Nehari manifold and existence of positive solutions to a class of quasilinear problems

C.O. Alves[a, *] and A. El Hamidi[b]

[a] Departamento de Matemática e Estatistica, UFCG,
CEP: 58109-970 , C. Grande -PB, Brazil

[b] Laboratoire de Math. & Applications,
Université de la Rochelle,17042 La Rochelle, France.

Abstract

In this paper, existence and multiplicity results to the following nonlinear elliptic equation

$$-\Delta_p u = \lambda |u|^{q-2}u + |u|^{p^*-2}u, \;\; u > 0 \;\text{ in }\; \Omega \subset \mathbb{R}^N,$$

together with mixed Dirichlet-Neumann or Neumann boundary conditions, are established. Here, $\Delta_p u$ denotes the p-Laplacian operator, $1 < q < p < N$, $p^* = \frac{Np}{N-p}$ and λ is a positive real parameter. The study is based on the extraction of Palais-Smale sequences in the Nehari manifold.

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1 Introduction

In this paper, we deal with the existence of multiple solutions to the boundary value problems

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{p^*-2}u \quad \text{in} \ \ \Omega, \\ u = 0 \quad \text{on} \ \ \Gamma, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \ \Sigma, \end{cases} \tag{1.1}$$

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{p^*-2}u \quad \text{in} \ \ \Omega, \\ \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = -a(x)|u|^{p-2}u \quad \text{on} \ \ \partial\Omega, \end{cases} \tag{1.2}$$

with respect to the real parameter λ. Here, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where Γ, Σ are smooth (N-1)-dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$, Δ_p is the p-Laplacian, $\frac{\partial}{\partial \nu}$ is the outer normal derivative, and $p^* = \frac{Np}{N-p}$. Throughout this paper, the function a is assumed to be in $L^\infty(\partial\Omega)$, $a(s) \geq a_0 > 0$ almost everywhere on a subset of $\partial\Omega$ with positive measure.

In a recent paper, K. J. Brown & Y. Zhang [6] have studied a subcritical semilinear elliptic equation with a sign-changing weight function and a bifurcation real parameter in the case $p = 2$ and Dirichlet boundary conditions. Exploiting the relationship between the Nehari manifold and fibering maps (i.e., maps of the form $t \mapsto J_\lambda(tu)$ where J_λ is the Euler function associated with the equation), they gave an interesting explanation of a well known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value. In Tarantello [16], using the same type of approach the critical case has been studied also assuming $p = 2$, $q = 2$ and Neumann boundary conditions.

In this work, we exploit similar facts to show the existence of multiple nontrivial positive solutions to (1.1) and (1.2). The idea of our approach can be summarized as follows: Let I_λ (resp. J_λ) the Euler functional associated to Problem (1.1) (resp. Problem (1.2)) defined on $W^{1,p}_\Gamma(\Omega)$ (resp. on $W^{1,p}(\Omega)$), where

$$W_\Gamma^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega); \;\; u|_\Gamma = 0\}$$

is the closure of $C_0^1(\Omega \cup \Gamma)$ with respect to the norm of $W^{1,p}(\Omega)$ (we refer the reader to the paper by Colorado & Peral [8] for a complete study in the case $p = 2$).

For each $u \in W_\Gamma^{1,p}(\Omega) \setminus \{0\}$ (resp. $W^{1,p}(\Omega) \setminus \{0\}$) $\lambda > 0$, we determine the real values of t (in terms of u and λ) such that tu belongs to the Nehari manifold:

$$\mathcal{N}_{I_\lambda} = \left\{ v \in W_\Gamma^{1,p}(\Omega) \setminus \{0\} \;:\; I'_\lambda(v)(v) = 0 \right\}$$

$$(\text{resp. } \mathcal{N}_{J_\lambda} = \left\{ v \in W^{1,p}(\Omega) \setminus \{0\} \;:\; J'_\lambda(v)(v) = 0 \right\}).$$

Then, the variable t is substituted by these special values to obtain new Euler functionals defined on the Nehari manifold. On shows easily that these functionals are bounded below, which allows to find possible critical points by minimization. Moreover, this approach allows the simultaneous construction of Palais-Smale sequences, in the Nehari manifold, giving directly existence and multiplicity results [10].

Let us mention that in the case $p = 2$ and with a subcritical concave-convex nonlinearity, this problem was studied recently by Colorado & Peral [8]. The authors showed that there is a special value Λ of the parameter λ such that Problem (1.1) admits at least two positive solutions for $\lambda \in (0, \Lambda)$, admits at least one positive solution for $\lambda = \Lambda$ and admits no positive solution for $\lambda > \Lambda$. In our opinion, an interesting question which is not treated in our paper, is to characterize Λ as a bifurcation value via the dynamic of the Nehari manifold with respect to the parameter λ (see [6]). Some results involving the p-Laplacian operator, concave-convex nonlinearity and Dirichlet boundary conditions can be found in the papers of Garcia Azorero & Peral Alonso [11] and Ambrosetti, Garcia Azorero & Peral Alonso [4].

