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Influence of continuous spectrum on ballooning instabilities in plasmas with shear-flow

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The influence of shear-flow on stability of plasma ballooning modes is important for Tokamak experiments. In a static plasma, the growth rate of ballooning modes is readily determined using the ''ballooning transformation,'' but this is ineffective for plasmas with flow. One then has only the quasi-static approximation. This gives the growth rate in the limit that shear velocity $\Omega' \to 0$, but no other information on the effect of shear-flow. Furthermore, it is invalid in typical cases because of the intervention of the stable magnetohydrodynamic continuum. In this paper, a simple model is used to investigate the influence of shear-flow on ballooning modes. This shows that the intervention of the continuum leads to a reduction in the growth rate proportional to $|\Omega'|$ for small Ω'. This is in accord with some numerical simulations—but contrary to the $(\Omega')^2$ variation expected from a perturbation expansion. In fact, since the effect is nonanalytic in Ω', it cannot be obtained from a perturbation expansion in Ω' and an alternative formalism is first developed for dealing with this problem. [S1070-664X(99)00406-1]

I. INTRODUCTION

The effect of shear-flow on the stability of short wavelength perturbations (ballooning modes) in a toroidal plasma is an important factor in tokamak experiments. In the absence of shear-flow, the stability of ballooning modes can be calculated using the well-known ''ballooning transformation''.[1–5] However this method is not effective for problems that involve significant shear-flow.[6–9]

In the ideal magnetohydrodynamic (MHD) model, linear perturbations of an axi-symmetric toroidal plasma, with sheared toroidal rotation $\Omega(\psi)$, are described by an equation of the form[8]

$$L\left[\psi,\theta,\frac{\partial}{\partial\theta}-inq,\frac{1}{n}\frac{\partial}{\partial\theta},\frac{1}{n}\frac{\partial}{\partial\psi},\frac{\partial}{\partial t}-in\Omega'\right]\xi=0, \quad (1)$$

where the operator L is periodic in the poloidal angle θ, n is the toroidal mode number, and ψ labels a magnetic surface with inverse rotational transform $2\pi q(\psi)$. The ballooning transformation replaces the periodic coordinate θ by an ''extended poloidal coordinate'' η $(-\infty<\eta<\infty)$. Then, when $\Omega'=0$, a perturbation with large toroidal mode number n can be expressed in an eikonal form

$$\xi=\xi\exp(inq[\eta+S(\psi)]+\gamma t), \quad (2)$$

where $\mathbf{B}\cdot\nabla S=0$ and ξ and S vary slowly across magnetic surfaces. In the limit $n\to\infty$, this reduces Eq. (1) to an *ordinary* differential equation (ODE) on each magnetic surface

$$L_S\left[\psi,\eta,\frac{\partial}{\partial\eta},q'(\eta+k),\gamma\right]\xi=0, \quad (3)$$

where the ''ballooning phase angle'' $k\equiv dS/d\psi$.

The eigenfunctions of L_S are bounded as $\eta\to\pm\infty$ and the eigenvalue is periodic in k, $\gamma(k+2\pi)=\gamma(k)$. For any phase angle k, the spectrum of eigenvalues $\gamma(k)$ on each surface includes a stable continuum, as well as any discrete unstable eigenvalues.[2] In a typical case, there is a single unstable eigenvalue over part of the range of k, but for the remainder there is only the stable continuum. (We will refer to this as the unstable eigenvalue ''merging with the continuum'' as k changes.)

As shown in the full theory (including corrections of order $1/n$),[2] the stability of ballooning modes in a *stationary* plasma is determined by the maximum of $\gamma(k,\psi)$ so that merging of the unstable eigenvalue with the continuum does not pose any particular problem.

Unfortunately, the situation is quite different in the presence of shear-flow. If the eikonal S is to be slowly varying, it must then satisfy both $\mathbf{B}\cdot\nabla S=0$ and[9–12]

$$\left(\frac{\partial}{\partial t}+v\cdot\nabla\right)S=0. \quad (4)$$

For toroidal flow, this implies

$$S=inq[\eta+S(\psi)]-\Omega(\psi)t, \quad (5)$$

and in the limit $n\to\infty$ one obtains, in place of Eq. (3), a *partial* differential equation (PDE)

$$L_R\left[\psi,\eta,\frac{\partial}{\partial\eta},q'(\eta+k)-\Omega' t,\frac{\partial}{\partial t}\right]\xi(\eta,t)=0. \quad (6)$$

The operator L_R is periodic in t at fixed $\hat{\eta}\equiv(\eta-\Omega' t/q')$ and Eq. (6) has Floquet solutions[8–10,12]

$$\xi=f(\hat{\eta},t)\exp(\mu t), \quad (7)$$

with f periodic in t. (These are not the eigenmodes of the stationary flow problem, but the ''Floquet growth rate'' μ is the same as that of the eigenmode. The relation between the Floquet and eigenmode solutions is described in Ref. 8 and discussed in more detail in Ref. 13.)

