stable if for every direction $\underline{d} \in S$,

$$|\underline{\lambda}^{\underline{d}}| < 1.$$

Theorem 5: ([14, Th. 8],[23, Th. 4.4]) Let $\mathfrak{B} = \ker R \subset \mathcal{U}^q$ be a behavior, and let S be a stability cone. The following are equivalent:

1) $\mathfrak{B}$ is S-stable.
2) $\mathcal{V}(R)$ is finite and every $\underline{\lambda} \in \mathcal{V}(R)$ is S-stable.
3) Every $z \in \mathfrak{B}$ is a polynomial exponential with S-stable frequencies.

C. Behaviors with two types of variables

Since in this paper we are interested in considering different types of variables in a behavior (the to-be-controlled variables and the control variables), we introduce the notation $\mathfrak{B}_{(w,c)}$ for a behavior whose variable z is partitioned into two sub-variables w and c. Partitioning the corresponding representation matrix as $[R \;\; M]$, we can write

$$\begin{aligned}\mathfrak{B}_{(w,c)} &= \{(w,c) \in \mathcal{U}^{\mathtt{w}+\mathtt{c}} \mid R(\underline{\sigma},\underline{\sigma}^{-1})w + M(\underline{\sigma},\underline{\sigma}^{-1})c = 0\} \\ &= \ker[R \; M].\end{aligned}$$

In the case one is only interested in analyzing the evolution of one of the sub-variables, say, w, it is useful to eliminate the other one (c) and consider the projection of the behavior $\mathfrak{B}_{(w,c)}$ into $\mathcal{U}^{\mathtt{w}}$, defined as

$$\pi_w(\mathfrak{B}_{(w,c)}) = \{w \mid \exists\; c \text{ such that } (w,c) \in \mathfrak{B}_{(w,c)}\}.$$

The elimination theorem [11] guarantees that $\pi_w(\mathfrak{B}_{(w,c)})$ is also a (kernel) behavior, for which a representation can be constructed as follows: take a minimal left annihilator (MLA) E of M. Then $\pi_w(\mathfrak{B}_{(w,c)}) = \ker(ER)$, see [7, Cor. 2.38].

On the other hand given a behavior $\mathfrak{B} = \ker R \subset \mathcal{U}^{\mathtt{w}}$ we define the lifting of $\mathfrak{B}$ into $\mathcal{U}^{\mathtt{w}+\mathtt{c}}$ as

$$\mathfrak{B}^*_{(w,c)} := \{(w,c) \in \mathcal{U}^{\mathtt{w}+\mathtt{c}} \mid \;\; c \text{ is free and } w \in \mathfrak{B}\}. \quad (1)$$

Obviously $\mathfrak{B}^*_{(w,c)} = \ker[R \; 0]$. Analogous definitions can be given if the roles of w and c are interchanged. For the sake of brevity, if no confusion arises, we identify $\mathfrak{B}$ and $\mathfrak{B}^*_{(w,c)}$ and denote $\mathfrak{B}_w := \pi_w(\mathfrak{B}_{(w,c)})$ and $\mathfrak{B}_c := \pi_c(\mathfrak{B}_{(w,c)})$.

Definition 6: Given a behavior $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ we say that c is *observable* from w if $(w,c_1),(w,c_2) \in \mathfrak{B}_{(w,c)}$ implies $c_1 = c_2$. The weaker notion of detectability is defined along the same lines. Let S be a stability cone. We say that c is *S-detectable* from w if $(w,c_1),(w,c_2) \in \mathfrak{B}_{(w,c)}$ implies $c_1 - c_2$ tends to zero along every half line in S.

Usually, in control problems involving behaviors with two types of variables it is important to consider the set of variables that are not observable or *hidden* from the remaining set of variables, see [20], [19], [16]. Hence, given a behavior $\mathfrak{B}_{(w,c)}$ we shall define

$$\mathfrak{B}_{(0,c)} := \{c \in \mathcal{U}^{\mathtt{c}} \mid (0,c) \in \mathfrak{B}_{(w,c)}\},$$

as the behavior of the variables c that are not observable or “hidden”from w. Clearly, $\mathfrak{B}_{(0,c)} = \ker M$. Similarly we define $\mathfrak{B}_{(w,0)}$ as the set of w variables that are hidden from the variables c.

