Proceedings of the 45th IEEE Conference on Decision & Control
Manchester Grand Hyatt Hotel
San Diego, CA, USA, December 13-15, 2006

Design of observers for systems with rational output function

Domenico Di Martino   Alfredo Germani   Costanzo Manes   Pasquale Palumbo

Abstract— **This note presents an approach for the design of asymptotic state observers for systems characterized by output functions that are ratios of polynomials in the state. The case of linear and bilinear input-state dynamics is considered, and conditions for exponential error decay are provided. The first step towards the construction of the observer is to show that the dynamics of a system in the considered class can be embedded into the dynamics of a system of higher dimension, with time-varying linear state dynamics and linear output map. The construction of the observer here proposed exploits the structure of the extended system. The solution of a Riccati differential equation provides the observer gain.**

Index Terms— **State observers, Riccati equation, Kronecker algebra, bilinear systems.**

I. INTRODUCTION

In a previous paper [15] the authors considered the problem of asymptotic state observation for systems characterized by polynomial output transformation and linear input-state dynamics. The problem was solved through the embedding of the original system in a system of a larger dimension described by linear input-state and state-output equations. This paper demonstrate how a somewhat similar approach can be used for the design of observers for systems with rational output transformation. A second extension with respect to the paper [15] consists in considering systems described by a bilinear input-state dynamics.

Bilinear systems have been intensively studied starting from the '70's (see e.g. [1], [2], [3]). Most papers in the literature deal with stationary systems characterized by bilinear input-state dynamics and linear output functions. Many authors considered the state observation problem for this class of systems, pointing out that, differently form what happens to linear systems, the input function plays a role in the study of observability: the distinguishability of two states depends on the input applied. This property has led some authors to define the observability with respect to *classes* of input functions. In [4] conditions are given for observability for constant inputs, and a state observer is presented that requires the knowledge of some of the input derivatives. In [5] a sufficient condition for observability with respect to a suitable subset of the class of bounded piecewise continuous inputs is given. Also in [6] conditions are studied for the construction of observers with uniform properties with respect to a class of inputs. Conditions for the existence of bilinear observers for bilinear systems for any input are studied in [7]. The problem of finding input functions that ensure state observability is investigated in [8].

In the work [9] the problem of observer synthesis is studied using a differential-algebraic approach, and input derivatives are needed in the observer equation. Also in this case *good inputs* are required for the observability and for the observer construction.

Most papers in the literature deal with the case of linear output functions. Bilinear output functions are considered only in bilinear differential stochastic systems (see e.g. [10], [11]).

This paper deals with the state observation problem for systems with the state dynamics described by bilinear differential equations, and with output functions that are ratios of polynomials of the state. Such kind of output functions can be used to model the presence of saturations in the output process. It is shown that the dynamics of systems in the considered class can be embedded into the dynamics of suitably defined extended systems, with a larger state dimension. Such systems are constructed by considering an extended state made of the Kronecker powers of the state up to the maximal order appearing in the output ratios of polynomials. Moreover, a suitable output transformation is defined in order to achieve an extended systems with linear time-varying state dynamics and output map. The construction of the observer presented in this paper exploits the structure of the extended system. This observer can be constructed in all cases in which the system, together with the applied input, satisfies an integral condition based on the observability Gramian of the extended system.

The paper is organized as follows: section II introduces the class of systems considered and some representations for them. Also the formalism of Kronecker products and powers is introduced for the system description. In section III an extended state space and an extended system are defined. In section IV the observation algorithm is presented, whose construction is based on the extended system. Simulations results are reported in section V. Conclusions follow.

This work is supported by *MIUR* (Italian Ministry for Education and Research) and by *CNR* (National Research Council of Italy) and by *LNGS-INFN* (Italian National Institute of Nuclear Physics).

D. Di Martino is with Istituto Nazionale di Fisica Nucleare (LNGS-INFN), 67010 Assergi (AQ), Italy. E-mail: domdimar@lngs.infn.it

A. Germani and C. Manes are with the Department of Electrical and Information Engineering, University of L'Aquila, Poggio di Roio, 67040 L'Aquila, Italy. E-mail: germani@ing.univaq.it, manes@ing.univaq.it

P. Palumbo is with Istituto di Analisi dei Sistemi ed Informatica del CNR "A. Ruberti", Viale Manzoni 30, 00185 Roma, Italy. E-mail: palumbo@iasi.rm.cnr.it

1-4244-0171-2/06/$20.00 ©2006 IEEE.