At a fixed t, the ''instantaneous'' eigenvalues $\gamma(t)$ of the operator L_R are those for a static plasma at a ballooning phase angle $k-\Omega' t/q'$, and $\gamma(t)$ is thus periodic in t. In the ''quasi-static'' limit $\Omega'\to 0$, the growth rate μ is the average of $\gamma(t)$ over the Floquet period,[14,15] or, equivalently, the average of $\gamma(k)$ over the phase angle k, i.e.

$$\mu=\frac{1}{2\pi}\oint \gamma(k)dk. \tag{8}$$

Since it is independent of Ω', this quasi-static approximation gives no indication of how the growth rate varies with shear-flow. As implied by the symmetry under $\Omega'\to-\Omega'$, a perturbation expansion in Ω' leads to a correction to the quasi-static limit that is proportional to Ω^2. However, a remarkable feature of numerical simulations of ballooning modes[14] is that in some cases the growth rate decreases *linearly* with velocity shear over a considerable range. Another limitation of the quasi-static approximation is that it is valid only when the separation of $\gamma(k)$ from other eigenvalues is large compared to the Floquet frequency Ω'/q' (see Sec. III). This condition is clearly violated, *even for small* Ω', when $\gamma(k)$ merges into the continuum. Thus, the continuum is a serious problem for any theory of ballooning modes in plasmas with shear-flow.

This paper is an attempt to understand the effect of velocity shear and the continuum on ballooning modes. The model is described in Sec. II. The analysis is restricted to small Ω', but the problem is not amenable to a perturbation expansion in Ω' and an alternative formalism is, therefore, introduced in Sec. III. The application to ballooning modes is given in Sec. IV. It shows that merging of the unstable eigenvalue with the continuum leads to a reduction in the growth rate that is nonanalytic in the flow parameter Ω'. An interpretation of this result and some conclusions are presented in Sec. V.

II. THE $s-\alpha$ MODEL

The $s-\alpha$ model of ballooning modes was introduced in Ref. 1 and extended to include sheared plasma rotation by Miller *et al.*[14] It represents an annular region of a low β toroidal plasma in which the magnetic surfaces are displaced circles, $R=R_0+\Delta(r)+r\cos\theta$. The plasma pressure gradient is embodied in the parameter

$$\alpha=-r\Delta''=-2\frac{p}{B^2}q^2R_0\frac{\partial}{\partial r}\log p, \tag{9}$$

and the *magnetic* shear in the parameter $s=r\partial q/q\partial r$. Then, after the ballooning transformation, instabilities of a static plasma are governed by

$$\frac{\partial}{\partial\eta}(1+h_0^2)\frac{\partial X}{\partial\eta}+\Gamma X=\frac{\gamma^2}{\gamma_A^2}(1+h_0^2)X, \tag{10}$$

where $h_0=s(\eta+k)-\alpha\sin\eta$, $\Gamma=\alpha(\cos\eta+h_0\sin\eta)$, $\gamma_A=B/\rho^{1/2}Rq$.

Equation (10) is equivalent to the general Eq. (3). The first term represents the effect of field line bending and the second includes the effect of both toroidal and geodesic curvature. Thus, despite the idealizations involved in the $s-\alpha$ model, it does reproduce the main features of a more realistic configuration.

In the extension of the $s-\alpha$ model by Miller *et al.*[14] the centrifugal effect of plasma flow is ignored, but the crucial effect of velocity shear in the annulus is included. Miller *et al.* discussed a compressible plasma, but we simplify the problem further by assuming the perturbation is incompressible. Then the model equation for ballooning modes in a plasma with shear-flow becomes

$$\frac{\partial}{\partial\eta}(1+h^2)\frac{\partial X}{\partial\eta}+\Gamma X=\frac{1}{\gamma_A^2}\frac{\partial}{\partial t}(1+h^2)\frac{\partial X}{\partial t}, \tag{11}$$

where now $h=(s(\eta+k)-s_v t-\alpha\sin\eta)$, $\Gamma=\alpha(\cos\eta+h\sin\eta)$, and s_v is the *velocity* shear parameter, $s_v=r\partial\Omega/q\partial r$. Equation (11) is equivalent to the general Eq. (6) and reduces to Eq. (10) in the static case.

