

PRICING EQUATIONS IN JUMP-TO-DEFAULT MODELS

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We study pricing equations in jump-to-default models, and we provide conditions under which the option price is the unique classical solution, with a special focus on boundary conditions. In particular, we find precise conditions ensuring that the option price at the default boundary coincides with the recovery payment. We also study spatial convexity of the option price, and we explore the connection between preservation of convexity and parameter monotonicity.

Keywords: Jump-to-default model; credit risk; martingales; the Black–Scholes equation.

1. Introduction

Merton's (1974) first structural model for credit risk, followed by Black & Cox (1976), relies on an understanding of the fact that equity and debt of a firm are linked. This link motivates the need to price equity derivatives and credit derivatives consistently in a unified framework, enabling consistent risk management and hedging. In these first structural models only diffusion processes were used, and default happens when the firm's asset value hits a lower boundary, thus making default predictable. An alternative approach is represented by so-called reduced form models, which were introduced by Jarrow & Turnbull (1995). In such models, a default intensity is specified, which leads to unpredictable default times. Recent studies of reduced form models in which the stock price jumps to zero at default are performed by Linetsky (2006), Carr & Linetsky (2006), Carr & Wu (2010), Carr & Madan (2010), Mendoza-Arriaga *et al.* (2010), and Bayraktar & Yang (2011) (see also Merton (1976, p. 135) for an early reference in which the arrival rate of the default event is constant).

In the current paper, we study diffusion models extended with a possible jump to default. Here the stock price follows a diffusion process up to the default time, which is modeled as the first jump time of a doubly stochastic Poisson (Cox) process, with a jump intensity that is dependent on the current state of the underlying diffusion,

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compare Linetsky (2006), Carr & Linetsky (2006) and Carr & Madan (2010). At default the stock price drops to zero, reflecting the principle of absolute priority stipulating that stockholders do not receive any recovery. The jump-to-default model is thus an analytically tractable, yet flexible, hybrid model that includes both a stock price process and a default hazard process.

While the focus in Linetsky (2006) and Carr & Linetsky (2006) is on models allowing for explicit closed forms solutions for the standard financial contracts, we consider general jump-to-default extended diffusion models under weak assumptions on the jump intensity. In the absence of closed form solutions, numerical methods have to be employed when pricing, and for this, appropriate boundary conditions are needed. We provide a thorough study of the corresponding pricing equation for this type of models. The contracts we consider have payoffs that are functions of the underlying stock price at maturity combined with a recovery payment in the event of default. In particular, conditions are given under which the price is continuous up to the boundary where the stock price is zero. One may note that these conditions are also necessary in the sense that if they are not fulfilled, then the option price close to the boundary will not be given by the recovery in case of default.

We also provide convexity theory for jump-to-default models. In particular, we show that if the default rate is convex in the underlying stock price and the recovery payment, expressed in terms of currency units at maturity, is increasing in time to maturity, then the jump-to-default model is convexity preserving in the sense that if the pay-off at maturity is convex, then also the price of the contract is convex in the stock price. Further we show that preservation of convexity has important implications for parameter dependence. In fact, convexity preserving models are increasing in the default rate and in the volatility. For related results on preservation of convexity and parameter dependence in classical diffusion models and jump-diffusion models, see for example (Bergman *et al.*, 1996; Ekström & Tysk, 2007, 2008; El Karoui *et al.*, 1998; Hobson, 1998; Janson & Tysk, 2003).

The outline of the paper is as follows. In Sec. 2, we introduce the model together with general assumptions, and we formulate our main results. For the sake of simplicity, these results are formulated for time-independent models, but are easily extendable to more general models as described in that section. In Sec. 3, we prove that the option price is the classical solution to the pricing equation as formulated in Sec. 2. Finally, in Sec. 4, we provide conditions under which the model is convexity preserving and option prices are monotone in the default intensity and in the volatility.

2. Assumptions and Main Results

We model the pre-default stock price Y_t by $Y_0 = x \in [0, \infty)$ and

$$dY_t = (r - q + \lambda(Y_t))Y_t dt + \sigma(Y_t) dW_t. \quad (2.1)$$

Here the interest rate r and the dividend yield q are non-negative constants, the default intensity λ and the diffusion coefficient σ are given functions of

the pre-default stock price, and W is a standard Brownian motion. Denote by $\tau_0 = \inf\{t \geq 0 : Y_t \leq 0\}$ the first hitting time (possibly infinite) of zero, and define

$$A_t = \begin{cases} \int_0^t \lambda(Y_s) ds & t < \tau_0 \\ \infty & t \geq \tau_0. \end{cases}$$

Let θ be exponentially distributed with parameter 1 and such that θ and W are independent. Let

$$\tau = \inf\{t \geq 0 : A_t > \theta\}, \tag{2.2}$$

and define the stock price process X by

$$X_t := \begin{cases} Y_t & t < \tau \\ 0 & t \geq \tau. \end{cases}$$

Then 0 is the cemetery state for X . Moreover, the stochastic process X jumps to 0 at time τ , and the rate of default by a jump to default, conditional on $X_t > 0$, is $\lambda(X_t)$.