Remark 7: The definition of observability and detectability can be reformulated in terms of the hidden behaviors. Indeed, taking into account that we are dealing with linear behaviors, it is not difficult to verify that c is observable from w if and only if $\mathfrak{B}_{(0,c)}$ is the zero behavior. Moreover, c is S-detectable from w if and only if $\mathfrak{B}_{(0,c)}$ is S-stable. Similarly, w is observable (S-detectable) from c if and only if $\mathfrak{B}_{(w,0)}$ is the zero behavior (S-stable).

D. Control by regular partial interconnection

The behavioral approach to control rests on the basic idea that to control a system is to impose appropriate additional restrictions to its variables in order to obtain a new desired behavior. These additional restrictions are achieved by interconnecting the given system with another system called the controller. From the mathematical point of view, system interconnection corresponds to the intersection of the behavior to be controlled with the controller behavior.

Two situations have been considered in the literature. The first one is known as *full interconnection* and corresponds to the case where the controller is allowed to impose restrictions on all the system variables. In this case, the interconnection of a behavior to be controlled, $\mathfrak{B} \subset \mathcal{U}^{\mathtt{w}}$, with a controller behavior, $\mathcal{C} \subset \mathcal{U}^{\mathtt{w}}$, yields a controlled behavior given by

$$\mathcal{K} = \mathfrak{B} \cap \mathcal{C}, \quad (2)$$

or alternatively, in module terms, by $\mathrm{Mod}(\mathcal{K}) = \mathrm{Mod}(\mathfrak{B}) + \mathrm{Mod}(\mathcal{C})$. If (2) holds, we say that $\mathcal{K}$ is *implementable* by full interconnection from $\mathfrak{B}$.

A particular interesting type of interconnection corresponds to the case where the restrictions imposed by the controller do not overlap with the restrictions already active for the behavior to be controlled. Recalling that the elements of the modules associated with a behavior represent the corresponding equations (or restrictions), this means, in terms of the corresponding modules that

$$\mathrm{Mod}(\mathfrak{B}) \cap \mathrm{Mod}(\mathcal{C}) = \{0\},$$

(or, equivalently, that $\mathfrak{B} + \mathcal{C} = \mathcal{U}^{\mathtt{w}}$) and therefore

$$\mathrm{Mod}(\mathcal{K}) = \mathrm{Mod}(\mathfrak{B}) \oplus \mathrm{Mod}(\mathcal{C}).$$

In this case we say that the interconnection of $\mathfrak{B}$ and $\mathcal{C}$ is a *regular interconnection* and denote it by $\mathfrak{B} \cap_{reg} \mathcal{C}$.

The second situation corresponds to the case where the system variables are divided into two disjoint sets: the set of to-be-controlled variables, whose behavior we want to shape, and the set of control variables, on which the controller is allowed to act in order to achieve the desired result. With the purpose of making the notion of partial control more precise we introduce the following notation.

Consider a behavior $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ (the plant), where the w is the (vector of) *to-be-controlled variable(s)*, and c is the (vector of) *control variable(s)*. In order to interpret the interconnection of the plant $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ with the controller $\mathcal{C} \;\; \subset \mathcal{U}^{\mathtt{c}}$ in terms of behavior intersection, we first have to lift the controller behavior $\mathcal{C}$ and regard it as a behavior $\mathcal{C}^*_{(w,c)}$ in the extended variable (w,c). This yields the “extended”controlled behavior

$$\mathfrak{B}_{(w,c)} \cap \mathcal{C}^*_{(w,c)} = \{(w,c) \in \mathcal{U}^{\mathtt{w}+\mathtt{c}} \mid (w,c) \in \mathfrak{B}_{(w,c)},\; c \in \mathcal{C}\}.$$

For the sake of simplicity, whenever no confusion arises, we shall simply write $\mathfrak{B}_{(w,c)} \cap \mathcal{C}$ instead of $\mathfrak{B}_{(w,c)} \cap \mathcal{C}^*_{(w,c)}$.