It is convenient to put $\tau=s_v t$ and $\sigma=s_v/\gamma_A$; then Eq. (11) becomes

$$LX=\sigma^2\frac{\partial}{\partial t}M\frac{\partial X}{\partial t}, \tag{12}$$

where

$$L=\left(\frac{\partial}{\partial\eta}(1+h^2)\frac{\partial}{\partial\eta}+\Gamma\right),\quad M=(1+h^2), \tag{13}$$

$$h=s(\eta+k)-\tau-\alpha\sin\eta,\quad \Gamma=\alpha(\cos\eta+h\sin\eta). \tag{14}$$

The operators L, M are self-adjoint and in this form the Floquet period $(=2\pi)$ is independent of the velocity shear—which appears only through the parameter σ. Equation (12) forms the basis for the remainder of our discussion.

III. FORMALISM

We have already mentioned the quasi-static approximation. This can be obtained by writing

$$X(t)=\xi\exp\left(\frac{1}{\sigma}\int^t\lambda(t')dt'\right), \tag{15}$$

and expanding ξ for small σ. Then in lowest order, Eq. (12) gives

$$L(t)\xi^{(0)}=\lambda^2M(t)\xi^{(0)}, \tag{16}$$

confirming that $\lambda(t)$ is just the instantaneous growth rate in a static plasma at ballooning phase angle $k-\Omega' t/q'$, and $\xi^{(0)}$ is the corresponding eigenfunction. However, this does not yet determine $X(t)$ because, since Eq. (16) is linear, $\xi^{(0)}$ may be multiplied by an arbitrary function of time. This indeterminacy is resolved in next order which, after annihilation of $\xi^{(1)}$, gives

$$\frac{d}{dt}\langle\xi^{(0)}(t)M(t)\xi^{(0)}(t)\rangle=0, \tag{17}$$

where the angle bracket signifies integration over $-\infty<\eta<\infty$.

To improve on the quasi-static approximation, and to incorporate the continuum, one might try to expand $X(t)$ in the full set of instantaneous eigenfunctions X_n of Eq. (12), i.e.

$$X(t)=\sum a_n(t)X_n(t)\exp\left(i\int^t \frac{\omega_n(t')}{\sigma}dt'\right), \tag{18}$$

where

$$L(t)X_n(t)=-\omega_n^2 M(t)X_n(t), \tag{19}$$

and the continuum has been rendered discrete, but closely spaced, by confining η to a large but finite interval $-l<\eta<l$. [The density of states, $dN(\omega)/d\omega$, in the continuum is $\sim l/\pi$.]

Unfortunately, even if the X_n are assumed to be complete, there is no unique correspondence between $X(t)$ and the coefficients a_n. In fact, there are two coefficients a_n for each X_n, one associated with positive frequency and one with negative frequency. This is because Eq. (12) involves the second time derivative so that two independent functions must be specified to determine a solution—and hence are also required to determine the a_n.

To overcome this difficulty, it is convenient to introduce a two-component representation for Eq. (12). That is we define

$$G=\begin{pmatrix} 0 & 1/M \\ L & 0 \end{pmatrix}, \quad \phi=\begin{pmatrix} X \\ Y \end{pmatrix}. \tag{20}$$

Then Eq. (12) is equivalent to

$$G\phi=\sigma\frac{\partial\phi}{\partial t}. \tag{21}$$

The operator G is not self-adjoint (although its elements L and $1/M$ are each self-adjoint in η space). Consequently, we introduce an adjoint operator

$$G^\dagger=\begin{pmatrix} 0 & L \\ 1/M & 0 \end{pmatrix}, \tag{22}$$

such that

$$\langle\psi G\phi\rangle=\langle\phi G^\dagger\psi\rangle. \tag{23}$$

We also introduce the time-dependent adjoint equation

$$G^\dagger\phi^\dagger=-\sigma\frac{\partial\phi^\dagger}{\partial t}. \tag{24}$$