II. SYSTEMS WITH BILINEAR DRIFT AND RATIONAL OUTPUT

This paper considers the state observation problem for systems described by bilinear input-state dynamics and by output functions that are ratios of polynomials (BDRO Systems: Bilinear Drift-Rational Output). Such systems are described by equations of the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B_0(t)u(t) + \sum_{i=1}^{p} u_i(t)B_i(t)x(t),\\ y_k(t) &= \frac{n_k\big(t,x(t)\big)}{d_k\big(t,x(t)\big)}, \quad k=1,\dots q\end{aligned} \tag{1}$$

where $x(t) \in I\!R^n$ is the system state, $y(t) \in I\!R^q$ is the measured output, $u(t) \in I\!R^p$ is a known input, and $n_k(t,x)$ and $d_k(t,x)$ are polynomials of x, with possibly time-varying coefficients.

Polynomials of vector variables are conveniently represented by means of linear combinations of Kronecker powers. Recall that the Kronecker product of two matrices M and N of dimensions $r \times s$ and $p \times q$ respectively, is the $(r \cdot p) \times (s \cdot q)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1s}N \\ \vdots & \ddots & \vdots \\ m_{r1}N & \dots & m_{rs}N \end{bmatrix}, \tag{2}$$

where the m_{ij} are the entries of M. The Kronecker power of a matrix M is recursively defined as

$$M^{[0]} = 1, \quad M^{[i]} = M \otimes M^{[i-1]}, \quad i \geq 1. \tag{3}$$

Note that if $M \in I\!R^{a \times b}$, then $M^{[i]} \in I\!R^{a^i \times b^i}$. See the Appendix in [12] for a quick survey on the Kronecker algebra. Some properties of the Kronecker product used throughout the paper are the following:

$$\begin{aligned}(A+B)\otimes(C+D) =& A\otimes C + A\otimes D +\\ & B\otimes C + B\otimes D \end{aligned}\tag{4}$$

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C \tag{5}$$

$$(A \cdot C) \otimes (B \cdot D) = (A \otimes B) \cdot (C \otimes D) \tag{6}$$

In particular, repeated application of properties (5) and (II) provides the identity

$$(Ax)^{[i]} = A^{[i]}x^{[i]}, \tag{7}$$

intensively used throughout the paper. See [13] for more properties.

A q components polynomial function of degree m of a vector $x \in I\!R^n$ can be written as

$$\sum_{i=0}^{m} M_i x^{[i]}, \tag{8}$$

where $M_i \in I\!R^{q \times n^i}$ are the coefficients of the polynomial. Using the Kronecker powers of the state, the output of the system (1) can be written as

$$y_k(t) = \frac{n_{k,0}(t) + \sum_{j=1}^{r_k} n_{k,j}^T(t)x^{[j]}(t)}{d_{k,0}(t) + \sum_{j=1}^{s_k} d_{k,j}^T(t)x^{[j]}(t)} \quad k=1,\dots q, \tag{9}$$

where r_k and s_k are the degrees of the numerator and denominator polynomials of each scalar output, and $n_{k,j}(t) \in I\!R^{n^j}$ and $d_{k,j}(t) \in I\!R^{n^j}$ are the vector coefficients of the powers $x^{[j]}(t) \in I\!R^{n^j}$. Denoting with m the maximal degree of the polynomials, i.e. $m = \max_{k=1,\dots,q}(r_k, s_k)$, and defining a *polynomial extended* state vector $X_m(t)$ as follows

$$X_m(t) = \begin{pmatrix} x(t) \\ x^{[2]}(t) \\ \vdots \\ x^{[m]}(t) \end{pmatrix} \in I\!R^{b(n,m)}, \tag{10}$$

where $b(n,m) = \sum_{i=1}^{m} n^i$, the components of the system output can be written in the following compact form

$$y_k(t) = \frac{n_{k,0}(t) + N_k^T(t)X_m(t)}{d_{k,0}(t) + D_k^T(t)X_m(t)} \tag{11}$$

where $N_k(t) \in I\!R^{b(n,m)}$ and $D_k(t) \in I\!R^{b(n,m)}$ are defined as follows

$$\begin{aligned} N_k^T(t) &= [n_{k,1}^T(t) \ \cdots \ n_{k,m}^T(t)], \\ D_k^T(t) &= [d_{1,m}^T(t) \ \cdots \ d_{k,m}^T(t)]. \end{aligned} \tag{12}$$