This paper is organized as follows: Section 2 is devoted to Problem (1.1) and Section 3 is the subject of Problem (1.2).

2 The mixed Dirichlet-Neumann Problem (1.1)

It is well known that weak solutions of (1.1) correspond to critical points of the C^1 functional $I_\lambda : W^{1,p}_\Gamma(\Omega) \longrightarrow \mathbb{R}$, given by

$$I_\lambda(u) = \frac{1}{p}P(u) - \frac{\lambda}{q}Q(u) - \frac{1}{p^*}P^*(u),$$

where

$$P(u) = \int_\Omega |\nabla u|^p dx, \;\; Q(u) = \int_\Omega |u|^q dx \;\text{ and }\; P^*(u) = \int_\Omega |u|^{p^*} dx.$$

Using the fact that Γ has strictly positive measure, the Poincaré inequality is still available in the space $W^{1,p}_\Gamma(\Omega)$, hence it can endowed with the following norm

$$\|u\| = \left\{ \int_\Omega |\nabla u|^p dx \right\}^{\frac{1}{p}},$$

(see some commentaries in Kesavan's book [12, page 125], for the case p=2).

In the sequel, $|| \; ||$, $|| \; ||_q$ and $|| \; ||_{p^*}$ will denote the norms on $W^{1,p}_\Gamma(\Omega)$, $L^q(\Omega)$ and $L^{p^*}(\Omega)$ respectively. We introduce the modified functional $\widetilde{I}_\lambda$ defined on $\mathbb{R} \times W^{1,p}_\Gamma(\Omega)$ by $\widetilde{I}_\lambda(t,u) := I_\lambda(tu)$, (see [19, 9, 10]). For every $u \in W^{1,p}_\Gamma(\Omega)$, $\partial_t \widetilde{I}_\lambda(.,u)$ (resp. $\partial_{tt}\widetilde{I}_\lambda(.,u)$) is the first (resp. second) derivative of the real valued function: $t \mapsto \widetilde{I}_\lambda(t,u)$.

2.1 Preliminary results

Since the functional $\widetilde{I}_\lambda$ is even in t and that we are interested by the nontrivial solutions of (1.1), we limit our study for $t > 0$ and $u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}$.

Lemma 2.1 *For every $u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}$, there is a unique $\lambda(u) > 0$ such that the real valued function $t \mapsto \partial_t \widetilde{I}_\lambda(t,u)$ has exactly two positive zeros* (resp. *one positive zero*) *if* $0 < \lambda < \lambda(u)$ (resp. $\lambda = \lambda(u)$). *This function has no zero for $\lambda > \lambda(u)$.*

Proof. Let u be an arbitrary element of $W^{1,p}_{\Gamma}(\Omega) \setminus \{0\}$ and let us write

$$\partial_t \widetilde{I}_\lambda(t,u) = t^{q-1} H_\lambda(t,u), \text{ where } H_\lambda(t,u) = t^{p-q} P(u) - \lambda Q(u) - t^{p^*-q} P^*(u).$$

Then

$$\partial_{tt} \widetilde{I}_\lambda(t,u) = (q-1) t^{q-2} H_\lambda(t,u) + t^{q-1} \partial_t H_\lambda(t,u),$$

holds true, with

$$\partial_t H_\lambda(t,u) = t^{p-q-1} \left\{ (p-q) P(u) - (p^*-q) t^{p^*-p} P^*(u) \right\}.$$

The real valued function $H_\lambda(.,u)$ is increasing on $(0, t(u))$, decreasing on $(t(u), +\infty)$ and attains its unique maximum for $t = t(u)$, where

$$t(u) = \left(\frac{p-q}{p^*-q} \frac{P(u)}{P^*(u)} \right)^{\frac{1}{p^*-p}}. \tag{2.3}$$

Thus, the function $H_\lambda(.,u)$ has two positive zeros (resp. one positive zero) if $H_\lambda(t(u),u) > 0$ (resp. if $H_\lambda(t(u),u) = 0$) and has no zero if $H_\lambda(t(u),u) < 0$. On the other hand, a direct computation gives

$$H_\lambda(t(u),u) = \frac{p^*-p}{p-q} \left(\frac{p-q}{p^*-q} \frac{P(u)}{P^*(u)} \right)^{\frac{p^*-q}{p^*-p}} P^*(u) - \lambda Q(u).$$

Similarly, $H_\lambda(t(u),u) > 0$ (resp. $H_\lambda(t(u),u) < 0$) if $\lambda < \lambda(u)$ (resp. $\lambda > \lambda(u)$) and $H_{\lambda(u)}(t(u),u) = 0$, where

$$\lambda(u) = \Theta \frac{P^{\frac{p^*-q}{p^*-p}}(u)}{Q(u)\, P^{*\,\frac{p-q}{p^*-p}}(u)}, \tag{2.4}$$

with

$$\Theta = \frac{p^*-p}{p-q} \left(\frac{p-q}{p^*-q} \right)^{\frac{p^*-q}{p^*-p}}.$$

