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EXPOSITA NOTE

Conditional ordering extensions

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Abstract We produce a solution to the problem of extending a quasiordering conditional on a finite list of *ex-ante* comparisons between pairs. This constitutes yet another extension of the classical Szpilrajn's theorem. Some examples of use of our result follow.

Keywords Conditional ordering extension · Szpilrajn's theorem · Infinite utility stream

JEL Classification C60 · D90 · D63

1 Introduction

The celebrated Szpilrajn's theorem establishes that every strict partial ordering can be extended to a strict linear order. A direct descendant of this statement specifies that any quasiordering (reflexive, transitive) has an ordering extension (complete, transitive): cf. Fishburn (1973, Lemma 15.4) or Suzumura (1983, Theorem A(4) Chap. 1). These keystone results prompted several other approaches to extension theorems that underly

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many contributions to the theory of choice. Next, we list some significant variations on the Szpilrajn topic. Dushnik and Miller (1941) built on Szpilrajn's original paper to state that every strict partial ordering is the intersection of its strict linear order extensions. Likewise, Donaldson and Weymark (1998) prove that every quasiordering is the intersection of its ordering extensions (see also Weymark (1999) for an alternative proof). Duggan (1999) proved a general extension theorem from which all the prior results are deduced as special cases. We can name various applications of these results. Donaldson and Weymark (1998) illustrate their theorem with several examples from welfare economics. Weymark (2000) characterizes the relations representable as Pareto extension relations. Suzumura (1976), (1983, Appendix A to Chap. 2) makes use of extension theorems in order to give conditions for a choice function to be rationalizable by an ordering. Svensson (1980) uses Szpilrajn's theorem as stated by Fishburn or Suzumura to prove the existence of ethical welfare relations in the frame of aggregation of infinite utility streams, and Asheim and Tungodden (2004) proved that such relations are precisely the ordering extensions of a well-known quasiordering.

In this work we tackle the problem of extending a quasiordering conditional on a list of *ex-ante* feasible comparisons between pairs. For the case of finite lists of arbitrary constraints we produce a necessary and sufficient condition for that conditional Szpilrajn problem to have a solution. To illustrate its use some examples are provided.

We have organized our work as follows. Section 2 poses the conditional Szpilrajn extension problem and presents a general solution that stems from the preliminary study of a significant instance. Section 3 contains examples of use of that general result. In Sect. 3.1 we derive a basic revealed preference fact under a finite number of demand observations. In Sect. 3.2 our general result is conflated with a modelling of aggregation of intergenerational utilities. We summarize in Sect. 4. Appendix contains the proofs to two theorems.

2 The existence of conditional ordering extensions

Let $\mathbf{X}$ be a non-empty set. A binary relation R on $\mathbf{X}$ is a subset of $\mathbf{X} \times \mathbf{X}$. As is standard, $x\,R\,y$ is shorthand for $(x, y) \in R$. A reflexive and transitive relation is called a *quasiordering*. An *ordering* is a complete quasiordering.

The *asymmetric factor* P_R and the *symmetric factor* I_R of R are defined by

$$P_R = \{(x, y) \in X \times X \mid xRy \text{ and not } yRx\},$$
$$I_R = \{(x, y) \in X \times X \mid xRy \text{ and } yRx\}.$$

The shorthands P for P_R, $\widetilde{P}$ for $P_{\widetilde{R}}$, $\hat{P}$ for $P_{\hat{R}}, \ldots$ or I for I_R, $\widetilde{I}$ for $I_{\widetilde{R}}$, $\hat{I}$ for $I_{\hat{R}}, \ldots$ are common use.

If R and S are binary relations on $\mathbf{X}$ and $R \subseteq S$ then we say that R is *contained* or *included* in S. An *extension* of R binary relation on $\mathbf{X}$ is a binary relation S on $\mathbf{X}$ such that $R \subseteq S$ and $P_R \subseteq P_S$. Szpilrajn's theorem (1930) (also Aliprantis and Border 1999, Theorem 1.7) assures that every quasiordering can be extended to a complete quasiordering, i.e., it has an ordering extension.