The behavior of interest is now

$$\mathcal{K} = \pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{C}).$$

In contrast with the situation in which all variables are available for control, the *full* interconnection case, we refer to this situation as *partial* interconnection or *partial* control.

Also in the context of partial interconnections, regularity plays an important role. Given two behaviors $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \subset \mathcal{U}^{\mathtt{c}}$, we say that the interconnection $\mathfrak{B}_{(w,c)} \cap \mathcal{C}$ is *regular* if

$$\mathrm{Mod}(\mathfrak{B}_{(w,c)}) \cap \mathrm{Mod}(\mathcal{C}^*_{(w,c)}) = \{0\},$$

or equivalently if $\mathfrak{B}_{(w,c)} + \mathcal{C}^*_{(w,c)} = \mathcal{U}^{\mathtt{w}+\mathtt{c}}$. In this case, we denote the interconnection by $\mathfrak{B}_{(w,c)} \cap_{reg} \mathcal{C}^*_{(w,c)}$ or (in simplified notation) by $\mathfrak{B}_{(w,c)} \cap_{reg} \mathcal{C}$.

The following lemma presents some results about partial interconnections and hidden behaviors that will be used in the sequel.

Lemma 8: Let $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \subset \mathcal{U}^{\mathtt{c}}$ be two behaviors. Then, the following hold true.

1) $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{C}) = \pi_w(\mathfrak{B}_{(w,c)} \cap (\mathcal{C} + \mathfrak{B}_{(0,c)}))$.
2) $\mathfrak{B}_{(w,c)} \cap_{reg} \mathcal{C}$ if and only if $\mathfrak{B}_{(w,c)} \cap_{reg} (\mathcal{C} + \mathfrak{B}_{(0,c)})$.
3) $\mathfrak{B}_{(w,c)} \cap_{reg} \mathcal{C}$ if and only if $\mathfrak{B}_c \cap_{reg} \mathcal{C}$.

Proof: Let $\mathfrak{B}_{(w,c)} = \ker[R \;\; M]$ and $\mathcal{C} = \ker C$. Note that $\mathfrak{B}_{(0,c)} = \ker M \subset \mathcal{U}^{\mathtt{c}}$ and since $\mathfrak{B}_{(0,c)} \subset \mathcal{C} + \mathfrak{B}_{(0,c)}$, then $\mathcal{C} + \mathfrak{B}_{(0,c)} = \ker KM$ for some L-polynomial matrix K.

1. It is enough to show that $\pi_w(\mathfrak{B}_{(w,c)} \cap (\mathcal{C} + \mathfrak{B}_{(0,c)})) \subset \pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{C})$ since the other inclusion is trivial. Let $w \in \pi_w(\mathfrak{B}_{(w,c)} \cap (\mathcal{C} + \mathfrak{B}_{(0,c)}))$. Then, by definition of π_w there exists a c such that $(w,c) \in \mathfrak{B}_{(w,c)} \cap (\mathcal{C} + \mathfrak{B}_{(0,c)}) = \ker \begin{bmatrix} R & M \\ 0 & KM \end{bmatrix}$. Clearly, c must satisfy $KMc = 0$, i.e., $c \in \mathcal{C} + \mathfrak{B}_{(0,c)} = \ker KM$ and therefore $c = c^* + c^{**}$, where $c^* \in \mathcal{C}$ and $c^{**} \in \mathfrak{B}_{(0,c)} = \ker M$. Hence, as $(w,c) \in \ker[R \;\; M]$, $(w,c^*) \in \ker[R \;\; M]$ which implies that $(w,c^*) \in \ker \begin{bmatrix} R & M \\ 0 & C \end{bmatrix} = \mathfrak{B}_{(w,c)} \cap \mathcal{C}$, and therefore $w \in \pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{C})$.