It follows that if $\lambda \in]0, \lambda(u)[$, the real valued function $\partial_t \widetilde{I}_\lambda(.,u)$ has two positive zeros, denoted by $t_1(u,\lambda)$ and $t_2(u,\lambda)$, verifying $0 < t_1(u,\lambda) <$

$t(u) < t_2(u,\lambda)$.
Since, $H_\lambda(t_1(u,\lambda),u) = H_\lambda(t_2(u,\lambda),u) = 0$, $\partial_t H_\lambda(t,u) > 0$ for $t < t(u)$ and $\partial_t H_\lambda(t,u) < 0$ for $t > t(u)$, it follows that

$$\partial_{tt}\widetilde{I}_\lambda(t_1(u,\lambda),u) > 0 \text{ and } \partial_{tt}\widetilde{I}_\lambda(t_2(u,\lambda),u) < 0.$$

This means that the real valued function $\widetilde{I}_\lambda(.,u)$, defined for $t > 0$, achieves its unique local minimum (resp. unique local maximum) at $t = t_1(u,\lambda)$ (resp. $t = t_2(u,\lambda)$). □

Notice that for every $u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}$ and $\lambda \in (0,\lambda(u))$, $t_1(u,\lambda)u$ and $t_2(u,\lambda)u$ belong to the Nehari manifold [19] defined by

$$\mathcal{N}_{I_\lambda} := \{u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\} \,:\, I'_\lambda(u)u = 0\}.$$

Now, we introduce

$$\lambda_1^* := \inf\left\{\lambda(u) \,:\, u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}\right\}. \tag{2.5}$$

If $S_{\Gamma,q}$ (resp. S_Γ) denotes the best Sobolev constant of the embedding $W^{1,p}_\Gamma(\Omega) \subset L^q(\Omega)$ (resp. $W^{1,p}_\Gamma(\Omega) \subset L^{p^*}(\Omega)$), then

$$\lambda_1^* \geq \widehat{C} S^{q/p}_{\Gamma,q} S_\Gamma^{\frac{N}{p}(1-q/p)} > 0.$$

Since $\partial_t\widetilde{I}_\lambda(t_1(u,\lambda),u) = 0$ (resp. $\partial_t\widetilde{I}_\lambda(t_2(u,\lambda),u) = 0$) for every $u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}$, it follows that the functional $u \longmapsto \widetilde{I}_\lambda(t_1(u,\lambda),u)$ (resp. $u \longmapsto \widetilde{I}_\lambda(t_2(u,\lambda),u)$) is bounded below on $W^{1,p}_\Gamma(\Omega) \setminus \{0\}$. Thus, for every $\lambda \in (0,\lambda_1^*)$, we define

$$\alpha_1(\lambda) = \inf\left\{\widetilde{I}_\lambda(t_1(u,\lambda),u) \,:\, u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}\right\}, \tag{2.6}$$

$$\alpha_2(\lambda) = \inf\left\{\widetilde{I}_\lambda(t_2(u,\lambda),u) \,:\, u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}\right\}. \tag{2.7}$$

Remark 2.1 *For every real number $\gamma > 0$, we have*

$$\widetilde{I}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \widetilde{I}_\lambda(t,u),$$
$$\partial_t\widetilde{I}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \frac{1}{\gamma}\partial_t\widetilde{I}_\lambda(t,u),$$
$$\partial_{tt}\widetilde{I}_\lambda\left(\gamma t, \frac{u}{\gamma}\right) = \frac{1}{\gamma^2}\partial_{tt}\widetilde{I}_\lambda(t,u),$$

it follows that

$$t_1(u,\lambda) = \frac{1}{\gamma}t_1\left(\frac{u}{\gamma},\lambda\right), \tag{2.8}$$
$$t_2(u,\lambda) = \frac{1}{\gamma}t_2\left(\frac{u}{\gamma},\lambda\right). \tag{2.9}$$

Therefore, $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ can be rewritten as follows

$$\alpha_1(\lambda) = \inf_{u\in\mathbb{S}} \widetilde{E}_\lambda(t_1(u,\lambda),u), \tag{2.10}$$
$$\alpha_2(\lambda) = \inf_{u\in\mathbb{S}} \widetilde{E}_\lambda(t_2(u,\lambda),u), \tag{2.11}$$

where $\mathbb{S}$ is the unit sphere of $W^{1,p}_\Gamma(\Omega)$.

Lemma 2.2 *Let $(u_n) \subset \mathbb{S}$ be a minimizing sequence of* (2.10) (resp. *of* (2.11)) *and $U_n := t_1(u_n,\lambda)u_n$* (resp. $V_n := t_2(u_n,\lambda)u_n$). *Then*

(i) $\limsup_{n\to+\infty} ||U_n|| < +\infty \quad$ (resp. $\limsup_{n\to+\infty} ||V_n|| < +\infty$),

(ii) $\liminf_{n\to+\infty} ||U_n|| > 0 \quad$ (resp. $\liminf_{n\to+\infty} ||V_n|| > 0$).

Proof.