Example 1 We take $\mathbf{X} = \mathbb{R}^2$ and let R be the quasiordering on $\mathbf{X}$ defined by:

$$(x_1, x_2)\ R\ (y_1, y_2) \quad \text{if and only if} x_1 \geq y_1, x_2 \geq y_2.$$

Let $a = (1, \frac{1}{2})$, $b = (0, 1)$, $c = (0, 2)$, $d = (\frac{1}{2}, \frac{1}{2})$. We can produce $\tilde{R}$ ordering on $\mathbf{X}$ that extends R and satisfies $b\ \tilde{P}\ a$: let $(x_1, x_2)\ \tilde{R}\ (y_1, y_2)$ if and only if $x_1 + 4x_2 \geq y_1 + 4y_2$, which derives from the utility assignement $u(x_1, x_2) = x_1 + 4x_2$. We can also produce $\bar{R}$ ordering on $\mathbf{X}$ that extends R and satisfies $d\ \bar{P}\ c$: let $(x_1, x_2)\ \bar{R}\ (y_1, y_2)$ if and only if $4x_1 + x_2 \geq 4y_1 + y_2$, which derives from the utility assignement $v(x_1, x_2) = 4x_1 + x_2$. Nonetheless, some simple computations show that it is impossible to produce $\hat{R}$ ordering extension of R such that both $b\ \hat{P}\ a$ and $d\ \hat{P}\ c$ hold true.

The existence of $\tilde{R}$ or $\bar{R}$ in Example 1 is guaranteed by a general fact whose proof is implicit in the proof of Theorem 1 below: if R is a quasiordering on $\mathbf{X}$, then there is R' ordering extension of R that is bound by the constraint $y\ R'\ x$ if and only if $x\ R\ y$ is false. In our example both $a\ R\ b$ and $c\ R\ d$ are false; *nevertheless there is no $\hat{R}$ ordering extension of R such that both b $\hat{P}$ a and d $\hat{P}$ c hold true*, which shows that the aforementioned statement does not admit an immediate generalization when several constraints bound the extension. The question arises: what are the exact conditions under which we can extend a quasiordering to an ordering conditional on a finite list of predetermined comparisons? For expository convenience we first consider the particular case where all the comparisons are of the type "strict preference", and afterwards we produce a general characterizaction where indifferences can be imposed too. The key in our solution to such extension of Szpilrajn's theorem is the next concept.

Definition 1 Let $X_I = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ be an ordered list of possibly repeated elements of $\mathbf{X}$, and R a quasiordering on $\mathbf{X}$. The R^A relation associated with X_I and R is given by $a_i\ R^A\ a_j$ if and only if $a_i\ R\ b_j$.

Under the conditions of Definition 1, R^A is irreflexive if and only if $a_i R b_i$ is false for each $i = 1, \ldots, n$, because $a_i R b_i$ amounts to $a_i\ R^A\ a_i$. If $n = 1$ then R^A is acyclic if and only if $a_1 R b_1$ is false. And if $n = 2$ then R^A is acyclic if and only if the assertions $a_1\ R\ b_1$, $a_2\ R\ b_2$, and ($a_1\ R\ b_2$ plus $a_2\ R\ b_1$) are all false.

Theorem 1 *Let R be a quasiordering on a set* $\mathbf{X}$. *Let* $X_I = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ *be an ordered list of possibly repeated elements of X. The following statements are equivalent:*

(a) There is $\widetilde{R}$ ordering extension of R such that $b_i\ \widetilde{P}\ a_i$ for each $i = 1, \ldots, n$, where $\widetilde{P}$ denotes the asymmetric part of $\widetilde{R}$.
(b) R^A associated with $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ and R is acyclic.

Proof See Appendix. □

In the situation of Example 1 we have a cycle of the associated R^A on $X_I = \{a, c, b, d\}$, namely $a\ R^A\ c\ R^A\ a$, which makes it impossible to produce $\widetilde{R}$ ordering extension of R with the properties $b\ \widetilde{P}\ a$ and $d\ \widetilde{P}\ c$.

In comparing two alternatives x and y by an ordering $\widetilde{R}$, exactly one of the following three cases is true: either $x\,\widetilde{P}\,y$, or $y\,\widetilde{P}\,x$, or $x\,\widetilde{I}\,y$. Theorem 1 does not solve our problem in full since it only accounts for constraints of the two first instances, and we may be interested in extensions that display indifference between pairs of options as well. In order to characterize when this can be done we use two conditions that are defined below. Both depend on the next concept.