2. In terms of the corresponding modules we need to show that

$$\begin{array}{ccc} \mathrm{RM}([R \;\; M]) & \cap & \mathrm{RM}([0 \;\; C]) = \{0\} \\ & \Updownarrow & \\ \mathrm{RM}([R \;\; M]) & \cap & \mathrm{RM}([0 \;\; KM]) = \{0\}. \end{array}$$

As $\ker C = \mathcal{C} \subset \mathcal{C} + \mathfrak{B}_{(0,c)} = \ker KM$, $\mathrm{RM}(KM) \subset \mathrm{RM}(C)$ and the "only if"part is obvious. For the converse, let $(0,0) \neq (r,m) \in \mathrm{RM}([R \;\; M]) \cap \mathrm{RM}([0 \;\; C])$. Clearly r must be zero and then there exists an L-polynomial row s such that $s[R \;\; M] = (0,m) \neq (0,0)$, which implies $sM = m \in \mathrm{RM}(C) \cap \mathrm{RM}(M) = \mathrm{RM}(KM)$. Thus, $(0,m) \in \mathrm{RM}([R \;\; M]) \cap \mathrm{RM}([0 \;\; KM])$.

3. By [22, Cor.3], the proof of 3 amounts to showing that

$$\begin{array}{ccc} \mathrm{RM}([R \;\; M]) & \cap & \mathrm{RM}([0 \;\; C]) = \{0\} \\ & \Updownarrow & \\ \mathrm{RM}(LM) & \cap & \mathrm{RM}(C) = \{0\}, \end{array}$$

where L is an MLA of R. In order to prove the "if"part, let $(0,0) \neq (r,m) \in \mathrm{RM}([R \;\; M]) \cap \mathrm{RM}([0 \;\; C])$. It is easy to see that r must be zero and therefore there exists $s \in L$ such that $s[R \;\; M] = (0,m)$. Thus, $0 \neq sM = m \in \mathrm{RM}(LM) \cap \mathrm{RM}(C)$. To prove the converse implication suppose that $0 \neq m \in \mathrm{RM}(LM) \cap \mathrm{RM}(C)$. Then, $m = \alpha LM = \beta C$ for some L-polynomial rows α and β. This implies that $(0,m) = \alpha L[R \;\; M] = \beta[0 \;\; C]$ and therefore $(0,0) \neq (0,m) \in \mathrm{RM}([R \;\; M]) \cap \mathrm{RM}([0 \;\; C])$. □

Remark 9: Obviously, a behavior $\mathcal{K} \subset \mathcal{U}^{\mathtt{w}}$ is implementable from a given behavior $\mathfrak{B} \subset \mathcal{U}^{\mathtt{w}}$ by full (not necessarily regular) interconnection if and only if $\mathcal{K} \subset \mathfrak{B}$. This condition is however not enough in the partial interconnection case. Indeed, it was proven in [1], [16], [19] that $\mathcal{K}$ is implementable by partial (not necessarily regular) interconnection from $\mathfrak{B}_{(w,c)}$ if and only if

$$\mathfrak{B}_{(w,0)} \subset \mathcal{K} \subset \mathfrak{B}_w.$$

It is immediately apparent that the study of partial control problems requires additional tools with respect to full control problems. One such a tool is the notion of the *canonical controller* which has proved to be a key concept for solving many implementation problems by partial control, see for instance [19], [16], [13], [5], [4]. For a given control objective $\mathcal{K} \subset \mathcal{U}^{\mathtt{w}}$, the canonical controller associate with $\mathcal{K}$ is defined as follows:

$$\mathcal{C}^{can}(\mathcal{K}) := \{c \mid \exists\, w \text{ such that } (w,c) \in \mathfrak{B}_{(w,c)} \text{ and } w \in \mathcal{K}\}.$$

For the problem of stabilization, since the control objective is not unique (as we are interested in stability, but not require a specific behavior to be achieved), we shall define the set of all canonical controllers associate to the set of implementable S-stable behaviors:

$$\begin{aligned} \mathcal{C}^{can}_s &= \\ &= \{\mathcal{C}^{can}(\mathfrak{B}^s_w) \mid \mathfrak{B}^s_w \text{ is an implementable } S\text{-stable behavior}\} \\ &= \{\mathcal{C}^{can}(\mathfrak{B}^s_w) \mid \mathfrak{B}^s_w \text{ is an } S\text{-stable behavior and} \\ &\qquad \mathfrak{B}_{(w,0)} \subset \mathfrak{B}^s_w \subset \mathfrak{B}_w\}. \end{aligned}$$

III. Stabilization by Regular Partial Interconnection

In this section we establish necessary and sufficient conditions for the solvability of the problem of stabilizing a given behavior by a regular partial interconnection. Moreover, we show that, under certain conditions, we can derive a constructive solution to the problem and characterize the structure of the to-be-controlled behavior.