In this paper, we consider options specified by a terminal reward if no default happened together with a time-dependent rebate in case of default. More precisely, let $\phi : [0, \infty) \rightarrow \mathbb{R}$ and $g : [0, \infty) \rightarrow \mathbb{R}$ be two continuous functions, and define $u : [0, \infty)^2 \rightarrow \mathbb{R}$ by

$$u(x, t) := \mathbb{E}_x \left[e^{-rt} g(X_t) 1_{\{t < \tau\}} + e^{-r\tau} \phi(t - \tau) 1_{\{\tau \leq t\}} \right]. \tag{2.3}$$

Note that, in this equation, $t \geq 0$ represents the time left to maturity. Also note that $\phi(t)$ represents the rebate paid out *at default* in case default happens when the time left to maturity is t . If the contract instead would be specified so that the holder receives the rebate $\tilde{\phi}(t)$ in case default happens when the time left to maturity is t , but to be paid out *at maturity*, then the correct option price is obtained if $\phi(t) := e^{-rt} \tilde{\phi}(t)$.

Remark. The process $e^{-(r-q)t} X_t$ is a martingale under some mild growth conditions on the coefficients (specified in Hypothesis 2.1). A market consisting of a risk-free bank account with zero interest rate and the risky asset X is of course incomplete, and there are infinitely many equivalent martingale measures. We do not address the issue of picking a suitable pricing measure, but we rather assume that the model has been calibrated and specified directly under the measure used for pricing. In this way, u defined in (2.3) is, by definition, the option price.

The corresponding pricing equation is

$$\begin{cases} u_t = \mathcal{L}u(x, t) + \lambda(x)\phi(t) & (x, t) \in (0, \infty)^2 \\ u(x, 0) = g(x) \\ u(0, t) = \phi(t), \end{cases} \tag{2.4}$$

where

$$\mathcal{L}u = \alpha u_{xx} + \beta u_x - (r + \lambda)v$$

with $\alpha(x) := \sigma^2(x)/2$ and $\beta(x) = (r - q + \lambda(x))x$. If $u \in C([0, \infty)^2) \cap C^{2,1}((0, \infty)^2)$ satisfies all equalities in (2.4), then we say that u is a classical solution of the pricing equation.

Hypothesis 2.1. *The pay-off functions $g : [0, \infty) \rightarrow \mathbb{R}$ and $\phi : [0, \infty) \rightarrow \mathbb{R}$ are continuous and satisfy $g(0) = \phi(0)$, and g is of at most polynomial growth. The default rate $\lambda : (0, \infty) \rightarrow [0, \infty)$ is nonincreasing and locally Lipschitz continuous. The diffusion coefficient $\sigma : [0, \infty) \rightarrow [0, \infty)$ is locally Hölder (1/2) and satisfies the bounds*

$$0 < \sigma(x) \leq C(1 + x)$$

for all $x \in (0, \infty)$.

Remark. Note that the case $\sigma(0) > 0$ is allowed.

One of our main results provides conditions under which the option price is the unique classical solution of the pricing equation.

Theorem 2.2 (Classical solution). *Assume Hypothesis 2.1. Also assume that either $\lim_{x \rightarrow 0} \lambda(x) = \infty$ or that the process Y can reach zero. Then the function u defined in (2.3) is a classical solution of the pricing equation (2.4). Moreover, it is the unique classical solution of at most polynomial growth in the spatial variable.*

Remark. The classical Feller condition for explosion of diffusions shows that Y can reach zero if and only if

$$\int_0^1 \exp \left\{ \int_x^1 \frac{\beta(z)}{\alpha(z)} dz \right\} \int_x^1 \frac{1}{\alpha(y)} \exp \left\{ - \int_y^1 \frac{\beta(z)}{\alpha(z)} dz \right\} dy dx < \infty,$$

where $\alpha(z) = \frac{1}{2}\sigma^2(z)$ and $\beta(z) = (r - q + \lambda(z))z$.