Then if ϕ_1 and $\phi_2^\dagger$ are *any* solutions of Eqs. (21) and (24), respectively, (i.e. not necessarily related—this depends on their initial conditions)

$$\frac{d}{dt}\langle\phi_2^\dagger\phi_1\rangle=0. \tag{25}$$

We define the eigenfunctions of G and $G^\dagger$ by

$$G\phi_n=i\omega_n\phi_n, \quad G^\dagger\phi_n^\dagger=i\omega_n^\dagger\phi_n^\dagger, \tag{26}$$

corresponding to $\phi_n\sim\exp(+i\omega_n t/\sigma)$ and $\phi_n^\dagger\sim\exp(-i\omega_n^\dagger t/\sigma)$. Each ϕ_n is thus associated with a single frequency and, using Eq. (23), $\omega_n=\omega_n^\dagger$. The two-component eigenfunctions $\phi_n,\phi_n^\dagger$ of G and $G^\dagger$ can be expressed in terms of the scalar eigenfunctions $X_n(\eta)$ of L as

$$\phi_n=\begin{pmatrix} 1 \\ i\omega_n M \end{pmatrix}X_n, \tag{27}$$

$$\phi_n^\dagger=\begin{pmatrix} i\omega_n M \\ 1 \end{pmatrix}X_n. \tag{28}$$

It follows from Eq. (23) that $\phi_m^\dagger$ and ϕ_n are orthogonal, i.e. $(\omega_m-\omega_n)\langle\phi_m^\dagger\phi_n\rangle=0$, and it can be shown using Eqs. (27) and (28) that

$$\langle\phi_m^\dagger\phi_n\rangle=\langle\phi_m\phi_n^\dagger\rangle=2i\omega_n\langle X_n M X_n\rangle\delta_{m,n}. \tag{29}$$

If we expand the two-component vector ϕ in the eigenfunctions of G

$$\phi=\sum_m a_m\phi_m, \tag{30}$$

the coefficients a_m are now unique and

$$a_m=\langle\phi_m^\dagger\phi\rangle/\langle\phi_m^\dagger\phi_m\rangle. \tag{31}$$

IV. CALCULATION

Following the formalism of the previous section, we expand the solution $\phi(t)$ of Eq. (21) in the instantaneous eigenfunctions ϕ_m

$$\phi(\eta,t)=\sum_m a_m\phi_m\exp\left(i\int_0^t\left(\frac{\omega_m(t')}{\sigma}\right)dt'\right), \tag{32}$$

then the a_m satisfy

$$\dot a_m+\sum_n R_{mn}a_n\exp\left(i\int_0^t\frac{(\omega_n-\omega_m)}{\sigma}dt'\right)=0, \tag{33}$$

with

$$R_{mn}=\langle\phi_m^\dagger\dot\phi_n\rangle/\langle\phi_m^\dagger\phi_m\rangle. \tag{34}$$

An important feature of Eq. (32) is that the velocity shear appears only in the exponential factors. As $\sigma\rightarrow 0$ these factors oscillate rapidly so that the effective coupling between the a_m is small. In effect they become adiabatic invariants. In the same way, the quasi-static approximation becomes valid when σ is smaller than the separation of an unstable eigenvalue from the continuum.

Now consider the situation in which a single unstable eigenmode exists for part of the Floquet cycle but merges into the continuum for the remainder. This is shown schematically in Fig. 1. (We assume there is symmetry about the mid-point of the cycle.) If σ is small then, while the unstable eigenvalue is well separated from the continuum, the mode evolves slowly; its amplitude increases according to the quasi-static expression (15), and its coupling to the continuum is weak. However, as it merges into the continuum, it changes form rapidly and is strongly coupled to the continuum—so that a spectrum of continuum modes is excited. When the unstable mode re-emerges it has the same form as before it merged into the continuum, and again fol-

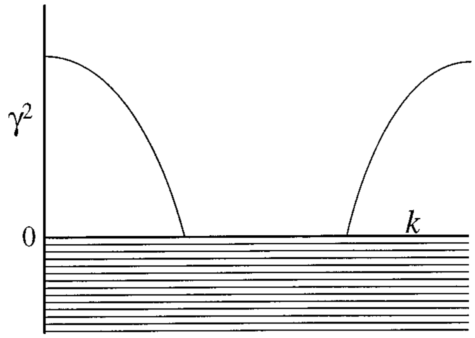

FIG. 1. Spectrum of instantaneous eigenvalues as function of ballooning phase angle *k*.

lows the quasi-static approximation—but its amplitude is reduced because some energy remains in the continuum modes.

