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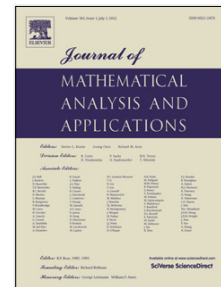
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*Manuscript

**EXISTENCE OF THREE SOLUTIONS FOR A FIRST-ORDER
PROBLEM WITH NONLINEAR NONLOCAL BOUNDARY
CONDITIONS**

DOUGLAS R. ANDERSON

ABSTRACT. Conditions for the existence of at least three positive solutions to the nonlinear first-order problem with a nonlinear nonlocal boundary condition given by

$$y'(t) - r(t)y(t) = \sum_{i=1}^m f_i(t, y(t)), \quad t \in [0, 1],$$

$$\lambda y(0) = y(1) + \sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j)), \quad \tau_j \in [0, 1],$$

are discussed, for sufficiently large $\lambda > 1$ and $r \geq 0$. The Leggett-Williams fixed point theorem is utilized.

1. INTRODUCTION

We are interested in the first-order boundary value problem with nonlinear nonlocal boundary condition given by

$$y'(t) - r(t)y(t) = \sum_{i=1}^m f_i(t, y(t)), \quad t \in [0, 1], \quad (1.1)$$

$$\lambda y(0) = y(1) + \sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j)), \quad \tau_j \in [0, 1], \quad (1.2)$$

where $r : [0, 1] \rightarrow [0, \infty)$ is continuous; the nonlocal points satisfy $0 \leq \tau_1 < \tau_2 < \dots < \tau_n \leq 1$; the nonlinear functions $\Lambda_j : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ satisfy

$$0 \leq y\psi_j(t, y) \leq \Lambda_j(t, y) \leq y\Psi_j(t, y), \quad t \in [0, 1], \quad y \in [0, \infty), \quad (1.3)$$

for some positive continuous (possibly nonlinear) functions $\psi_j, \Psi_j : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$; the scalar λ satisfies

$$\lambda > \left(1 + \sum_{j=1}^n \beta_j\right) \exp\left(\int_0^1 r(\eta) d\eta\right) > 1, \quad \beta_j := \max_{[0,1] \times [0,C]} \Psi_j(t, y) \quad (1.4)$$

for some real constant $C > 0$; and the nonlinear functions $f_i : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ are all continuous. We also set

$$\alpha_j := \min_{[0,1] \times [B, \lambda B]} \psi_j(t, y) \quad (1.5)$$

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for some constant real $B > 0$ for later reference. Note that by continuity and compactness α_j and β_j exist and satisfy $\beta_j > \alpha_j > 0$.

Some of the motivation for this paper and the study of problem (1.1), (1.2) is as follows. First-order equations with various boundary conditions, including multi-point and nonlocal conditions, are of recent interest. For example, we cite the following papers. Zhao and Sun [14] were concerned with the first-order PBVP (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) + r(t)y(t) = \lambda f(t, y(t)), \quad t \in [0, 1], \quad (1.6)$$

$$y(0) = y(1). \quad (1.7)$$

Tian and Ge [12] investigated the first-order three-point problem (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) + r(t)y(t) = \lambda f(t, y(t)), \quad t \in [0, 1], \quad (1.8)$$

$$y(0) - \alpha y(\eta) = \gamma y(1), \quad (1.9)$$

while Gao and Luo [2] were interested in the problem (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) + r(t)y(t) = \lambda f(t, y(t)), \quad t \in [0, 1], \quad (1.10)$$

$$y(0) = \sum_{j=1}^n \gamma_j y(t_j); \quad (1.11)$$

similarly Anderson [1] studied the first-order problem (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) + r(t)y(t) = \lambda f(t, y(t)), \quad t \in [0, 1], \quad (1.12)$$

$$y(0) = y(1) + \sum_{j=1}^n \gamma_j y(t_j). \quad (1.13)$$

In a related paper, Nieto and R. Rodríguez-López [8] considered

$$y'(t) + r(t)y(t) = \lambda f(t), \quad t \in [0, 1], \quad (1.14)$$

$$\lambda y(t_0) = \sum_{j=1}^n \gamma_j y(t_j). \quad (1.15)$$

Gilbert [3] looked at (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) = \lambda f(t, y(t)), \quad a.e. t \in [0, 1], \quad (1.16)$$

$$y(0) = y(1), \quad \text{or} \quad y(0) = y_0 \quad (1.17)$$

using measure theory and Δ -Carathéodory functions. Goodrich [4] analyzed the p -Laplacian problem (if $\mathbb{T} = \mathbb{R}$)