Moreover, defining the matrix function

$$B\big(t,u\big) = \sum_{i=1}^{p} B_i(t)u_i, \tag{13}$$

the BDRO system (1) can be written in the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B_0(t)u(t) + B\big(t,u(t)\big)x(t),\\ y_k(t) &= \frac{n_{k,0}(t) + N_k^T(t)X_m(t)}{d_{k,0}(t) + D_k^T(t)X_m(t),} \quad k=1,\dots q,\end{aligned} \tag{14}$$

Each scalar output in system (14) can put in the form

$$y_k(t)\Big(d_{k,0}(t) + D_k^T(t)X_m(t)\Big) = n_{k,0}(t) + N_k^T(t)X_m(t), \tag{15}$$

and therefore

$$y_k(t)d_{k,0}(t) - n_{k,0}(t) = N_k^T(t)X_m(t) - y_k(t)D_k^T(t)X_m(t). \tag{16}$$

Defining the transformed outputs $\tilde{y}_k(t)$

$$\tilde{y}_k(t) = y_k(t)d_{k,0}(t) - n_{k,0}(t), \quad k=1,\dots,q, \tag{17}$$

the following linear output functions can be written

$$\tilde{y}_k(t) = \Big(N_k^T(t) - y_k(t)D_k^T(t)\Big)X_m(t), \quad k = 1 \ \cdots, q. \tag{18}$$

These can be put in matrix form as follows

$$\tilde{y}(t) = \mathcal{C}_y(t)X_m(t) \tag{19}$$

where

$$\tilde{y}(t) = \begin{pmatrix} y_1(t)d_{1,0}(t) - n_{1,0}(t) \\ \vdots \\ y_q(t)d_{q,0}(t) - n_{q,0}(t) \end{pmatrix}, \tag{20}$$

$$\mathcal{C}_y(t) = \begin{pmatrix} N_1^T(t) - y_1(t)D_1^T(t) \\ \vdots \\ N_q^T(t) - y_q(t)D_q^T(t). \end{pmatrix} \tag{21}$$

Defining the matrices

$$A_u(t) = A(t) + B\big(t, u(t)\big), \quad B_u(t) = B_0(t)u(t). \tag{22}$$

The BDRO system (14) with the transformed output (20) can written in the following compact form

$$\begin{aligned} \dot{x}(t) &= A_u(t)x(t) + B_u(t) \\ \tilde{y}(t) &= \mathcal{C}_y(t)X_m(t) \end{aligned} \tag{23}$$

III. THE EXTENDED SYSTEM

The previous section has shown that the BDRO system (1) admits the representations (14) and (23), this last after the definition of a transformed output $\tilde{y}(t)$, given by (20). This section shows that a BDRO system can be *embedded* into a system of larger dimension characterized by a linear time-varying dynamics. The main result consists in showing that the dynamics of the polynomial extended state $X_m(t)$ defined in (10) obeys the time varying-linear equation presented by the following lemma:

Lemma III.1. *Consider the bilinear input-state equation of system* (1) *and its representation given by the first of* (23)*, with $A_u(t)$ and $B_u(t)$ defined in* (22)*. The dynamics of the polynomial extended state $X_m(t)$ defined in* (10) *is governed by the differential equation:*

$$\dot{X}_m(t) = \mathcal{A}_u(t)X_m(t) + \mathcal{B}_u(t), \tag{24}$$

where matrix $\mathcal{A}_u$ has the following structure

$$\begin{bmatrix} \mathcal{A}_{1,1} & O & \cdots & 0 & 0 \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \cdots & 0 & 0 \\ O & \mathcal{A}_{3,2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{m,m-1} & \mathcal{A}_{m,m} \end{bmatrix}, \tag{25}$$

where matrices $\mathcal{A}_{i,i},\ i = 1, \ldots, m$ and $\mathcal{A}_{i,i-1},\ 2 \le i \le m$ are recursively defined by the equations

$$\begin{aligned} \mathcal{A}_{1,1} &= A_u(t), \\ \mathcal{A}_{i,i} &= \mathcal{A}_{1,1} \otimes I_{n^{i-1}} + I_n \otimes \mathcal{A}_{i-1,i-1}, \quad i > 1, \\ \mathcal{A}_{2,1} &= B_u(t), \\ \mathcal{A}_{i,i-1} &= \mathcal{A}_{2,1} \otimes I_{n^{i-1}} + I_n \otimes \mathcal{A}_{i-1,i-2}, \quad i > 2. \end{aligned} \tag{26}$$