Definition 2 Let $X_I = \{a_1, \ldots, a_p, b_1, \ldots, b_p\}$ be an ordered list of possibly repeated elements of $\mathbf{X}$. For each $x \in X_I$, let

$$\delta(x) = \begin{cases} b_i & \text{when } x = a_i \\ a_i & \text{when } x = b_i \end{cases}$$

The next two definitions are referred to $X_I = \{a_1, \ldots, a_p, b_1, \ldots, b_p\}$ ordered list of possibly repeated elements of $\mathbf{X}$, R quasiordering on $\mathbf{X}$ and $n \leq p$. We denote $X_I^n = \{a_{n+1}, \ldots, a_p, b_{n+1}, \ldots, b_p\}$, thus when $n = p$ one has $X_I^n = X_I^p = \varnothing$.

Definition 3 The R^I relation associated with R and X_I is defined by:

$$\text{for each } x, y \in X_I : x\, R^I\, y \Leftrightarrow x\, R\, \delta(y).$$

We say that R^I is δ-*cyclic* along X_I^n when $x_1\, R^I\, x_2\, R^I \ldots R^I\, x_k\, R^I\, x_1$ implies that $\delta(x_1)\, R^I\, \delta(x_k)\, R^I \ldots R^I\, \delta(x_2)\, R^I\, \delta(x_1)$ holds too, provided that $x_1, \ldots, x_k \in X_I^n$.

Because R is reflexive one has $x R^I \delta(x)$ for each possible $x \in X_I$. Besides, if R^I is δ-cyclic along X_I^n then $a_i R b_i \Leftrightarrow b_i R a_i$ for each possible $i = n+1, \ldots, p$. The reason is that such equivalence amounts to $a_i R^I a_i \Leftrightarrow b_i R^I b_i \Leftrightarrow \delta(a_i) R^I \delta(a_i)$ for $i = n+1, \ldots, p$.

Definition 4 The R^G relation associated with R, $n \leq p$ and X_I is defined by:[1] for each $i, j \in \{1, \ldots, p\}$,

$$a_i\, R^G\, a_j \Leftrightarrow \begin{cases} a_i\, R^A\, a_j, \text{ i.e., } a_i\, R\, b_j = \delta(a_j) \\ \text{or} \\ a_i\, R\, \delta(y_1^t),\, y_{k_t}^t\, R\, b_j, \text{ where } y_1^t R^I y_2^t R^I \ldots y_{k_t}^t \text{ and } y_1^t, y_2^t, \ldots, y_{k_t}^t \in X_I^n. \end{cases}$$

(Observe that $a_i\, R\, \delta(y_1^t),\, y_1^t\, R\, b_j$ ensures $a_i\, R^G\, a_j$ whenever $y_1^t \in X_I^n$).

We say that R^G is δ-*consistent* with X_I and $n \leq p$ if:

$$a_{i(1)}\, R^G\, a_{i(2)}\, R^G \ldots R^G\, a_{i(k)}\, R^G\, a_{i(1)} \text{ entails } i(t) > n \quad \text{for some } t = 1, \ldots, k.$$

We are ready to state an extension of Szpilrajn's Theorem that accounts for all kind of feasible *ex-ante* comparisons. Its proof is in Appendix.

Theorem 2 *Let R be a quasiordering on a set* $\mathbf{X}$. *Let $X_I = \{a_1, \ldots, a_p, b_1, \ldots, b_p\}$ be an ordered list of possibly repeated elements of* $\mathbf{X}$ *and let $n \leq p$. The following statements are equivalent:*

[1] Observe that $n = p$ produces R^A.

(a) There is $\widetilde{R}$ ordering extension of R such that $b_i \, \widetilde{P} \, a_i$ for each $i = 1, \ldots, n$, and $b_i \, \widetilde{I} \, a_i$ for each $i = n + 1, \ldots, p$.
(b) R^G is δ-consistent with X_I and n, and R^I is δ-cyclic along X_I^n.

Of course Theorem 2 implies Theorem 1: if $n = p$ then δ-cyclicity of R^I holds vacuously, $R^G = R^A$, and δ-consistency of R^G with X_I and n amounts to acyclicity of R^A. The other extreme instance of Theorem 2, namely $n = 0$, is explicited as a corollary. For illustrative purposes we sketch a direct proof of it.