$$\phi_p(y'(t)) = h(t)f(y(t)), \quad t \in [0, 1], \quad (1.18)$$

$$y(0) = \Psi(y) \quad \text{or} \quad y(0) = B_0(y'(1)) \quad \text{or} \quad y(0) = (y'(1))^m, \quad (1.19)$$

while Graef and Kong [6] explored the related p -Laplacian problem (if $\mathbb{T} = \mathbb{R}$)

$$\phi_p(y'(t)) = f(t, y(t)), \quad t \in [0, 1], \quad (1.20)$$

$$y(0) = B(y'(1)). \quad (1.21)$$

Otero-Espinar and Vivero [9] gave a general view of different kinds of weak first-order discontinuous boundary value problems on an arbitrary time scale with nonlinear functional boundary value conditions. Shu and Deng [11] proved the existence of three positive solutions to (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) = \lambda f(y(t)), \quad t \in [0, 1], \quad (1.22)$$

$$y(0) = \gamma y(1). \quad (1.23)$$

Zhao [13] applied a monotone iteration method to the problem (if $\mathbb{T} = \mathbb{R}$)

$$y'(t) + r(t)y(t) = \lambda f(t, y(t)), \quad t \in [0, 1], \quad (1.24)$$

$$y(0) = g(y(1)), \quad (1.25)$$

where g denotes a nonlinear boundary condition. Precup and Trif [10] dealt with the existence, localization and multiplicity of positive solutions to non-local problems for first order differential systems of the form

$$y'(t) = f(t, y(t)), \quad t \in [0, 1], \quad (1.26)$$

$$y(0) = \alpha[y], \quad (1.27)$$

where α is linear and continuous. In a second-order problem, Goodrich [5] was concerned with (if $\mathbb{T} = \mathbb{R}$)

$$y'' = -\lambda f(t, y(t)), \quad t \in [0, 1], \quad (1.28)$$

$$y(0) = \psi(y), \quad y(1) = 0. \quad (1.29)$$

where ψ is a nonlocal boundary condition. One can see from these representative works that problem (1.1) with the nonlinear nonlocal boundary condition (1.2) is new.

2. AT LEAST THREE POSITIVE SOLUTIONS

In this section we establish the existence of at least three positive solutions for the boundary value problem (1.1), (1.2), by finding fixed points for operators on cones in a Banach space; the theorem we will use below is the Leggett-Williams fixed point theorem [7].

A nonempty closed convex set P contained in a real Banach space S is called a cone if it satisfies the following two conditions:

- (i) if $y \in P$ and $\zeta \geq 0$ then $\zeta y \in P$;
- (ii) if $y \in P$ and $-y \in P$ then $y = 0$.

The cone P induces an ordering \leq on S by $x \leq y$ if and only if $y - x \in P$. An operator K is said to be completely continuous if it is continuous and compact (maps bounded sets into relatively compact sets). A map θ is a nonnegative continuous concave functional on P if it satisfies the following conditions:

- (i) $\theta : P \rightarrow [0, \infty)$ is continuous;

(ii) $\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y)$ for all $x, y \in P$ and $0 \leq t \leq 1$.

Let

$$P_C := \{y \in P : \|y\| < C\} \quad (2.1)$$

and

$$P(\theta, A, B) := \{y \in P : A \leq \theta(y), \|y\| \leq B\}. \quad (2.2)$$

Theorem 2.1 (Leggett-Williams). *Let P be a cone in the real Banach space S , $K : \overline{P_C} \rightarrow \overline{P_C}$ be completely continuous, and θ be a nonnegative continuous concave functional on P with $\theta(y) \leq \|y\|$ for all $y \in \overline{P_C}$. Suppose there exist constants $0 < A < B < B^\dagger \leq C$ such that the following conditions hold:*

- (i) $\{y \in P(\theta, B, B^\dagger) : \theta(y) > B\} \neq \emptyset$ and $\theta(Ky) > B$ for all $y \in P(\theta, B, B^\dagger)$;
- (ii) $\|Ky\| < A$ for $\|y\| \leq A$;
- (iii) $\theta(Ky) > B$ for $y \in P(\theta, B, C)$ with $\|Ky\| > B^\dagger$.