To assess the reduction in amplitude, suppose that before merging into the continuum the perturbation follows the quasi-static approximation with instantaneous unstable eigenfunction Ψ_0. After it merges with the continuum, the perturbation becomes $\psi_F(t)$—a combination of many continuum modes. Then, after a time T, the unstable instantaneous eigenmode reappears. At this point the expansion of $\psi_F(T)$ in instantaneous eigenfunctions will include both the unstable mode Ψ_0 and the continuum, i.e. $\psi_F(T)=\alpha\Psi_0+\Sigma\, a_m\phi_m$, and, according to Eq. (31), the coefficient α is

$$\alpha=\frac{\langle\Psi_0^\dagger\psi_F(T)\rangle}{\langle\Psi_0^\dagger\Psi_0\rangle}. \tag{35}$$

Thereafter, the perturbation again follows the quasi-static approximation, but with preexponential factor $\alpha\Psi_0$ instead of Ψ_0.

We must now express α in a more useful form. To do so we note that just as $\psi_F(t)$ is the vector that develops *from* Ψ_0 at $t=0$, so we may introduce $\psi_R^\dagger(t)$ as the vector that develops *into* $\Psi_0^\dagger$ at $t=T$. Then, since $\langle\psi_R^\dagger\psi_F\rangle$ is constant [Eq. (25)], the coefficient α may be written

$$\alpha=\langle\psi_R^\dagger(t)\psi_F(t)\rangle/\langle\Psi_0^\dagger\Psi_0\rangle, \tag{36}$$

where t is any time between entry and exit from the continuum. In order to exploit symmetry, we take $t=T/2$ and write α as

$$\alpha=\langle\psi_R^\dagger(T/2)\psi_F(T/2)\rangle/\langle\psi_R^\dagger(T)\psi_F(0)\rangle. \tag{37}$$

The forward function $\psi_F(t)$ can be expanded in instantaneous eigenfunctions as in Eq. (32), with the a_m satisfying Eq. (33) and the initial condition $a_m(0)=\langle\phi_m^\dagger\Psi_0\rangle/\langle\phi_m^\dagger\phi_m\rangle$. Similarly the reverse function $\psi_R^\dagger(t)$ can be expanded as

$$\psi_R^\dagger(t)=\sum_m b_m\phi_m^\dagger\exp\left(-i\int_T^t\frac{\omega_m(t')}{\sigma}dt'\right), \tag{38}$$

with the b_m satisfying

$$\dot{b}_m+\sum_m b_n\hat{R}_{mn}\exp\left(-i\int_T^t\frac{(\omega_n-\omega_m)}{\sigma}dt'\right)=0, \tag{39}$$

and the ‘‘final’’ condition

$$b_m(T)=\langle\phi_m\Psi^\dagger\rangle/\langle\phi_m\phi_m^\dagger\rangle=a_m(0), \tag{40}$$

[In Eq. (39), $\hat{R}_{mn}=\langle\phi_m\dot{\phi}_n^\dagger\rangle/\langle\phi_m\phi_m^\dagger\rangle=R_{mn}$.]

Then

$$\alpha=\frac{\Sigma\, a_m(T/2)b_m(T/2)\exp(i\int_0^T\omega_m/\sigma)}{\Sigma\, a_m(0)b_m(T)}. \tag{41}$$

Because of symmetry about the mid-point of the Floquet cycle, $\omega_m(t)=\omega_m(T-t)$ and $R_{mn}(t)=-R_{mn}(T-t)$. Thus, if $p=T-t$, $b(t)=\hat{b}(p)$ we have

$$\frac{d\hat{b}_m(p)}{dp}+\sum_n R_{mn}(p)\hat{b}_n(p)\exp\left(i\int_0^p\frac{(\omega_n-\omega_m)}{\sigma}dt'\right)=0, \tag{42}$$

and $\hat{b}_m(p=0)=a_m(0)$. Therefore, $\hat{b}_m$ satisfies the same equation and initial condition as a_m, so that $b_m(t)=a_m(T-t)$ and α becomes

$$\alpha=\frac{\Sigma(a_m(T/2))^2\exp(i\int_0^T\omega_m/\sigma dt')}{\Sigma(a_m(0))^2}. \tag{43}$$

We now need only to estimate the spectrum a_m^2 of excited modes. As noted earlier, the exponential factor in Eq. (33) oscillates rapidly as $\sigma\to0$ and the effective coupling between modes is therefore small, unless their frequency difference $(\omega_m-\omega_n)$ is itself $\lesssim\sigma$. Consequently, as the unstable mode merges with the continuum it excites only modes with $\omega_m\lesssim\sigma$. More specifically, if we ignore the variation of factors other than the rapid exponential, the amplitude of the excited modes is of order

$$\frac{a_m^2}{a_0^2}\sim\frac{\sigma^2R_{m0}^2}{\omega_m^2}. \tag{44}$$

Of course this is correct only when the change in the driving mode amplitude a_0^2 can be ignored and when $a_m^2<a_0^2$, but it shows that the excitation of continuum modes is large (i.e., independent of σ as $\sigma\to0$) in a frequency band $\omega_m\lesssim\sigma$ and small (i.e., $\to0$ as $\sigma\to0$) outside this band.