Corollary 1 *Let R be a quasiordering on a set $\mathbf{X}$. Let $X_I = \{a_1, \ldots, a_p, b_1, \ldots, b_p\}$ be an ordered list of possibly repeated elements of $\mathbf{X}$. The following statements are equivalent:*

(a) There is $\widetilde{R}$ ordering extension of R such that $b_i \, \widetilde{I} \, a_i$ for each $i = 1, \ldots, p$.
(b) R^I is δ-cyclic along X_I.

Sketch of proof of the Corollary Implication $a) \Rightarrow b)$ is routine. To check $b) \Rightarrow a)$ let $x \, \overline{R} \, y$ if and only if $x \, R \, y$ or ($x \, R \, x_1$,$x_k \, R \, y$ with $x_1 \, R^I \, x_2 \, R^I \, \ldots R^I \, x_k$ and $x_i \in X_I$ for each i). Then $\overline{R}$ is a reflexive and transitive *extension* of R. Besides $x_i \, \overline{I} \, \delta(x_i)$ for each $x_i \in X_I$. Thus any complete ordering extension of $\overline{R}$ satisfies the thesis. □

3 Some examples

The next sections provide us with examples of use of the results we have presented.

3.1 Revealed preference

Recall that the data set $\{(p^1, x^1), \ldots, (p^n, x^n)\}$ can be rationalized if there is $\succcurlyeq$ complete ordering on X such that $x^i \succ y$ if $p^i x^i \geq p^i y$ and $x^i \neq y$, for each $i = 1, \ldots, n$.[2]

Proposition 1 *The data set $\{(p^1, x^1), \ldots, (p^n, x^n)\}$ can be rationalized if and only if the relation $x \, S \, y \Leftrightarrow x^i = x \neq y, \; p^i x^i \geq p^i y$ for some $i = 1, \ldots, n$ is acyclic.*

Proof Necessity is direct. In order to prove sufficiency, let R be the quasiordering on X given by $x R y \Leftrightarrow$ either $x = y$ or ($x \in \{x^1, \ldots, x^n\}, y \notin \{x^1, \ldots, x^n\}$). For each $i, j \in \{1, \ldots, n\}$, let $x^i \, R^A \, x^j \;\Leftrightarrow\; (x^i \, R \, x^k$ and $x^k S x^j$ for some $k = 1, \ldots, n$). By Theorem 1, R can be extended to $\succcurlyeq$ complete ordering on X with $x^i S x^j \Rightarrow x^i \succ x^j$ if and only if R^A is acyclic. This holds because R^A coincides with S on $\{x^1, \ldots, x^n\}$. □

3.2 Intergenerational utility with initial constraints

In this section $\mathbf{X}$ denotes $Y^{\mathbb{N}}$, where $Y \subseteq \mathbb{R}$ has at least two different numbers. The elements of $\mathbf{X}$ are infinite utility streams, and we denote $\mathbf{x} = (x_n) =$

[2] For more advanced results in this line we address to Afriat (1967), Lee and Wong (2000), Matzkin and Richter (1991), and Varian (1983), among others.

$(x_1, \ldots, x_n, \ldots) \in \mathbf{X}$. They capture the utilities that are allocated to the future generations. By $\mathbf{x} \geq \mathbf{y}$ we mean $x_i \geq y_i$ for each $i = 1, 2, \ldots$, and $\mathbf{x} > \mathbf{y}$ holds for $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. A quasi-ordering (or an ordering) R on $\mathbf{X}$ is *Paretian* if and only if for each $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} > \mathbf{y} \Rightarrow \mathbf{x}\, P_R\, \mathbf{y}$, and it is *equitable* if and only if for all $\mathbf{x} \in \mathbf{X}$ and σ finite permutation of $\mathbb{N}$ one has $\mathbf{x}\, I_R\, \sigma(\mathbf{x})$. We say that R is *ethical* if it is Paretian and equitable. We also use the term *ethical welfare relation* for an ethical ordering. It is an ethical device that a decision maker could use, e.g., to assess the aggregate result of different policies that endow individual generations with utilities (cf. Asheim et al. (2006) for an example within the setting of a simple linear technology).