(i) Let $(u_n) \subset \mathbb{S}$ be a minimizing sequence of (2.10). Since $\partial_t\widetilde{I}_\lambda(t_1(u_n,\lambda),u_n) = 0$, it follows that

$$||U_n||^p = \lambda||U_n||_q^q + ||U_n||_{p^*}^{p^*}. \tag{2.12}$$

Similarly, since $\partial_{tt}\widetilde{I}_\lambda(t_1(u_n,\lambda),u_n) > 0$, it follows that

$$(p-1)||U_n||^p - \lambda(q-1)||U_n||_q^q - (p^*-1)||U_n||_{p^*}^{p^*} > 0. \tag{2.13}$$

Combining (2.12) and (2.13), we get $I_\lambda(U_n) < 0$, for every n.

Suppose that there is a subsequence of (U_n), still denoted by (U_n) such that $\lim\limits_{n\to+\infty} ||U_n|| = +\infty$. It is well known that there is some constant $C_{p^*,q}$ such that $||U_n||_q \le C_{p*,q}||U_n||_{p^*}$ for every n, then $\lim\limits_{n\to+\infty} ||U_n||_{p^*} = +\infty$. Using the fact that $0 < q < p^*$ we get $||U_n||_q^q = \mathrm{o}_n\left(||U_n||_{p^*}^{p^*}\right)$, and consequently

$$||U_n||^p = ||U_n||_{p^*}^{p^*}(1 + \mathrm{o}_n(1)).$$

Thus,

$$I_\lambda(U_n) = ||U_n||_{p^*}^{p^*}\left(\frac{1}{N} + \mathrm{o}_n(1)\right),$$

which implies that $I_\lambda(U_n)$ tends to $+\infty$ as n goes to $+\infty$ and this is impossible. Hence, we conclude that $\limsup\limits_{n\to+\infty} ||U_n|| < +\infty$.
The same arguments with a minimizing sequence (u_n) of (2.11) show that $\limsup\limits_{n\to+\infty} ||V_n|| < +\infty$.
(ii) Let $(u_n) \subset \mathbb{S}$ be a minimizing sequence of (2.10) and suppose that there is a subsequence of (U_n), still denoted by (U_n) such that $\lim\limits_{n\to+\infty} ||U_n|| = 0$. It follows that $\lim\limits_{n\to+\infty} I_\lambda(U_n) = 0$ *i.e.* $\alpha_1(\lambda) = 0$, which is impossible since $\widetilde{I}_\lambda(t_1(u_n,\lambda),u_n) < 0$ for every n.

Let $(u_n) \subset \mathbb{S}$ be a minimizing sequence of (2.11). Since $\partial_t\widetilde{I}_\lambda(t_2(u_n,\lambda),u_n) = 0$ and $\partial_{tt}\widetilde{I}_\lambda(t_2(u_n,\lambda),u_n) < 0$ it follows that

$$\begin{cases} ||V_n||^p - \lambda||V_n||_q^q - ||V_n||_{p^*}^{p^*} = 0, \\ (p-1)||V_n||^p - \lambda(q-1)||V_n||_q^q - (p^*-1)||V_n||_{p^*}^{p^*} < 0. \end{cases}$$

Combining the two last inequalities we obtain, for every n,

$$(p-q)||V_n||^p < (p^*-q)||V_n||_{p^*}^{p^*} \le (p^*-q){S_\Gamma}^{p^*/p}||V_n||^{p^*},$$

via the continuous embedding $W^{1,p}_\Gamma(\Omega) \subset L^{p^*}(\Omega)$. Then $(p-q) \le (p^* - q){S_\Gamma}^{p^*/p}||V_n||^{p^*-p}$. Now, suppose that there is a subsequence of (V_n), still denoted by (V_n) such that $\lim\limits_{n\to+\infty} ||V_n|| = 0$. This implies that $p - q \le 0$, which is impossible. $\square$

Lemma 2.3 *Let* $(u_n) \subset \mathbb{S}$ *be a minimizing sequence of* (2.10) (resp. *of* (2.11)). *Then,* $(U_n) := (t_1(u_n,\lambda)u_n)$ (resp. $(V_n) := (t_2(u_n,\lambda)u_n)$) *are Palais-Smale sequences for the functional* I_λ.

Proof. We will show this lemma only for the sequence (U_n), the proof for (V_n) can be done in the same way.
First, according to the previous lemma, it is clear that (U_n) is bounded in $W^{1,p}_\Gamma(\Omega)$. On the other hand, notice that for every $u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}$ and $\lambda \in (0, \lambda_1^*)$, we have $\partial_t \widetilde{I}_\lambda(t_1(u,\lambda), u) = 0$ and $\partial_{tt}\widetilde{I}_\lambda(t_1(u,\lambda),u) \ne 0$. The implicit function theorem implies that $t_1(u,\lambda)$ is C^1 with respect to u since $\widetilde{I}$ is. Let us introduce the C^1 functional $\mathcal{I}_\lambda$ defined on $\mathbb{S}$ by

$$\mathcal{I}_\lambda(u) = \widetilde{I}_\lambda(t_1(u,\lambda),u) \equiv I_\lambda(t_1(u,\lambda)u).$$