Remark. If the pre-default price Y cannot reach zero, and if the local default rate $\lambda(x)$ is bounded, then the boundary condition at $x = 0$ specified in (2.4) is not correct. As an example, it is straightforward to check that if $g(x) = x$, $r = q = 0$, σ is linear, $\lambda(x) = 1$ and $\phi(t) = t$, then $u(x, t) = x + t - 1 + e^{-t}$. Thus $u(0+, t) = t - 1 + e^{-t} \neq t = \phi(t)$. We believe that in such cases, the appropriate boundary condition at $x = 0$ is always obtained by formally inserting $x = 0$ in the pricing equation, thus obtaining

$$\begin{cases} u_t(0, t) = \lambda(0)\phi(t) - (r + \lambda(0))u(0, t) & t > 0 \\ u(0, 0) = \phi(0). \end{cases} \quad (2.5)$$

Solving (2.5) then gives the explicit boundary condition

$$u(0, t) = e^{-(r+\lambda(0))t} \left(\lambda(0) \int_0^t e^{(r+\lambda(0))s} \phi(s) ds + \phi(0) \right).$$

We also study preservation of spatial convexity and parameter monotonicities.

Theorem 2.3 (Preservation of convexity). *In addition to Hypothesis 2.1, also assume that g and λ are convex and that the function $e^{rt}\phi(t)$ is increasing. Then $x \mapsto u(x, t)$ is convex for any fixed $t \geq 0$.*

Theorem 2.4 (Parameter monotonicities). *Assume that g is convex and $e^{rt}\phi(t)$ is increasing, and let $\lambda_i : (0, \infty) \rightarrow [0, \infty)$, $\sigma_i : [0, \infty) \rightarrow [0, \infty)$, $i = 1, 2$ be functions such that $\lambda_1 \leq \lambda_2$ and $\sigma_1 \leq \sigma_2$. If λ_i and σ_i satisfy the conditions of Hypothesis 2.1 and if λ_1 or λ_2 is convex, then the corresponding option prices satisfy $u_1 \leq u_2$.*

Remark. Theorems 2.3 and 2.4 also hold if the assumption $\phi(0) = g(0)$ is replaced with the weaker condition $\phi(0) \geq g(0)$. Indeed, in that case one may approximate g by $g_n \geq g$ such that g_n is convex and $g_n(0) = \phi(0)$.

The proof of Theorem 2.2 is given in Sec. 3, and the proofs of Theorems 2.3 and 2.4 are given in Sec. 4. For similar results in jump models with fixed jump intensity, but with level-dependent relative jump sizes, see Ekström & Tysk (2007).

Remark. The results of this section are readily extendable to the case of time-dependent models. Let $\sigma = \sigma(x, t)$ and $\lambda = \lambda(x, t)$ be measurable in time. For the results above to hold, we need σ and λ to satisfy the conditions in the relevant theorems in the spatial variable for each fixed t . Moreover, if these conditions hold locally uniformly in t , the proofs in the upcoming sections are more or less directly applicable. For instance, if the process Y can reach zero, it is enough if $\lambda(x, t) \leq \bar{\lambda}(x)$ and $\sigma(x, t) \geq \underline{\sigma}(x)$ where $\bar{\lambda}$ and $\underline{\sigma}$ satisfy the relevant conditions in the theorems above as well as the Feller condition. In the case of exploding default rates, a locally uniform lower bound suffices.

Remark. In Table 1, we list a few choices of intensity functions λ and volatilities σ/x that have been studied in the literature. All the last three models fulfill the model assumptions of Theorems 2.2–2.4. In Madan & Unal (1998) the authors

Table 1. Some examples of hybrid models from the literature. Here c , δ and p are positive constants, and a and b are functions of time.

Model	Intensity	Volatility
Madan & Unal (1998)	$\frac{c}{(\ln(X_t/\delta))^2}$	Constant
Linetsky (2006)	αx^{-p}	Constant
Carr & Linetsky (2006)	$b(t) + cx^{-2p}$	$a(t)x^{-p}$
Carr & Madan (2010)	$b(t)x^{-p}$	General local vol.

assume a positive default level $\delta > 0$. Clearly, a simple translation of our results shows that the model in Madan & Unal (1998) can be considered a special case of our general set-up, and the assumptions of Theorems 2.2–2.4 are then satisfied.

We end this section with a few examples of commonly used contracts.

Example. European call options with no recovery in case of default (i.e. $g(x) = (x - K)^+$, $\phi(t) = 0$) and European put options with the recovery equal to the strike price, paid out at maturity (i.e. $g(x) = (K - x)^+$, $\phi(t) = K^{-rt}$), are covered by Theorems 2.2–2.4.

Example. A defaultable bond with no recovery (i.e. $g(x) = 1$, $\phi(t) = 0$) is covered by versions of Theorems 2.3–2.4 for concave contracts. Thus, if the remaining assumptions of these theorems are fulfilled, then the value is concave in x and decrease in the jump intensity and in the volatility.