A particular quasi-ordering that is crucial in this analysis is the Suppes-Sen *Grading Principle* R_S. It is defined by the expression

$$\mathbf{x} R_S \mathbf{y} \Leftrightarrow \mathbf{x} \geq \sigma(\mathbf{y}) \quad \text{for some } \sigma \text{ finite permutation of } \mathbb{N}.$$

Its asymmetric and symmetric factors are made explicit in Asheim and Tungodden (2004, p. 223). Not only R_S is an ethical quasi-ordering on $\mathbf{X}$, but also any ordering extension of it via Szpilrajn's theorem is an ethical welfare relation (cf. Svensson 1980, Theorems 1, 2) and conversely, i.e., any ethical welfare relation is an ordering extension of R_S (cf. Asheim and Tungodden 2004, p. 223).

Now suppose that we need to perform a simultaneous finite list of ethical decisions between pairs of streams. The problem would be trivial if we had an ethical welfare relation to avail ourselves of. But *no ethical welfare relation can be "explicitly described"* (cf. Zame 2007). Nonetheless the next result proves that we can determine if a given list of *ex-ante* comparisons is ethical or not, without being concerned with which concrete ethical welfare relation implements such comparisons.

Proposition 2 *Let* $\{\mathbf{x_i}\}_{i=1,\ldots,p}$ *and* $\{\mathbf{y_i}\}_{i=1,\ldots,p}$ *be two lists of streams of* $\mathbf{X}$. *The following statements are equivalent*:

(a) There exists R ethical welfare relation that declares $\mathbf{y_i}\, P_R\, \mathbf{x_i}$ *for each* $i = 1, \ldots, n$ *and* $\mathbf{y_i}\, I_R\, \mathbf{x_i}$ *for each* $i = n+1, \ldots, p$.

(b) R_S^G *is* δ*-consistent with* $X_I = \{\mathbf{x_1}, \ldots, \mathbf{x_p}, \mathbf{y_1}, \ldots, \mathbf{y_p}\}$ *and n, and* R_S^I *is* δ*-cyclic along* X_I^n.[3]

Proof Implication $(a) \Rightarrow (b)$ stems directly from Theorem 2 because any ethical welfare relation is an ordering extension of R_S. In order to check $(b) \Rightarrow (a)$ apply Theorem 2 and recall that the ordering extension thus obtained is ethical as Svensson showed. □

4 Conclusion

We have stated a result that elucidates the existence of Szpilrajn extensions of a quasiordering conditional on any finite list of comparisons between pairs of options. Because ordering (or Szpilrajn) extensions have a wide range of applications in mathematical

[3] If $n = p$, this condition means: the R_S^A relation associated with $X_I = \{\mathbf{x_1}, \ldots, \mathbf{x_n}, \mathbf{y_1}, \ldots, \mathbf{y_n}\}$ and R_S is acyclic.

economics, that result prompts many potential derivations. As examples of use we have addressed two side issues. A simple revealed preference result is derived when the number of demand observations is finite. And we specified a direct reformulation of Theorem 2 to the field of aggregation of intergenerational utility.

Our study is admittedly limited: its extension to infinite lists of comparisons suggests itself as a question for further research. In particular, considering comparisons between an element and a subset can provide interesting insights about the problem.

5 Appendix

Proof of Theorem 1 Implication $a) \Rightarrow b)$ follows from the fact that $\widetilde{R}$ contains R only. Suppose that $a_{i(1)}\ R^A\ a_{i(2)}\ R^A\ \ldots\ R^A a_{i(k)}\ R^A\ a_{i(1)}$ is a cycle of R^A. Then $a_{i(1)}\ R\ b_{i(2)},\ a_{i(2)}\ R\ b_{i(3)}, \ldots, a_{i(k)}\ R\ b_{i(1)}$. Because $R \subseteq \widetilde{R}$ we obtain $b_{i(1)} \widetilde{P} a_{i(1)} \widetilde{R} b_{i(2)} \widetilde{P} a_{i(2)} \ldots a_{i(k)} \widetilde{R} b_{i(1)}$, which entails $b_{i(1)} \widetilde{P} b_{i(1)}$. This contradicts reflexivity of $\widetilde{R}$.

To check $b) \Rightarrow a)$ we proceed in two steps. First we construct recursively $R_0 := R \subseteq R_1 \subseteq R_2 \subseteq \ldots \subseteq R_n$ that satisfy the following requirements:

1. For each $k = 1, \ldots, n$, R_k is reflexive and transitive and it extends R_{k-1}.
2. R_k^A is acyclic on $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$.
3. For each $k = 1, \ldots, n$: $b_i\ P_k\ a_i$ whenever $i = 1, \ldots, k$.