Then

$$\alpha_1(\lambda) = \inf_{u\in\mathbb{S}} \mathcal{I}_\lambda(u) \ \text{ and } \ \lim_{n\to+\infty} \mathcal{I}_\lambda(u_n) = \alpha_1(\lambda).$$

Using the Ekeland variational principle on the complete manifold $(\mathbb{S}, ||\ ||)$ to the functional $\mathcal{I}_\lambda$, we conclude that

$$|\mathcal{I}'_\lambda(u_n)(\varphi_n)| \le \frac{1}{n}||\varphi_n||, \ \text{ for every } \ \varphi_n \in \mathrm{T}_{u_n}\mathbb{S},$$

where $\mathrm{T}_{u_n}\mathbb{S}$ is the tangent space to $\mathbb{S}$ at the point u_n. Moreover, for every $\varphi_n \in \mathrm{T}_{u_n}\mathbb{S}$, one has

$$\begin{aligned} \mathcal{I}'_\lambda(u_n)(\varphi_n) &= \partial_t \widetilde{I}_\lambda(t_1(u_n,\lambda),u_n)t'_1(u_n,\lambda)(\varphi_n) + \partial_u \widetilde{I}_\lambda(t_1(u_n,\lambda),u_n)(\varphi_n), \\ &= \partial_u \widetilde{I}_\lambda(t_1(u_n,\lambda),u_n)(\varphi_n), \end{aligned}$$

since $\partial_t\widetilde{I}_\lambda(t_1(u_n,\lambda),u_n)\equiv 0$, where $t_1'(u_n,\lambda)$ denotes the derivative of $t_1(.,\lambda)$ with respect to its first variable at the point (u_n,λ).
Furthermore, let

$$\begin{array}{rcll}\pi\ :\ W_\Gamma^{1,p}(\Omega)\setminus\{0\} & \longrightarrow & (0,+\infty)\times\mathbb{S} & \\ u & \longmapsto & \left(||u||,\frac{u}{||u||}\right) := (\pi_1(u),\pi_2(u)). & \end{array}$$

Applying Hölder's inequality, we get for every $(u,\varphi)\in\left(W_\Gamma^{1,p}(\Omega)\setminus\{0\}\right)\times W_\Gamma^{1,p}(\Omega)$:

$$\left\{\begin{array}{lcl}|\pi_1'(u)(\varphi)| & \le & ||\varphi||,\\ ||\pi_2'(u)(\varphi)|| & \le & 2\frac{||\varphi||}{||u||}.\end{array}\right.$$

From Lemma 2.2, there is a positive constant C such that

$$t_1(u_n,\lambda)\ge C,\ \ \forall\ n\in\mathbb{N}.$$

Then for every $\varphi\in W_\Gamma^{1,p}(\Omega)$, there are $\varphi_n^1\in\mathbb{R}$ and $\varphi_n^2\in \mathrm{T}_{u_n}\mathbb{S}$ such that $|\varphi_n^1|\le||\varphi||$, $||\varphi_n^2||\le\frac{2}{C}||\varphi||$ and

$$\begin{array}{rcl}I_\lambda'(t_1(u_n,\lambda)u_n)(\varphi) & = & \partial_t\widetilde{I}_\lambda(t_1(u_n,\lambda),u_n)(\varphi_n^1)+\partial_u\widetilde{I}_\lambda(t_1(u_n,\lambda),u_n)(\varphi_n^2),\\ & = & \partial_u\widetilde{I}_\lambda(t_1(u_n,\lambda),u_n)(\varphi_n^2),\\ & = & \mathcal{I}_\lambda'(u_n)(\varphi_n^2).\end{array}$$

Therefore,

$$\begin{array}{rcl}I_\lambda'(t_1(u_n,\lambda)u_n)(\varphi) & \le & \dfrac{1}{n}||\varphi_n^2||\\ & \le & \dfrac{2}{nC}||\varphi||.\end{array}$$

We easily conclude that

$$\lim_{n\to\infty}||I_\lambda'(U_n)||_*=0,$$

where $||\,||_*$ denotes the norm in the dual space of $W_\Gamma^{1,p}(\Omega)$. □

Remark 2.2 *Until now, the minimizing sequences we consider are not nonnegative. Notice that for every* $u \in W^{1,p}_\Gamma(\Omega) \setminus \{0\}$ *and* $0 < \lambda < \lambda_1^*$, *one has* $\widetilde{I}_\lambda(t, |u|) = \widetilde{I}_\lambda(t, u)$, $t_1(|u|, \lambda) = t_1(u, \lambda)$ *and* $t_2(|u|, \lambda) = t_2(u, \lambda)$. *Thus, every minimizing sequence* $(u_n) \subset \mathbb{S}$ *of* (2.10) *or* (2.11) *can be considered as a sequence of nonnegative functions.*

Hereafter, we will assume the sequences U_n and V_n, defined in Lemma 2.3, to be nonnegative.