The problem of stabilization by regular partial interconnection can be formally stated as follows: Given a behavior $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ and a stability cone S, find conditions for the existence of a controller behavior $\mathcal{C} \subset \mathcal{U}^{\mathtt{c}}$ such that

$$\pi_w(\mathfrak{B}_{(w,c)} \cap_{reg} \mathcal{C}) \text{ is an } S\text{-stable behavior.}$$

Assumption: We assume in the sequel that $\mathfrak{B}_{(w,0)}$ is an S-stable behavior. This entails no loss of generality since, as follows from Remark 9 it is a necessary condition for the stabilization of $\mathfrak{B}_{(w,c)}$ by regular partial interconnection. Note that this means that w is S-detectable from c in $\mathfrak{B}_{(w,c)}$, a condition that already appears in [1] for the 1D case.

Next, we present a result that characterizes the situation in which the to-be-controlled variables of a given behavior $\mathfrak{B}_{(w,c)}$ are stable with respect to a stability cone.

Lemma 10: Let S be a stability cone and $\mathfrak{B}_{(w,c)} = \ker[R\ M]$ a behavior. Let L and E be an MLA of R and M respectively. Then, the following are equivalent:

1) $\mathfrak{B}_w$ is S-stable.
2) $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$ is S-stable.
3) $\ker \begin{bmatrix} L \\ E \end{bmatrix}$ is S-stable.
4) There exists an S-stable behavior $\mathfrak{B}^s$ such that $\mathfrak{B}_c = \mathfrak{B}^s + \mathfrak{B}_{(0,c)}$.

Proof: (1) $\Rightarrow$ (2) Let (w, c) such that $R(\underline{\sigma}, \underline{\sigma}^{-1})w = -M(\underline{\sigma}, \underline{\sigma}^{-1})c$. It is easy to see that if w is S-stable, then $R(\underline{\sigma}, \underline{\sigma}^{-1})w$, and therefore also $M(\underline{\sigma}, \underline{\sigma}^{-1})c$, is S-stable.

(2) $\Rightarrow$ (1) By definition $\mathfrak{B}_w = \{w \mid \exists\ c \text{ such that } Rw = -Mc\} = \{w \mid \exists\ v \in M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c\ :\ Rw = v\}$. Since $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$ is S-stable, v is a polynomial exponential trajectory with S-stable frequencies. Thus, if w is such that $Rw = v$, then

$$w = w^* + w^0,$$

with $w^0 \in \ker R$ and w^* a polynomial exponential trajectory whose frequencies are contained in the frequencies of v [23]. Thus, the frequencies of w are among those of w^* and w^0. Since $\ker R = \mathfrak{B}_{(w,0)}$ is assumed to be S-stable, together with the condition that v is S-stable, this implies that w is S-stable, i.e., $\mathfrak{B}_w$ is S-stable.

(2 $\Leftrightarrow$ 3) It follows from the fact that applying [15, Lemma 2.13] we obtain

$$\ker \begin{bmatrix} L \\ E \end{bmatrix} = M(\underline{\sigma}, \underline{\sigma}^{-1})\ker LM = M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c.$$