Example. The floating leg of a credit default swap (CDS) pays a fixed amount at default if default happens before maturity, and if no default occurs, nothing is paid out ($g(x) = 0$, $\phi(t) = 1$). Thus Theorems 2.3–2.4 apply (compare the remark after Theorem 2.4). The fixed leg of a CDS can be valued as a portfolio of defaultable bonds with no recovery, compare the previous example.

3. The Option Price is a Classical Solution

We begin the proof of Theorem 2.2 by showing uniqueness by employing classical maximum principle arguments.

Proposition 3.1 (Uniqueness). *There exists at most one classical solution of (2.4) of at most polynomial growth.*

Proof. Assume that u_1 and u_2 are two classical solutions of (2.4) of at most polynomial growth. Let $T_0 < \infty$, and let C and $n \geq 1$ be positive constants such that $u := u_2 - u_1$ satisfies

$$|u(x, t)| \leq C(1 + x^n)$$

for all $(x, t) \in [0, \infty) \times [0, T_0]$. Let $h(x, t) = e^{Mt}(1 + x^{n+1})$, where the constant M is chosen large enough so that h satisfies

$$\frac{\partial h}{\partial t} > \mathcal{L}h$$

for all $(x, t) \in [0, \infty) \times [0, T_0]$. Then $h(0, t) > u(0, t) = 0$ and h dominates u at spatial infinity. Using standard arguments to prove the maximum principle, see (Lieberman, 1996, Chapter II), one can show that $u \leq \epsilon h$ for all $(x, t) \in [0, \infty) \times [0, T_0]$ and any $\epsilon > 0$. Consequently, $u \leq 0$, and by symmetry $u = 0$. Thus $u_1 \equiv u_2$ on $[0, \infty) \times [0, T_0]$. Since T_0 is arbitrary, this finishes the proof. \square

We next show that the option price u is of at most polynomial growth.

Proposition 3.2 (Polynomial moments). *For each $k \geq 0$ and $T > 0$, there exists a constant $C > 0$ such that $\mathbb{E}_x X_\gamma^k \leq C(1 + x^k)$ for all $x \geq 0$ and all stopping times $\gamma \leq T$.*

Proof. Denote by Y the process in (2.1), absorbed at zero if ever reached. By construction, $0 \leq X_t \leq Y_t$ for all t , so it is enough to prove the statement for Y . By redefining γ as $\gamma \wedge \inf\{t \geq 0 : Y_t \geq M\}$ we can assume that $Y_{t \wedge \gamma}$ is bounded. We shall see that our bound does not depend on M and we can thus let $M \rightarrow \infty$ to obtain the result. Let $h \in C^2([0, \infty))$ be positive and satisfy

- $h(x) = 1$ for $x \leq 1/2$,
- $h(x) \geq x^k$ everywhere,
- $h(x) = x^k$ for $x \geq 2$.

The assumptions in Hypothesis 2.1 guarantee the existence of a constant C_1 such that

$$\alpha h'' + \beta h' \leq C_1 h$$

for all $x \in [0, \infty)$. Define $f(t) = \mathbb{E}_x[h(Y_{t \wedge \gamma})]$. Then, by Ito's formula,

$$\begin{aligned} f(t) &= h(x) + \mathbb{E}_x \left[\int_0^{t \wedge \gamma} \alpha(Y_s) h''(Y_s) + \beta(Y_s) h'(Y_s) ds \right] \\ &\leq h(x) + C_1 \mathbb{E}_x \left[\int_0^{t \wedge \gamma} h(Y_s) ds \right] \leq h(x) + C_1 \int_0^t f(s) ds. \end{aligned}$$

Consequently, Gronwall's lemma yields

$$\mathbb{E}_x[Y_{t \wedge \gamma}^k] \leq \mathbb{E}_x[h(Y_{t \wedge \gamma})] \leq h(x) e^{C_1 t} \leq C(1 + x^k),$$

which finishes the proof. \square

To prove continuity of the value function u , we first want to establish continuity properties of the stock price X and of the default time τ . To emphasize the dependence of the initial point, we write X^x and τ^x , respectively. Similarly, the pre-default stock price is denoted by Y^x . Recall that the pre-default stock price process Y^x is defined in (2.1) using the same Brownian motion W regardless of the initial point x . Similarly, the default time τ^x in (2.2) is defined in terms of the same exponentially distributed random variable θ , regardless of x . In this way, the process X^x is a martingale with respect to the completion $\mathbb{F}^x = (\mathcal{F}_t^x)_{t \geq 0}$ of the filtration $\sigma(W_s, I_s^x; s \leq t)$, where the default indicator process I^x is defined by $I_t^x = 1_{\{\tau^x \leq t\}}$. (However, one may note that X^x is not a martingale with respect to \mathbb{F}_t^y , $y \neq x$.)