Since we consider mixed Dirichlet-Neumann boundary conditions in Problem (1.1), we will need the following estimate, due to Cherrier [7]:

Lemma 2.4 *For each* $\tau > 0$, *there exists* $M_\tau > 0$ *such that*

$$\left[\frac{S}{2^{\frac{p}{N}}} - \tau\right] \|u\|^p_{p^*} \leq \|\nabla u\|^p_p + M_\tau \|u\|^p_p, \quad \forall u \in W^{1,p}(\Omega).$$

At this stage, we will state a version of the Concentration Compactness Lemma due P. L. Lions [13, 14], which follows using similar arguments explored in the case $W^{1,p}_0(\Omega)$ together with the Cherrier's inequality. In the $W^{1,p}(\Omega)$ case, we can refer the reader to [15] by Medeiros.

Lemma 2.5 *Let* $\{u_n\}$ *be a weakly convergent sequence in* $W^{1,p}_\Gamma(\Omega)$ *with weak limit* u, *and such that:*
i) $\|\nabla u_n\|^p_p \to \mu$ *weakly-* in the sense of measure,*
ii) $\|u_n\|^{p^*}_{p^*} \to \nu$ *weakly-* in the sense of measure.*
Then, for some finite index set I *we have:*

$$\begin{cases} 1)\ \nu = \|u\|^{p^*}_{p^*} + \sum_{j \in I} \nu_j \delta_{x_j}, \quad \nu_j > 0, \\ 2)\ \mu \geq \|\nabla u\|^p_p + \sum_{j \in I} \mu_j \delta_{x_j}, \quad \mu_j > 0, \\ 3)\ if\ x_j \in \Omega\ \ then\ \ S\nu_j^{\frac{p}{p^*}} \leq \mu_j, \\ 4)\ if\ x_j \in \Sigma\ \ then\ \ \frac{S}{2^{\frac{p}{N}}}\nu_j^{\frac{p}{p^*}} \leq \mu_j. \end{cases}$$

Finally, adapting well know arguments found in [2], [11], and the previous lemma, we can prove the following lemma.

Lemma 2.6 *If $\{u_n\} \subset W_\Gamma^{1,p}(\Omega)$ is a Palais-Smale Sequence to I_λ with $u_n \rightharpoonup u$ in $W_\Gamma^{1,p}(\Omega)$, then the set $I^* = \{i;\ x_i \in \Omega\} \subset I$ given in Lemma 2.5 is finite or empty and for some subsequence*

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e. in } \Omega.$$

Now, we establish that the Euler functional I_λ satisfies the Palais-Smale condition under some condition on the level of Palais-Smale sequences.

Lemma 2.7 *There exists a constant K depending only on p, q, N and Ω such that for every $\lambda > 0$, the functional I_λ satisfies the Palais-Smale condition in the interval $(-\infty, \frac{1}{2N}S^{\frac{N}{p}} - K\lambda^{\frac{p^*}{p^*-q}})$.*

Proof. Let $\{u_n\} \subset W_\Gamma^{1,p}(\Omega)$ be a Palais-Smale sequence for I_λ. Using standard arguments it follows that the sequence $\{u_n\}$ is bounded. Thus, from the above lemmas there exists a subsequence still denoted by $\{u_n\}$ and a function $u \in W_\Gamma^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$. Using the same arguments explored in Alves [3], there is a constant K depending only on p, q, N and Ω such that

$$I_\lambda(u) \geq -K\lambda^{\frac{p^*}{p^*-q}}.$$

Let $v_n = u_n - u$. Then by Brézis & Lieb [5], we have

$$\|v_n\|^p = \|u_n\|^p - \|u\|^p + o_n(1),$$

$$\|v_n\|_{p^*}^{p^*} = \|u_n\|_{p^*}^{p^*} - \|u\|_{p^*}^{p^*} + o_n(1),$$

and by Sobolev embedding

$$\int_\Omega |u_n|^q dx \to \int_\Omega |u|^q dx.$$

The above limits imply

$$\|v_n\|^p - \|v_n\|_{p^*}^{p^*} = o_n(1)$$

and

$$\frac{1}{p}\|v_n\|^p - \frac{1}{p^*}\|v_n\|_{p^*}^{p^*} = c - I_\lambda(u) + o_n(1).$$

Since the sequence $(v_n)_n$ is bounded in $W_\Gamma^{1,p}(\Omega)$, there exist $l \geq 0$ and a subsequence, still denote by $\{v_n\}$, verifying

$$\|v_n\|^p \to l.$$

Hence,

$$\|v_n\|_{p^*}^{p^*} \to l.$$

Using Cherrier's inequality and passing to the limit $n \to \infty$, we obtain

$$\left[\frac{S}{2^{\frac{p}{N}}} - \tau\right] l^{\frac{p}{p^*}} \leq l \ \ \forall \tau > 0,$$

that is,

$$\frac{S}{2^{\frac{p}{N}}} l^{\frac{p}{p^*}} \leq l.$$

Now, we claim that $l = 0$. Indeed in one hand, if $l > 0$ the last inequality implies

$$l \geq \frac{S^{\frac{N}{p}}}{2}.$$

On the other hand,

$$\frac{1}{N} l = c - I_\lambda(u),$$

and then

$$c \geq \frac{1}{2N} S^{\frac{N}{p}} - K\lambda^{\frac{p^*}{p^*-q}},$$

which contradicts the hypothesis. Therefore, $l = 0$ and we conclude that

$$u_n \to u \ \text{ in } \ W_\Gamma^{1,p}(\Omega).$$

□

Lemma 2.8 *Let $\beta := \frac{p^*}{p^*-q}$. There exist $v \in W^{1,p}_{\Gamma}(\Omega)$ and $\lambda_2^* > 0$ such that for $\lambda \in (0, \lambda_2^*)$, we have*

$$\sup_{t\geq 0} I_\lambda(tv) < \frac{1}{2N} S^{\frac{N}{p}} - K\lambda^\beta.$$