(2) $\Rightarrow$ (4) By Lemma 3, $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$ is finite dimensional and let $\{w_1, ..., w_r\}$ be a basis for $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$ where each w_i is an S-stable polynomial exponential trajectory. For each w_i there exists an S-stable polynomial exponential trajectory c_i such that

$$w_i = M(\underline{\sigma}, \underline{\sigma}^{-1})c_i,$$

where $c_i \in \mathfrak{B}_c$, $i = 1, \dots, r$. Define $\mathfrak{B}^s$:=span$<c_1, ..., c_r>$, where the span is considered over $\mathbb{R}[\underline{\sigma}, \underline{\sigma}^{-1}]$. Note that $\mathfrak{B}^s$ is an S-stable (kernel) behavior (thus linear and shift invariant) contained in $\mathfrak{B}_c$. For all $w = \sum \alpha_i w_i \in M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$, $\alpha_i \in \mathbb{R}$, we have that $c = \sum \alpha_i c_i$ satisfies $w = M(\underline{\sigma}, \underline{\sigma}^{-1})c$. This implies that $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c \subset M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}^s$. The reciprocal is obvious, taking into account that $\mathfrak{B}^s \subset \mathfrak{B}^c$. So,

$$M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c = M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}^s.$$

This implies that

$$\mathfrak{B}_c = \mathfrak{B}^s + \ker M.$$

Indeed, if $c \in \mathfrak{B}_c$, then $M(\underline{\sigma}, \underline{\sigma}^{-1})c = M(\underline{\sigma}, \underline{\sigma}^{-1})c^s$ for some $c^s \in \mathfrak{B}^s$. Thus, $c - c^s \in \ker M$, i.e., $c \in c^s + \ker M$ and therefore $c \in \mathfrak{B}^s + \ker M$, proving that $\mathfrak{B}_c \subset \mathfrak{B}^s + \ker M$. On the other hand, both $\mathfrak{B}^s$ and $\ker M$ are contained in $\mathfrak{B}_c$ (the former by construction and the latter since $\mathfrak{B}_c = \ker LM$), and so $\mathfrak{B}_c \supset \mathfrak{B}^s + \ker M$.

(4) $\Rightarrow$ (2) Obvious. □

The following Theorem, whose proof we omit, provides necessary and sufficient conditions for the solvability of the problem of stabilization by regular partial interconnection.

Theorem 11: Let S be a stability cone and $\mathfrak{B}_{(w,c)} = \ker[R\ M]$ a behavior. Let L, E and $[F_1\ F_2]$ be an MLA of R, M and $\begin{bmatrix} L \\ E \end{bmatrix}$ respectively and denote $\mathfrak{B}_1 := \ker \begin{bmatrix} L \\ E \end{bmatrix}$. Then, the following are equivalent:

1) $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection.
2) There exists a controller behavior $\mathcal{C}$ such that $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap_{reg} \mathcal{C})$ is S-stable.
3) There exists a behavior $\mathfrak{B}_2$ such that
$$\mathfrak{B}_1 + \mathfrak{B}_2 = \ker E \quad \text{and} \quad \mathfrak{B}_1 \cap \mathfrak{B}_2 \text{ is } S\text{-stable.}$$
4) There exist matrices $\overline{A}$ and $\overline{K}$ such that
$$\left[\begin{array}{cc|c} 0 & I & \overline{A} \\ F_1 & F_2 & \end{array}\right] \text{ is an MLA of } \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix}$$
and
$$\ker \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix} \text{ is } S\text{-stable.}$$
5) There exists a $\mathcal{C} \in \mathcal{C}_s^{can}$ that is implementable by regular *full* interconnection from $\mathfrak{B}_c$.

It is worth pointing out that since S-stable behaviors are finite dimensional, statement 3 of Theorem 11 can be further analyzed using the results of Bisiacco and Valcher in [3] on the problem of decomposing a 2D behavior into the sum of two sub-behaviors (one of which is fixed) having finite dimensional intersection. Unfortunately, [3, Th. 5.7] shows that the conditions for the existence of such decomposition are far from being constructive. Statement 5 reduces the problem to the implementation of *a* canonical controller $\mathcal{C}$ by regular full interconnection. Although such condition is not difficult to test for a given $\mathcal{C}$ through a direct summand condition, see [2], [15], [18], it becomes uneasy as $\mathcal{C}_s^{can}$ contains, in general, infinite number of elements. However, it can be proved that under certain conditions we are able to obtain a rather simple equivalent condition for the problem solvability.