Moreover, by standard comparison results (see for example (Revuz & Yor, 1999, Theorem IX.3.7)) we have that $x_1 \leq x_2$ implies $Y^{x_1} \leq Y^{x_2}$. Since λ is non-increasing, we also find that $\tau^{x_1} \leq \tau^{x_2}$ and $X^{x_1} \leq X^{x_2}$.

Proposition 3.3. *Let the assumptions of Theorem 2.2 hold. Then $\tau^x \rightarrow 0$ a.s. as $x \rightarrow 0$.*

Proof. First assume that $\lambda(0+) = \infty$. Let $t > 0$ and $\epsilon > 0$. For $k > 1$, let $A_k := \{X_s \leq kx, s \in [0, t]\}$, and let $B := \{X_s > 0, s \in [0, t]\}$. By the martingale inequality,

$$\mathbb{P}_x(A_k) \geq 1 - 1/k.$$

Thus

$$\begin{aligned} \mathbb{P}_x(B) &\leq \mathbb{P}_x(B \cap A_k) + \mathbb{P}_x(A_k^c) \\ &= \mathbb{P}_x\left(\left\{\int_0^t \lambda(Y_s) ds < \theta\right\} \cap B \cap A_k\right) + \mathbb{P}_x(A_k^c) \\ &\leq \mathbb{P}_x(t\lambda(kx) < \theta) + 1/k. \end{aligned}$$

Since $\lambda(y) \rightarrow \infty$ as $y \rightarrow 0$, we may choose $k = 2/\epsilon$ and $\delta > 0$ small enough so that $\mathbb{P}(t\lambda(k\delta) < \theta) \leq \epsilon/2$, which finishes the proof in the case that λ explodes.

Now assume that the process Y can reach zero in finite time. Denote by φ the decreasing and positive solution of the ordinary differential equation

$$\alpha\varphi'' + \beta(x)\varphi' - \varphi = 0.$$

The function φ is uniquely determined up to multiplication with a positive constant, see (Borodin & Salminen, 2002, pp. 18–19). Since Y can reach zero, general diffusion theory yields $\varphi(0) < \infty$. Recall the notation $\tau_0 = \inf\{t : Y_t \leq 0\}$, and let $M_t := e^{-t\wedge\tau_0}\varphi(Y_{t\wedge\tau_0})$. By Ito's formula, $\{M_t, 0 \leq t < \infty\}$ is a local martingale. Since M is bounded, $M_\infty = \lim_{t \rightarrow \infty} M_t$ exists, and $\{M_t, 0 \leq t \leq \infty\}$ is a martingale. By optional sampling,

$$\varphi(x) = \mathbb{E}_x M_{\tau_0} = \mathbb{E}_x e^{-\tau_0} \varphi(Y_{\tau_0}) = \varphi(0) \mathbb{E}_x e^{-\tau_0}.$$

Given $t > 0$ and $\epsilon > 0$, choose $\delta > 0$ so that

$$\frac{\varphi(x)}{\varphi(0)} \geq 1 - \epsilon(1 - e^{-t})$$

for $x \leq \delta$. Then $\mathbb{P}_x(\tau_0 > t) \leq \epsilon$ for such x . Since $\tau \leq \tau_0$, the result follows. \square

Proposition 3.4. *Let $t \in [0, \infty)$ and $x, y \in [0, \infty)$. Then $(X_t^y, \tau^y) \rightarrow (X_t^x, \tau^x)$ a.s. as $y \rightarrow x$.*

Proof. First recall that if $x < y$, then $X_t^x \leq X_t^y$. Therefore

$$\mathbb{E}|X_t^x - X_t^y| = |x - y|$$

by the martingale property. Consequently, $X_t^y \rightarrow X_t^x$ a.s. as $y \rightarrow x$.

For the convergence of τ^y to τ^x , first assume that $y < x$. Then $\tau^y \leq \tau^x$, and it follows from above that there exists a set of full measure such that $X_t^y \rightarrow X_t^x$ for all rational t almost surely. It follows that $\tau^y \rightarrow \tau^x$ almost surely as $y \uparrow x$.