Then K has at least three fixed points y_1, y_2 , and y_3 in $\overline{P_C}$ satisfying:

$$\|y_1\| < A, \quad \theta(y_2) > B, \quad A < \|y_3\| \text{ with } \theta(y_3) < B.$$

In the next theorem we use the following notation. Let

$$G(t, s) = \frac{\exp\left(\int_s^t r(\eta) d\eta\right)}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \times \begin{cases} \lambda & : 0 \leq s < t \leq 1, \\ \exp\left(\int_0^1 r(\eta) d\eta\right) & : 0 \leq t \leq s \leq 1, \end{cases} \quad (2.3)$$

denote a function that will be shown (in the proof of Theorem 2.2 below) to be the corresponding Green function for our boundary value problem,

$$M := \frac{1}{\int_0^1 G(1, s) ds} \left(1 - \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \sum_{j=1}^n \beta_j}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \right) > 0 \quad (2.4)$$

for β_j from (1.4), and

$$N := \frac{1}{\int_0^1 G(0, s) ds} \left(1 - \frac{\sum_{j=1}^n \alpha_j}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \right) > 0 \quad (2.5)$$

for α_j from (1.5). Both M and N are positive by (1.4).

Theorem 2.2. *Suppose that (1.3) and (1.4) hold, and suppose that there exist constants $0 < A < B < \lambda B \leq C$ such that*

- (F₁) $f_i(t, y) < MA/m$ for $t \in [0, 1]$, $y \in [0, A]$,
- (F₂) $f_i(t, y) > NB/m$ for $t \in [0, 1]$, $y \in [B, \lambda B]$,
- (F₃) $f_i(t, y) \leq MC/m$ for $t \in [0, 1]$, $y \in [0, C]$,

for $i = 1, 2, \dots, m$, where M and N are given above by (2.4) and (2.5), respectively. Then the boundary value problem (1.1), (1.2) has at least three positive solutions y_1, y_2, y_3 satisfying

$$\|y_1\| = y_1(1) < A, \quad B < \theta(y_2) = y_2(0), \quad \|y_3\| = y_3(1) > A \quad \text{with} \quad \theta(y_3) = y_3(0) < B,$$

where θ is given below in (2.6).

Proof. Let the Banach space $S = C[0, 1]$ be endowed with the sup norm, and define the cone $P \subset S$ by

$$P = \{y \in S : y \geq 0, y \text{ increasing}\}.$$

Let the nonnegative continuous concave functional $\theta : P \rightarrow [0, \infty)$ be defined by

$$\theta(y) := \min_{t \in [0, 1]} y(t) = y(0), \quad y \in P. \quad (2.6)$$

Using (2.3), define the operator $K : P \rightarrow S$ by

$$\begin{aligned} Ky(t) &= \sum_{i=1}^m \int_0^1 G(t, s) f_i(s, y(s)) ds + \frac{\exp\left(\int_0^t r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \\ &= \sum_{i=1}^m \int_0^t \frac{\lambda \exp\left(\int_s^t r(\eta) d\eta\right)}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} f_i(s, y(s)) ds \\ &\quad + \sum_{i=1}^m \int_t^1 \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \exp\left(\int_s^t r(\eta) d\eta\right)}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} f_i(s, y(s)) ds \\ &\quad + \frac{\exp\left(\int_0^t r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)}. \end{aligned}$$

Note that if $y \in P$, the fact that the f_i are positive, (1.3), and (1.4) imply that $Ky(t) \geq 0$ for $t \in [0, 1]$. We claim that fixed points of K are solutions of (1.1), (1.2). If $y = Ky$, then

$$\begin{aligned} \lambda y(0) - y(1) &= \sum_{i=1}^m \int_0^1 \frac{\lambda \exp\left(\int_s^1 r(\eta) d\eta\right)}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} f_i(s, y(s)) ds + \frac{\lambda \sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \\ &\quad - \sum_{i=1}^m \int_0^1 \frac{\lambda \exp\left(\int_s^1 r(\eta) d\eta\right)}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} f_i(s, y(s)) ds \\ &\quad - \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \\ &= \sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j)), \end{aligned}$$

so the boundary condition (1.2) is satisfied. Moreover,

$$y'(t) = (Ky)'(t) = r(t)y(t) + \sum_{i=1}^m f_i(t, y(t)) > 0, \quad t \in [0, 1], \quad (2.7)$$

implies that y is increasing and (1.1) is met. Thus we see that $Ky \in P$, that is to say, $K : P \rightarrow P$. In addition, K is completely continuous using a standard argument.