Interaction between modes within the continuum does not appreciably alter this picture. The change in amplitude of one mode due to interaction with another is of order

$$\frac{\delta a_m^2}{a_n^2}\sim\frac{\sigma^2R_{mn}^2}{(\omega_m-\omega_n)^2}, \tag{45}$$

which is again small unless $(\omega_m-\omega_n)\lesssim\sigma$. However, because of the orthogonality of continuum modes, the matrix element R_{mn} vanishes as $m\to n$ and consequently as $\omega_m\to\omega_n$. Hence

$$\frac{\delta a_m^2}{a_n^2}\sim\sigma^2(R'_{mn})^2, \tag{46}$$

which is small as $\sigma\to0$ *irrespective* of the frequency difference between the interacting modes.

From this discussion we see that the spectrum of continuum modes at $T/2$ is largely determined by excitation as the unstable mode merges into the continuum and it extends over a frequency range $0 < \omega \lesssim \sigma$. Equation (44) suggests that the spectrum has the form

$$a_m^2 \sim \frac{R^2}{(1+\omega_m^2/\sigma^2)}, \tag{47}$$

but the only property we need is that a_m^2 is a function of σ/ω_m and is small when $\omega_m/\sigma \gg 1$.)

Incorporating these features into Eq. (43) and replacing the summation by integration,

$$\alpha = \frac{\int a_m^2(\omega/\sigma)\exp(i\oint \omega/\sigma)\,d\omega}{\int a_m^2(\omega/\sigma)\,d\omega}, \tag{48}$$

which is independent of σ. Thus, *the effect of the continuum is to reduce the amplitude of ballooning modes by a constant factor per Floquet period.*

V. CONCLUSIONS

In a stationary toroidal plasma, ballooning modes are described by a simple ODE in which the eigenvalue $\gamma(k)$ is a periodic function of the phase angle k and stability is determined by max $\gamma(k)$. In a plasma with sheared toroidal rotation, ballooning modes are described by a more complex PDE in which the phase angle increases linearly with time at a rate proportional to the velocity shear Ω'. When $\Omega' \to 0$ the growth rate is equal to the average of the (periodic) instantaneous eigenvalue $\gamma(k(t))$, i.e.,

$$\mu = \frac{1}{2\pi} \oint \gamma(k)\,dk. \tag{49}$$

However, this ''quasi-static'' approximation is inadequate in the typical case that the stationary plasma is stable for part of the range of phase angle k [so that $\gamma(k)$ merges into the stable continuum]. We have shown that in this case there is a form of ''continuum damping'' that reduces the mode amplitude by a constant factor per cycle of $\gamma(t)$, irrespective of the rate (or sign) of the velocity shear. Since the number of cycles per second is proportional to velocity shear Ω', the true growth rate becomes

$$\mu = \frac{1}{2\pi} \oint \gamma(k)\,dk - \kappa_1 |\Omega'|, \tag{50}$$

where $\kappa_1 \sim \log(1/\alpha)$ and is independent of Ω'.

The constant reduction in amplitude per cycle can be understood as follows. When the velocity shear is increased, a wider spectrum of continuum modes is excited—but the time spent in the continuum is correspondingly reduced. Consequently, the total phase variation across the excited spectrum is unchanged. It is this overall phase dispersion that controls the continuum damping.

Equation (50) shows that the effect of velocity shear and the continuum is nonanalytic in the shear velocity Ω' and cannot be obtained from a perturbation expansion in Ω'. As noted earlier, such an expansion would give a contribution to the growth rate $\sim \kappa_2(\Omega')^2$, which for small Ω' is negligible compared to $\kappa_1|\Omega'|$. This may explain the otherwise puzzling observation [as in Fig. 9 of Ref. 14] of a ballooning mode growth rate that decreases linearly with shear velocity. It may also be relevant that where this linear dependence is absent [as in Fig. (10) of Ref. 14] the values of α and s are very small. In such a case, the growth rate $\gamma(k)$ is almost independent of k and an unstable mode may not run into the continuum.

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