Now consider the case when $y > x$. Since $X_t^y \rightarrow X_t^x$ a.s. as $y \searrow x$, and since $X_t^x = 0$ on $\{\tau^x \leq t\}$ we have that $X_t^y 1_{\{\tau^x \leq t\}} \rightarrow 0$ a.s. It follows from Proposition 3.3

that $\lim_{y \searrow x} \tau^y \leq t$ on $\{\tau^x \leq t\}$. Consequently, $\tau^y \searrow \tau^x$ a.s. as $y \searrow x$, thus finishing the proof. \square

Proposition 3.5. *Under the assumptions of Theorem 2.2, u is continuous.*

Proof. We claim that $X_u^y \rightarrow X_t^x$ in probability as $y \rightarrow x$ and $u \rightarrow t$. We assume that $u \leq t$ (a similar argument applies if $u \geq t$). Then

$$\begin{aligned} \mathbb{P}(|X_u^y - X_t^x| > \varepsilon) &\leq \mathbb{P}(|X_u^y - X_u^x| > \varepsilon/2) + \mathbb{P}(|X_u^x - X_t^x| > \varepsilon/2) \\ &\leq \frac{2}{\varepsilon}|x - y| + \mathbb{P}(|Y_u^x - Y_t^x| > \varepsilon/2) + \mathbb{P}(u < \tau_x \leq t). \end{aligned}$$

The first term obviously tends to zero as $y \rightarrow x$, and the second term tends to zero as $u \rightarrow t$ since the paths of Y_u^y are almost surely continuous. The third term also tends to zero since the distribution of τ has no point masses, thus demonstrating the claim.

Now, the function u is defined by (2.3). The continuity of the boundary data given by g and ϕ , the uniform integrability provided by Proposition 3.2 and the convergence in probability of (X_u^y, τ^y) to (X_t^x, τ^x) demonstrated above and in Proposition 3.4 give the desired continuity of u . \square

Proposition 3.6. *The option price u is a classical solution of (2.4).*

Proof. Let \mathcal{R} be a strip $(x_1, x_2) \times (0, \infty)$ with $x_1 > 0$, and consider the parabolic problem

$$\begin{cases} v_t = \mathcal{L}v + \lambda\phi & (x, t) \in \mathcal{R} \\ v = u & (x, t) \in \partial_p \mathcal{R}, \end{cases} \quad (3.1)$$

where $\partial_p \mathcal{R} = \{x_1, x_2\} \times [0, \infty) \cup (x_1, x_2) \times \{0\}$ is the parabolic boundary of \mathcal{R} . Since u is continuous by Proposition 3.5, there exists a unique solution $v \in C(\overline{\mathcal{R}}) \cap C^{2,1}(\mathcal{R})$ to (3.1). Let $(x_0, t_0) \in \mathcal{R}$ and define

$$\tau_{\mathcal{R}} := \inf\{t \geq 0 : (X_t, t_0 - t) \notin \mathcal{R}\}.$$

Then the process

$$M_t := e^{-r(t \wedge \tau_{\mathcal{R}})} v(X_{t \wedge \tau_{\mathcal{R}}}, t_0 - t \wedge \tau_{\mathcal{R}}) 1_{\{t \wedge \tau_{\mathcal{R}} < \tau\}} + e^{-rt} \phi(t_0 - \tau) 1_{\{\tau \leq t \wedge \tau_{\mathcal{R}}\}}$$

is a martingale. By the strong Markov property, $u = v$ on \mathcal{R} . Consequently, $u \in C^{2,1}((0, \infty)^2)$ and u satisfies $u_t = \mathcal{L}u + \lambda\phi$. Consequently, u is a classical solution of (2.4). \square

Theorem 2.2 now follows from Propositions 3.1, 3.2 and 3.6.

4. Convexity and Parameter Monotonicities

In this section, we prove Theorems 2.3 and 2.4 about convexity of the option price and related monotonicity properties with respect to volatility and the default rate.

We begin with a convexity result under certain additional assumptions on the parameters. The key ingredient in the proof is a type of maximum principle argument developed for preservation of convexity, compare Janson & Tysk (2004).

Lemma 4.1. *In addition to the assumptions in Theorem 2.3, also assume that λ and α are in $C^2([0, \infty))$ and satisfy $\lambda(0) < \infty$ and $\alpha(0) > 0$. Furthermore, assume that λ is constant on $[B, \infty)$ and $\alpha = Ax^2$ on $[B, \infty)$ for some constants $A, B > 0$. Then $x \mapsto u(x, t)$ is convex for $t \geq 0$.*