We now show that all of the conditions of Theorem 2.1 are satisfied, where B^\dagger is replaced by λB . For all $y \in P$ we have $y(0) = \theta(y) \leq \|y\| = y(1)$. Using (2.1), if $y \in \overline{P_C}$, then $\|y\| \leq C$ and assumption (F_3) implies $f_i(t, y(t)) \leq MC/m$ for $t \in [0, 1]$ and $i = 1, 2, \dots, m$. By (2.7) we know that Ky is increasing; as a result,

$$\begin{aligned} \|Ky\| &= Ky(1) \\ &= \sum_{i=1}^m \int_0^1 G(1, s) f_i(s, y(s)) ds + \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \\ &\leq \sum_{i=1}^m \int_0^1 G(1, s) \frac{MC}{m} ds + \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \sum_{j=1}^n y(\tau_j) \Psi_j(\tau_j, y(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \\ &\leq MC \int_0^1 G(1, s) ds + \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) C \sum_{j=1}^n \Psi_j(\tau_j, y(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \\ &\leq C \left(M \int_0^1 G(1, s) ds + \frac{\exp\left(\int_0^1 r(\eta) d\eta\right) \sum_{j=1}^n \beta_j}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \right) = C, \end{aligned}$$

where we used (1.3), (1.4), and (2.4). Therefore $K : \overline{P_C} \rightarrow \overline{P_C}$. In the same way, if $y \in \overline{P_A}$, then assumption (F_1) yields $f_i(t, y(t)) < MA/m$ for $t \in [0, 1]$ and $i = 1, 2, \dots, m$. If we mimic the argument above, then it follows that $K : \overline{P_A} \rightarrow P_A$, as the maximum for Ψ_j on $[0, 1] \times [0, A]$ is less than or equal to β_j given in (1.4) since $A < C$, for $j = 1, 2, \dots, n$. Hence, condition (ii) of Theorem 2.1 is satisfied.

To check condition (i) of Theorem 2.1, choose $y_P(t) \equiv \lambda B$ for $t \in [0, 1]$. Using (2.2), then $y_P \in P(\theta, B, \lambda B)$ and $\theta(y_P) = y_P = \lambda B > B$, so that $\{y \in P(\theta, B, \lambda B) : \theta(y) > B\} \neq \emptyset$. Consequently, if $y \in P(\theta, B, \lambda B)$, then $B \leq y(t) \leq \lambda B$ for $t \in [0, 1]$. From assumption (F_2) we have that $f_i(t, y(t)) > NB/m$ for $t \in [0, 1]$ and $i = 1, 2, \dots, m$; by the definitions of θ in (2.6) and the cone P , we have

$$\begin{aligned} \theta(Ky) &= Ky(0) \\ &= \sum_{i=1}^m \int_0^1 G(0, s) f_i(s, y(s)) ds + \frac{\sum_{j=1}^n \Lambda_j(\tau_j, y(\tau_j))}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} \\ &> NB \int_0^1 G(0, s) ds + \frac{B \sum_{j=1}^n \alpha_j}{\lambda - \exp\left(\int_0^1 r(\eta) d\eta\right)} = B, \end{aligned}$$

for N in (2.5) and α_j in (1.5). As a result we have

$$\theta(Ky) > B, \quad y \in P(\theta, B, \lambda B),$$

so that condition (i) of Theorem 2.1 holds.

Lastly we consider Theorem 2.1 (iii). Suppose $y \in P(\theta, B, C)$ with $\|Ky\| > \lambda B$. Then, using (1.4), (2.3), and the definition of θ in (2.6), we see that

$$\theta(Ky) = Ky(0) \geq \frac{1}{\lambda}Ky(1) = \frac{\|Ky\|}{\lambda} > \frac{\lambda B}{\lambda} = B.$$

Consequently all the conditions hold in Theorem 2.1, and its conclusion follows. \square

3. CLOSING COMMENTS

Although we deal with problem (1.1), (1.2) on the real unit interval, the boundary value problem and accompanying techniques introduced in this work can be readily extended to related difference equations and dynamic equations on time scales.

The main conditions here are (1.3) on the nonlinear functions, and on λ in (1.4). Note that these assumptions are fairly mild. We do not assume that the Λ_j are completely separable, nor that they are asymptotically linear. As for λ , we merely have it bounded below, leaving an unbounded range of possible values that it may assume.

Future authors may wish to explore (1.1), (1.2) using monotone iteration methods, upper and lower solution techniques, existence of solutions under singularity, or semi-positone conditions, and so on. We leave the details to interested readers.

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