To do this, we define the R_k's as follows:

For each $k = 1, \ldots, n$: $x\ R_k\ y$ if and only if $x\ R_{k-1}\ y$ or ($x\ R_{k-1}\ b_k$ and $a_k\ R_{k-1}\ y$).

We now proceed recursively: first we prove that R_1 satisfies 1–3 and then prove that if R_{t-1} satisfies 1–3 then so does R_t.

It is routine to check that R_1 is reflexive and transitive. R_1 extends R because xRy implies xR_1y and if furthermore yRx is false then yR_1x yields the contradiction $a_1 RxRyRb_1$ since R is transitive and $a_1\ R\ b_1$ is false. This accounts for 1.

In order to prove 3, because $b_1 R_1 a_1$ holds we only need to check that $a_i\ R_1\ b_i$ is false for each $i = 1, \ldots, n$. Suppose on the contrary $a_i\ R_1\ b_i$. This means that either $a_i\ R\ b_i$ (hence $a_i\ R^A\ a_i$) or $a_i\ R\ b_1$ and $a_1\ R\ b_i$ (hence $a_i\ R^A\ a_1\ R^A\ a_i$), contradicting a).

Finally, let us show that R_1^A is acyclic. Suppose $a_{i(1)}\ R_1^A\ a_{i(2)}\ R_1^A \ldots R_1^A\ a_{i(k)}\ R_1^A\ a_{i(1)}$, i.e., $a_{i(1)}\ R_1\ b_{i(2)}, a_{i(2)}\ R_1\ b_{i(3)}, \ldots, a_{i(k)}\ R_1\ b_{i(1)}$. Each statement $a_{i(j)}\ R_1\ b_{i(j+1)}$—we identify $k+1$ with 1—translates into either $a_{i(j)}\ R\ b_{i(j+1)}$ or $a_{i(j)}\ R\ b_1$ and $a_1\ R\ b_{i(j+1)}$). By substituting the adequate instance in each statement we produce a cycle of $R^A = R_0^A$, contradicting b).

Now suppose that R_{k-1} satisfies 1–3. Then we can check that R_k satisfies 1 and 2 exactly as we did above. In order to check 3, observe that when $i < k$ we can assure $b_i\ P_{k-1}\ a_i$ which in turn entails $b_i\ P_k\ a_i$ because $P_{k-1} \subseteq P_k$. Since $b_k\ R_k\ a_k$ holds by definition, and 2 yields that $a_i\ R_k\ b_i$ is false for each $i = 1, \ldots, n$, we obtain $b_k\ P_k\ a_k$. This shows that R_k satisfies 3.

In our second step, an appeal to Szpilrajn's theorem ensures that R_n (and therefore R) can be extended to an ordering $\widetilde{R}$ on $\mathbf{X}$. For each $i = 1, \ldots, n$ one has $b_i\ P_{\widetilde{R}}\ a_i$, because $b_i\ P_n\ a_i$ and $\widetilde{R}$'s asymmetric part extends R_n's asymmetric part. Thus $\widetilde{R}$ satisfies the thesis. □

Proof of Theorem 2 We first prove $(a) \Rightarrow (b)$.

To prove δ-consistency, take $a_{i(1)}\, R^G\, a_{i(2)}\, R^G \ldots R^G\, a_{i(k)}\, R^G\, a_{i(1)}$. By using $R \subseteq \widetilde{R}$, thus $I \subseteq \widetilde{I}$, we readily obtain $a_{i(t)}\, \widetilde{R}\, b_{i(t+1)}$ for each possible t—we keep the convention $k+1=1$. Also, for each $t \in \{1,\ldots,k\}$ we know that $b_{i(t)}\, \widetilde{R}\, a_{i(t)}$ by (a). Therefore if $i(t) \leq n$ for some $t \in \{1,\ldots,k\}$ we obtain the absurd conclusion $b_{i(t)}\, \widetilde{P}\, b_{i(t)}$ as in the proof of Theorem 1 because $b_{i(t)}\, \widetilde{P}\, a_{i(t)}$.