Proof. We assume that g and ϕ are smooth functions, $\phi'(t) > -r\phi(t)$ and that g, g' and g'' are of at most polynomial growth. The general case follows easily by approximation since the pointwise limit of a sequence of convex functions is again convex. Under the current hypotheses, the equation holds up to the boundary $x = 0$. Using $u(0, t) = \phi(t)$ we find that

$$\alpha(0)u_{xx}(0, t) = \phi'(t) + r\phi(t) > 0. \tag{4.1}$$

Consequently, u is strictly convex in a neighborhood of the boundary $x = 0$. Moreover, u_{xx} is of at most polynomial growth in the spatial variable. In fact, this can be shown using explicit representation formulas for geometric Brownian motion involving the initial condition g , the continuous lateral boundary condition $u(B, t)$ and a nonhomogeneous term ϕ . Let $T > 0, C > 0$ and $n \geq 0$ be constants so that

$$|u_{xx}| \leq C(1 + x^n) \tag{4.2}$$

for $(x, t) \in [0, \infty) \times [0, T]$. Define a convex function $h(x, t) := e^{Mt}(x^2 + x^{n+3})$ for some constant M chosen large enough so that

$$h_{txx} > (\mathcal{L}h)_{xx}$$

at all points $(x, t) \in (0, \infty) \times [0, T]$. In fact, the existence of such a constant is guaranteed by the assumptions that λ is constant and $\alpha = Ax^2$ for large x . Now consider the function $u^\epsilon := u + \epsilon h$ for fixed $\epsilon > 0$. Note that by (4.1) and (4.2) there exists some constant $\delta > 0$ such that u^ϵ is strictly convex on $[0, \delta] \times [0, T]$ and on $(\delta^{-1}, \infty) \times [0, T]$. Assume (to reach a contradiction) that

$$t_0 := \inf\{t \in [0, T] : u_{xx}^\epsilon(x, t) < 0 \text{ for some } x \in (0, \infty)\} \tag{4.3}$$

is finite. By compactness, the infimum is attained for some point $(x_0, t_0) \in [\delta, \delta^{-1}] \times (0, T]$. At this point, $u_{xx}^\epsilon = 0$. Moreover, $x \mapsto u_{xx}^\epsilon(x, t_0)$ has a minimum point at $x = x_0$, so $u_{xxx}^\epsilon = 0$ and $u_{xxx}^\epsilon \geq 0$. Similarly, $u_{txx}^\epsilon \leq 0$ at (x_0, t_0) . Consequently,

$$\begin{aligned} 0 &\geq u_{txx}^\epsilon = (\mathcal{L}u + \lambda\phi)_{xx} + \epsilon h_{txx} > (\mathcal{L}u^\epsilon + \lambda\phi)_{xx} \\ &\geq \lambda_{xx}(xu_x^\epsilon - u^\epsilon + \phi) \geq 0, \end{aligned}$$

where the last inequality follows since λ is convex and since $x \mapsto u^\epsilon(x, t_0) - \phi(t_0)$ is convex and is zero for $x = 0$. This is a contradiction, which shows that u^ϵ is convex. Since ϵ is arbitrary, u is also convex in x for $t \in [0, T]$. Since T is arbitrary, this finishes the proof. \square

We next demonstrate that preservation of convexity implies that the option price is increasing in the default rate and in the volatility.

Proposition 4.2. *Assume that g , ϕ , α_i and λ_i , $i = 1, 2$ satisfy the assumptions of Hypothesis 2.1 and that $\alpha_1(x) \leq \alpha_2(x)$ and $\lambda_1(x) \leq \lambda_2(x)$ for all $x \in [0, \infty)$. Also assume that g is convex and that either (α_1, λ_1) or (α_2, λ_2) is convexity preserving. Then the corresponding option prices u_1 and u_2 satisfy $u_1 \leq u_2$.*

Proof. Let $n \geq 1$ be such that

$$\max\{|u_1(x, t)|, |u_2(x, t)|\} \leq C(1 + x^n) \quad (4.4)$$

for all $(x, t) \in (0, \infty) \times [0, T]$. Denote by \mathcal{L}_i the operator corresponding to α_i and λ_i , $i = 1, 2$, and choose M large enough so that the function $h(x, t) := e^{Mt}(1 + x^{n+1})$ satisfies

$$h_t > \mathcal{L}_2 h \quad (4.5)$$

for all $x > 0$ and $t \in [0, T]$. To see that such M can be found, observe that

$$\begin{aligned} e^{-Mt}(h_t - \mathcal{L}_2 h) &= M(1 + x^{n+1}) + (r + \lambda_2)(1 - nx^{n+1}) + q(n+1)x^{n+1} \\ &\quad - \alpha_2 n(n+1)x^{n-1} \\ &\geq M(1 + x^{n+1}) + (r + \lambda_2)(1 - nx^{n+1}) - C(x^{n-1} + x^{n+1}), \end{aligned}$$

for some $C > 0$. For all small enough x the coefficient for $r + \lambda_2$ is positive and we can clearly find M so that $M(1 + x^{n+1}) - C(x^{n-1} + x^{n+1}) \geq 1$. On the other hand, for large x we have that λ_2 is bounded and thus the negative terms can be cancelled out by letting M be large enough.