Theorem 12: Let $\mathfrak{B}_{(w,c)} = \ker[R\ \ M]$ be a behavior, S a stability cone and L, E an MLA of R and M respectively. Assume that $[R\ \ M]$ has full row rank. Then $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection if and only if $\ker L$ is rectifiable (i.e., L is ℓZP).

When dealing with *partial* control problems, one normally seeks for reducing them to equivalent problems in the context of *full* control, as happens for instance, for the problem of implementation by regular partial interconnection, see [13], [16], [19], [4], [5]. This is so because full interconnection problems can be handled more easily and in many cases there exist computational effective solutions. However, for the problem under consideration a characterization in terms of stabilization by regular full control seems to be impossible. Nevertheless, by imposing a condition on the hidden behavior $\mathfrak{B}_{(0,c)}$, we can obtain a characterization in the context of full control. The following results treat this issue.

Theorem 13: Let $\mathfrak{B}_{(w,c)}$ be a behavior and S a stability cone. Assume that $\mathfrak{B}_{(0,c)}$ is S-stabilizable by regular full interconnection from $\mathfrak{B}_c$. Then, the following are equivalent:

1) $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection.
2) $\mathfrak{B}_c$ is S-stabilizable by regular full interconnection.

Proof: (1) $\Rightarrow$ (2) Assume $\overline{\mathcal{K}}$ stabilizes $\mathfrak{B}_{(w,c)} = \ker[R \;\; M]$ by regular partial interconnection and let E and L be an MLA of M and R, respectively. Define $\mathcal{K} := \overline{\mathcal{K}} + \mathfrak{B}_{(0,c)}$. As $\ker M = \mathfrak{B}_{(0,c)} \subset \mathcal{K}$, $\mathcal{K} = \ker KM$, for some L-polynomial matrix K. According to Lemma 8, $\mathcal{K}$ also stabilizes $\mathfrak{B}_{(w,c)}$ by regular partial interconnection, i.e., $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{K})$ is S-stable, and $\mathrm{RM}[R \;\; M] \cap \mathrm{RM}[0 \;\; KM] = \{0\}$, which implies that $\mathrm{RM}(LM) \cap \mathrm{RM}(KM) = \{0\}$. By assumption there exists a behavior $\widehat{\mathcal{K}}$ such that $\mathfrak{B}_{(0,c)} \cap_{reg} \widehat{\mathcal{K}}$ is S-stable. Take

$$\mathcal{C} := \mathcal{K} \cap \widehat{\mathcal{K}} = \ker \begin{bmatrix} KM \\ \widehat{K} \end{bmatrix}$$

where $\widehat{K}$ is a representation of $\widehat{\mathcal{K}}$. We claim that $\mathcal{C}$ stabilizes $\mathfrak{B}_c$ by regular full interconnection.

Denote $\widetilde{\mathfrak{B}}_{(w,c)} := \mathfrak{B}_{(w,c)} \cap \mathcal{C} = \ker \begin{bmatrix} R & M \\ 0 & KM \\ 0 & \widehat{K} \end{bmatrix}$.
Applying the equivalence of 1, 2 and 4 of Lemma 10 to $\widetilde{\mathfrak{B}}_{(w,c)}$, together with the fact that $\pi_c(\widetilde{\mathfrak{B}}_{(w,c)}) = \mathfrak{B}_c \cap \mathcal{C}$ and that $\widetilde{\mathfrak{B}}_{(0,c)} = \mathfrak{B}_{(0,c)} \cap \mathcal{C} = \ker \begin{bmatrix} M \\ KM \\ \widehat{K} \end{bmatrix}$ is S-stable (since it is contained in $\mathfrak{B}_{(0,c)} \cap_{reg} \widehat{K}$), we obtain that

$$\begin{aligned} \pi_w(\widetilde{\mathfrak{B}}_{(w,c)}) \Leftrightarrow \begin{bmatrix} M(\underline{\sigma}, \underline{\sigma}^{-1}) \\ KM(\underline{\sigma}, \underline{\sigma}^{-1}) \\ \widehat{K}(\underline{\sigma}, \underline{\sigma}^{-1}) \end{bmatrix} (\mathfrak{B}_c \cap \mathcal{C}) \text{ is } S\text{-stable} \\ \Leftrightarrow \mathfrak{B}_c \cap \mathcal{C} \text{ is } S\text{-stable.} \end{aligned} \tag{3}$$