Let us now check that R^I is δ-cyclic along X_I^n. Suppose $x_1 R^I x_2 R^I \ldots x_k R^I x_1$, but $\delta(x_2) R^I \delta(x_1)$ is false, and $x_1,\ldots x_k \in X_I^n$. The proof will be finished if we derive a contradiction. We have assumed that $\delta(x_2)\, R\, x_1$ is false. Because $x_1\, R^I\, x_2$ means $x_1\, R\, \delta(x_2)$ we obtain $x_1\, P\, \delta(x_2)$ and thus $x_1\, \widetilde{P}\, \delta(x_2)$. Also, $x_2\, \widetilde{R}\, \delta(x_3)\, \widetilde{I}\, x_3\, \widetilde{R} \ldots \widetilde{I}\, x_1$ yields $x_2\, \widetilde{R}\, x_1$. We have reached the contradiction $x_1\, \widetilde{P}\, \delta(x_2)\, \widetilde{I}\, x_2\, \widetilde{R}\, x_1$.

To prove $b) \Rightarrow a)$ we first define an auxiliary binary relation and then we obtain the desired extension. To that purpose we let $\overline{R}$ be defined by the expression:

$$x\, \overline{R}\, y \ \Leftrightarrow\ \text{either } x\, R\, y \text{ or there are } y_1 R^I y_2 R^I \ldots R^I y_k,\ \ y_1,\ldots,y_k \in X_I^n \\ \text{such that } x\, R\, \delta(y_1) \text{ and } y_k\, R\, y. \tag{1}$$

Observe that $x\, R\, \delta(y_1)$, $y_1\, R\, y$ and $y_1 \in X_I^n$ together ensure $x\, \overline{R}\, y$. Furthermore, $a_i\, \overline{R}\, b_j \Leftrightarrow a_i\, R^G\, a_j$ for each $i, j = 1,\ldots,p$.

It is routine to check that $\overline{R}$ is reflexive and transitive. We denote $\overline{R}$'s symmetric part by $\overline{I}$. For $i = n+1,\ldots,p$, one obtains that $a_i\, \overline{I}\, b_i$, i.e., $a_i\, \overline{R}\, b_i\, \overline{R} a_i$, by using $a_i\, R\, a_i = \delta(b_i)$ and $b_i\, R\, b_i = \delta(a_i)$. Now let us prove that $\overline{R}$ extends R. Observe that only $P \subseteq \overline{P}$ must be checked since $\overline{R}$ includes R. Suppose on the contrary that there are x, y such that $x\, P\, y$ (thus $x\, R\, y$ and $x \overline{R} y$ hold true, $y\, R\, x$ is false) and $y \overline{R} x$. According to Eq. (1), there must be $y_1 R^I y_2 R^I \ldots R^I y_k$, $y_1,\ldots,y_k \in X_I^n$ such that $y\, R\, \delta(y_1)$, $y_k\, R\, x$. Coupled with $x R y$ this entails $y_k\, R\, y\, R\, \delta(y_1)$ and $y_k\, R\, \delta(y_1)$ because R is transitive. We have obtained $y_1\, R^I\, y_2\, R^I \ldots y_k\, R^I y_1$, and now assumption (a) yields $\delta(y_1)\, R^I\, \delta(y_k)\, R^I \ldots \delta(y_2)\, R^I \delta(y_1)$. We deduce $\delta(y_1)\, R\, y_k\, R\, x$, which yields the contradiction $y\, R\, x$ because $y\, R\, \delta(y_1)$.

Now two cases arise. If $n = 0$ then any ordering extension of $\overline{R}$ fulfils the thesis. If $n > 0$ then Theorem 1 applies: $\overline{R}^A$ has no cycle on ${}^nX_I = \{a_1,\ldots,a_n,b_1,\ldots,b_n\}$ because $\overline{R}^A = R^G$ $(a_i\, \overline{R}^A\, a_j \Leftrightarrow a_i\, \overline{R}\, b_j)$ and R^G is δ-consistent with X_I and n. By Theorem 1 there is $\widetilde{R}$ ordering extension of $\overline{R}$ such that $b_i\, \widetilde{P}\, a_i$ for each $i = 1,\ldots,n$. Besides, $a_i\, \widetilde{I}\, b_i$ for each $i = n+1,\ldots,p$ because $\widetilde{R}$ contains $\overline{R}$ and $a_i\, \overline{I}\, b_i$ for each $i = n+1,\ldots,p$. Therefore $\widetilde{R}$ satisfies the thesis. □

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