In particular,

$$\alpha_2(\lambda) < \frac{1}{2N} S^{\frac{N}{p}} - K\lambda^\beta,$$

where K is the constant found in Lemma 2.7.

Proof. Let us denote by $\{w_\varepsilon\}$ the family of functions given by

$$w_\varepsilon(x) = C_N \varepsilon^{\frac{N-p}{p^2}} \left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{p-N}{p}}$$

which attains the best constant S of the Sobolev embedding

$$D^{1,p}(I\!R^N) \hookrightarrow L^{p^*}(I\!R^N).$$

Without loss of generality, we can consider that $0 \in \Sigma$. Moreover, the set $\partial\Omega$ satisfies the following property (see more details in Adimurthi, Pacella and Yadava [1]):
There exist $\delta > 0$, an open neighborhood $\mathcal{V}$ of 0 and a diffeomorphism $\Psi : B_\delta(0) \longrightarrow \mathcal{V}$ which has a jacobian determinant equal to one at 0, with $\Psi(B_\delta^+) = \mathcal{V} \cap \Omega$, where $B_\delta^+ = B_\delta(0) \cap \{x \in \mathbb{R}^N \, : \, x_N > 0\}$.

Let $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ in a neighborhood of the origin. We define $u_\varepsilon(x) = \phi(x) w_\varepsilon(x)$. Taking $v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{p^*}}$ and using the same type of arguments developed in Medeiros [15], we get the following estimates (see Tarantello [16] and Wang [18] for the case $p = 2$)

$$\|\nabla v_\varepsilon\|_p^p = \begin{cases} \frac{S}{2^{\frac{p}{N}}} - C\varepsilon^{\frac{p-1}{p}} + o(\varepsilon^{\frac{p-1}{p}}) + \mathrm{O}\left(\varepsilon^{\frac{\mathrm{N-p}}{\mathrm{p}}}\right) & \text{if } \; N \geq p^2 \\ \\ \frac{S}{2^{\frac{p}{N}}} - C\varepsilon^{\frac{N-p}{p}} f(\varepsilon) + \mathrm{O}\left(\varepsilon^{\frac{\mathrm{N-p}}{\mathrm{p}}}\right) & \text{if } \; N < p^2 \end{cases}$$

where C is a positive constant and $\lim_{\varepsilon\to 0} f(\varepsilon) = +\infty$. Let $\delta_2 > 0$ be such that

$$\frac{1}{2N} S^{\frac{N}{p}} - K\lambda^{\beta} > 0, \ \ \forall\, \lambda \in (0, \delta_2).$$

Using the definition of I_λ , we get

$$I_\lambda(tv_\varepsilon) \le \frac{t^p}{p} \|v_\varepsilon\|_p^p, \ \ \forall\, t \ge 0,$$

which implies that there exists $t_0 \in (0, 1)$ satisfying

$$\sup_{0\le t\le t_0} I_\lambda(tv_\varepsilon) < \frac{1}{2N} S^{\frac{N}{p}} - K\lambda^{\beta}, \ \ \forall\, \lambda \in (0, \delta_2).$$

Analyzing the case $N \ge p^2$, we have

$$I_\lambda(tv_\varepsilon) \le \frac{1}{2N} S^{\frac{N}{p}} - C\varepsilon^{\frac{p-1}{p}} + o(\varepsilon^{\frac{p-1}{p}}) + O\left(\varepsilon^{\frac{N-p}{p}}\right) - \frac{\lambda t^q}{q} \int v_\varepsilon^q, \ \ \forall\, t > 0.$$

Therefore,

$$\sup_{t\ge t_0} I_\lambda(tv_\varepsilon) \le \frac{1}{2N} S^{\frac{N}{p}} - C\varepsilon^{\frac{p-1}{p}} + o(\varepsilon^{\frac{p-1}{p}}) + O(\varepsilon^{\frac{N-p}{p}}) - \frac{\lambda t_0^q}{q} \int v_\varepsilon^q.$$

Hence,

$$\sup_{t\ge t_0} I_\lambda(tv_\varepsilon) < \frac{1}{2N} S^{\frac{N}{p}} - C\varepsilon^{\frac{p-1}{p}} + o(\varepsilon^{\frac{p-1}{p}}) + \mathrm{O}\left(\varepsilon^{\frac{\mathrm{N-p}}{\mathrm{p}}}\right) - \mathrm{K}\lambda^{\beta}, \ \ \forall\, \lambda \in (0, \delta_3),$$

where

$$\delta_3 = \left(\frac{t_0^q \int v_\varepsilon^q}{2Kq}\right)^{\frac{1}{\beta-1}}.$$