Clearly $\pi_w(\widetilde{\mathfrak{B}}_{(w,c)})$ is S-stable as it is contained in $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{K})$ which is S-stable, and therefore $\mathfrak{B}_c \cap \mathcal{C}$ is S-stable. We are thus reduced to proving that the interconnection of $\mathfrak{B}_c = \ker LM$ and $\mathcal{C} = \ker \begin{bmatrix} KM \\ \widehat{K} \end{bmatrix}$ is regular, i.e., $\mathrm{RM}(LM) \cap (\mathrm{RM}(\widehat{K}) + \mathrm{RM}(KM)) = \{0\}$. Note that $\mathrm{RM}(\widehat{K}) \cap (\mathrm{RM}(LM) + \mathrm{RM}(KM)) = \{0\}$ since $\mathrm{RM}(\widehat{K}) \cap \mathrm{RM}(M) = \{0\}$. Thus,

$$\mathrm{RM}(LM) \cap (\mathrm{RM}(\widehat{K}) + \mathrm{RM}(KM)) = \mathrm{RM}(LM) \cap \mathrm{RM}(KM),$$

which in particular implies that

$$\mathfrak{B}_c \cap_{reg} \mathcal{C} \text{ if and only if } \mathfrak{B}_{(w,c)} \cap_{reg} \mathcal{K}.$$

By assumption $\mathfrak{B}_{(w,c)} \cap_{reg} \mathcal{K}$ and so $\mathfrak{B}_c \cap_{reg} \mathcal{C}$.

(2) $\Rightarrow$ (1) Let $\mathcal{C}$ be such that $\mathfrak{B}_c \cap_{reg} \mathcal{C}$ is S-stable. Then, $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c) \cap_{reg} \mathcal{C})$ is also S-stable and therefore part 2 of Theorem 11 is satisfied. □

Note that the assumption that $\mathfrak{B}_{(0,c)}$ is S-stabilizable by regular full interconnection from $\mathfrak{B}_c$ is a relaxation of the condition that $\mathfrak{B}_{(0,c)}$ is S-stable, or in other words, that c is S-detectable from w in $\mathfrak{B}_{(w,c)}$. This leads to the following corollary.

Corollary 14: Let $\mathfrak{B}_{(w,c)}$ be a behavior and S a stability cone. If c is S-detectable from w, then $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular *partial* interconnection if and only if $\mathfrak{B}_c$ is S-stabilizable by regular full interconnection.

This result, can be compared with [1, Th.6] on the stabilization of 1D behaviors by regular partial interconnection, with the difference that now the conditions are given on the behavior of the control variable c rather than on the behavior of the system variable w. This is not unexpected since, as shown in [13], [16], contrary to what happens in the 1D case, the partial implementation conditions for nD behaviors are equivalent to full implementation conditions on the behaviors of c rather than of w.

However, the following theorem shows that if $\mathfrak{B}_{(0,c)}$ is controllable, a necessary condition for stabilization of $\mathfrak{B}_{(w,c)}$ by regular partial control is the stabilization of $\mathfrak{B}_w$ by regular full interconnection. In turn, as shown in [6], this amounts to saying that

$$\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^s, \tag{4}$$

where $\mathfrak{B}^c$ (the controllable part of $\mathfrak{B}$) is rectifiable and $\mathfrak{B}^s$ is an S-stable behavior.

Theorem 15: Let $\mathfrak{B}_{(w,c)}$ be a behavior and S a stability cone. Assume that $\mathfrak{B}_{(0,c)}$ is controllable. If $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection, then the following two equivalent conditions hold:

1) $\mathfrak{B}_w$ is S-stabilizable by regular full interconnection.
2) There exists an S-stable behavior $\mathfrak{B}^s$ such that

$$\mathfrak{B}_w = \mathfrak{B}_w^c \oplus \mathfrak{B}^s, \tag{5}$$

where $\mathfrak{B}_w^c$ (the controllable part of $\mathfrak{B}_w$) is rectifiable.

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