Now put $u^\varepsilon := u_2 + \varepsilon h$, and suppose that the set

$$E := \{(x, t) \in (0, \infty) \times [0, T] : u^\varepsilon < u_1\}$$

is nonempty. By (4.4), E is contained in $(\delta, \delta^{-1}) \times [0, T]$ for some $\delta > 0$, so \bar{E} is compact. Thus there exists a point $(x_0, t_0) \in \bar{E}$ such that

$$t_0 = \inf\{t \in [0, T] : (x, t) \in \bar{E} \text{ for some } x \in (0, \infty)\}.$$

By continuity, $u^\varepsilon(x_0, t_0) = u_1(x_0, t_0)$, so $u^\varepsilon(x, 0) - u_1(x, 0) \geq \varepsilon h(x, 0) > 0$ gives $t_0 > 0$. From this, it is clear that

$$\partial_t(u^\varepsilon - u_1) \leq 0 \text{ at } (x_0, t_0). \quad (4.6)$$

At (x_0, t_0) we also have $u_x^\varepsilon = (u_1)_x$ and $u_{xx}^\varepsilon \geq (u_1)_{xx}$. If (α_1, λ_1) is convexity preserving, then by (4.5) we have

$$\begin{aligned} \partial_t(u^\varepsilon - u_1) &= \mathcal{L}_2 u^\varepsilon - \mathcal{L}_1 u_1 + \lambda_2 \phi - \lambda_1 \phi + \varepsilon(h_t - \mathcal{L}_2 h) \\ &> \alpha_2(u_{xx}^\varepsilon - (u_1)_{xx}) + (\alpha_2 - \alpha_1)(u_1)_{xx} \\ &\quad + (\lambda_2 - \lambda_1)(x_0(u_1)_x - (u_1 - \phi)) \\ &\geq 0 \end{aligned} \quad (4.7)$$

since u_1 is convex with $u_1(0, t_0) - \phi(t_0) = 0$. Similarly, if (α_2, λ_2) is convexity preserving, then

$$\begin{aligned} \partial_t(u^\varepsilon - u_1) &> \alpha_1(u_{xx}^\varepsilon - (u_1)_{xx}) + (\alpha_2 - \alpha_1)u_{xx}^\varepsilon \\ &\quad + (\lambda_2 - \lambda_1)(x_0 u_x^\varepsilon - (u^\varepsilon - \phi)) \\ &\geq 0 \end{aligned} \tag{4.8}$$

since u^ε is convex.

But (4.7) and (4.8) contradict (4.6) so we conclude that E is empty. Now the result follows from letting $\varepsilon \rightarrow 0$. \square

We now remove the additional assumptions in Lemma 4.1, thus completing the proof of Theorem 2.3.

Proof of Theorem 2.3. First assume, in addition to the assumptions in Theorem 2.3, that α satisfies the conditions specified in Lemma 4.1. For a given default rate $\lambda(x)$, approximate λ with smooth and decreasing $\lambda_n \leq \lambda$ such that $\lambda_n(0) < \infty$, λ_n is constant for large x , $\lambda_n \uparrow \lambda$ pointwise, and with locally bounded Hölder(1/2)-norms uniformly in n . Let u and u_n be the corresponding option prices. By Lemma 4.1 and Proposition 4.2, each u_n is convex in the spatial variable, and $u_n \leq u_{n+1} \leq u$. Define

$$\hat{u}(x, t) := \lim_{n \rightarrow \infty} u_n(x, t) \leq u(x, t).$$

By interior Schauder estimates, \hat{u} solves $\hat{u}_t = \mathcal{L}\hat{u}$ in the interior. Since $u_n \leq \hat{u} \leq u$, the function \hat{u} also satisfies the appropriate boundary conditions, so \hat{u} is a classical solution of (2.4). By uniqueness, $\hat{u} = u$. Since \hat{u} is the pointwise limit of a sequence of spatially convex functions, \hat{u} is also spatially convex.

To treat the general case of Theorem 2.3, approximate α with smooth $\alpha_k \geq \alpha$ such that $\alpha_k(0) > 0$, $\alpha_k = A_k x^2$ for $x \geq B_k$, $\alpha_k \downarrow \alpha$ pointwise as $k \rightarrow \infty$, and with locally bounded Hölder(1/2)-norms uniformly in k . Let u and u_k be the corresponding option prices. As shown above, u_k is spatially convex and decreasing in k . Define

$$\tilde{u}(x, t) := \lim_{k \rightarrow \infty} u_k(x, t) \geq u(x, t).$$

As above, \tilde{u} is a classical solution of the pricing equation, so $\tilde{u} = u$ by uniqueness. Consequently, u is spatially convex, which finishes the proof. \square

Proof of Theorem 2.4. Theorem 2.4 is an immediate consequence of Theorem 2.3 and Proposition 4.2. \square

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