We fix $\varepsilon > 0$ such that

$$-C\varepsilon^{\frac{p-1}{p}} + o(\varepsilon^{\frac{p-1}{p}}) + \mathrm{O}\left(\varepsilon^{\frac{\mathrm{N-p}}{\mathrm{p}}}\right) < 0,$$

this is possible since $\frac{N-p}{p} - \frac{p-1}{p} \ge \frac{(p-1)^2}{p} > 0$. If we set $\lambda_2^* = \min\{\delta_2, \delta_3\}$, we obtain

$$\sup_{t\ge 0} I_\lambda(tv_\varepsilon) < \frac{1}{2N} S^{\frac{N}{p}} - K\lambda^{\beta}, \ \ \forall\, \lambda \in (0, \lambda_2^*),$$

and finally

$$\alpha_2(\lambda) < \frac{1}{2N} S^{\frac{N}{p}} - K\lambda^{\beta}, \ \ \forall \, \lambda \in (0, \lambda_2^*).$$

The case $N < p^2$ follows with the same type of arguments. □

Theorem 2.1 *Let $1 < q < p$ and $\widehat{\lambda} = \min\{\lambda_1^*, \lambda_2^*\}$. Then for $\lambda \in (0, \widehat{\lambda})$, problem (1.1) has at least two nonnegative solutions.*

Proof. Using the above lemmas, there exist two sequences of positive functions $\{U_n\}$ and $\{V_n\}$ in $W_\Gamma^{1,p}(\Omega)$ such that

$$I_\lambda(U_n) \to \alpha_1(\lambda), \quad ||I_\lambda'(U_n)||_* \to 0 \ \text{ as } \ n \to +\infty$$

and

$$I_\lambda(V_n) \to \alpha_2(\lambda), \quad ||I_\lambda'(V_n)||_* \to 0 \ \text{ as } \ n \to +\infty.$$

Notice that for every $\lambda \in (0, \widehat{\lambda})$, one has

$$\alpha_1(\lambda) \leq \alpha_2(\lambda) < \frac{1}{2N} S^{\frac{N}{p}} - K\lambda^{\beta}.$$

Then, there exist two nonnegative functions $U_\lambda, V_\lambda \in W_\Gamma^{1,p}(\Omega)$ verifying

$$U_n \longrightarrow U_\lambda \ \text{ in } \ W_\Gamma^{1,p}(\Omega) \ \text{ as } \ n \to \infty$$

and

$$V_n \longrightarrow V_\lambda \ \text{ in } \ W_\Gamma^{1,p}(\Omega) \ \text{ as } \ n \to \infty.$$

Finally, the solutions $\{U_\lambda\}$ and $\{V_\lambda\}$ satisfy the inequalities

$$\partial_{tt}\widetilde{I}_\lambda(1, U_\lambda) > 0 \ \text{ and } \ \partial_{tt}\widetilde{I}_\lambda(1, V_\lambda) < 0,$$

which imply that $U_\lambda \neq V_\lambda$. Finally, applying the Harnack's inequality (see Trudinger [17]), we conclude that $\{U_\lambda\}$ and $\{V_\lambda\}$ are positive in Ω. This achieves the proof. □.

3 The Neumann Problem (1.2)

In this section, we will state similar results for the Neumann problem (1.2):

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{p^*-2}u \ \text{ in } \ \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = -a(x)|u|^{p-2}u \ \text{ on } \ \partial\Omega. \end{cases}$$

Let us recall that the function a is assumed to be in $L^\infty(\partial\Omega)$, $a(s) \geq a_0 > 0$ almost everywhere on on a subset of $\partial\Omega$ with positive measure.
The Euler functional $J_\lambda : W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ related to the above problem is given by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{p}\int_{\partial\Omega} a(x)|u|^p - \frac{\lambda}{q}\int_\Omega |u|^q - \frac{1}{p^*}\int_\Omega |u|^{p^*}.$$

As in the previous section, for solutions of (1.2) we understand critical points of the $C^1(W^{1,p}(\Omega))$ functional J_λ. Hereafter, we will denote by $\| \ \|$ the following norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p + \int_{\partial\Omega} a(x)|u|^p\Big)^{\frac{1}{p}}$$

on $W^{1,p}(\Omega)$. As in the previous section, P Q and P^* stand for the following functionals

$$P(u) = \|u\|^p, \ \ Q(u) = \int_\Omega |u|^q dx \ \text{ and } \ P^*(u) = \int_\Omega |u|^{p^*} dx.$$

Now, we are able to state the following

Theorem 3.1 *Let $1 < q < p$, there exists $\widehat{\lambda}$ such that Problem (1.2) has at least two positive solutions for $\lambda \in (0, \widehat{\lambda})$.*

Proof. With slight changes in the proofs of the last section we obtain the result. □

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