(I_{n^i} denotes the identity matrix of dimension n^i) and

$$\mathcal{B}_u(t) = \begin{bmatrix} B_u(t) \\ O \\ \vdots \\ O \end{bmatrix}. \tag{27}$$

(Although all matrices $\mathcal{A}_{i,i}$ and $\mathcal{A}_{i,i-1}$ in (25) *and* (26) *are functions of t and $u(t)$, such dependence is omitted for brevity.)*

Proof: Taking into account the definitions (25) and (27), the state dynamics (24) is equivalent to the following equations

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}_{1,1}x(t) + B_u(t), \\ \frac{d}{dt}x^{[i]}(t) &= \mathcal{A}_{i,i}x^{[i]}(t) + \mathcal{A}_{i,i-1}x^{[i-1]}(t), \\ & i = 2, \ldots, m. \end{aligned} \tag{28}$$

The first of (28) is readily proved by observing that, by definition, $\mathcal{A}_{1,1} = A_u(t)$. The second equation, for $i = 2$, is proved with the following passages, by using the properties of the Kronecker product listed in (4)–(II):

$$\begin{aligned} \frac{d}{dt}x^{[2]}(t) &= x \otimes \dot{x} + \dot{x} \otimes x \\ &= x \otimes \big(A_u x + B_u\big) + \big(A_u x + B_u\big) \otimes x \\ &= \big(A_u \otimes I_n + I_n \otimes A_u\big)\, x^{[2]} \\ &\quad + \big(B_u \otimes I_n + I_n \otimes B_u\big)\, x \end{aligned} \tag{29}$$

By definitions (26)

$$\begin{aligned} \mathcal{A}_{2,2} &= \mathcal{A}_{1,1} \otimes I_n + I_n \otimes \mathcal{A}_{1,1}, \\ \mathcal{A}_{2,1} &= \mathcal{A}_{2,1} \otimes I_n + I_n \otimes \mathcal{A}_{2,1}, \end{aligned} \tag{30}$$

where $\mathcal{A}_{1,1} = A_u$ and $\mathcal{A}_{2,1} = B_u$, so that the second of (28) is proved for $i = 2$.

Now, proceed by induction: assume that (28) is true for a given $i \ge 2$ and prove that it is also true for $i + 1$. Indeed

$$\begin{aligned} \frac{d}{dt}x^{[i+1]}(t) &= x \otimes \frac{d}{dt}x^{[i]} + \dot{x} \otimes x^{[i]} \\ &= x \otimes \big(\mathcal{A}_{i,i}x^{[i]} + \mathcal{A}_{i,i-1}x^{[i-1]}\big) + \big(A_u x + B_u\big) \otimes x^{[i]} \\ &= \big(A_u \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i}\big)\, x^{[i+1]} + \\ &\quad \big(B_u \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i-1}\big)\big)x^{[i]}. \end{aligned} \tag{31}$$

From definitions (26) it follows

$$\begin{aligned} \mathcal{A}_{i+1,i+1} &= A_{1,1} \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i} \\ \mathcal{A}_{i+1,i}(u) &= A_{2,1} \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i-1}(u) \end{aligned} \tag{32}$$

so that

$$\frac{d}{dt}x^{[i+1]}(t) = \mathcal{A}_{i+1,i+1}x^{[i+1]}(t) + \mathcal{A}_{i+1,i}\big(u(t)\big)x^{[i]}(t), \tag{33}$$

and the induction is proved.

Remark III.2. The state dynamics of system (1) is said to be *embedded* into the dynamics

$$\dot{X}(t) = \mathcal{A}_u(t)X(t) + \mathcal{B}_u(t), \tag{34}$$

whose state $X(t) \in I\!R^{b(m,n)}$ is of larger dimension than $x(t)$. Such an embedding is to be intended as follows: if $X(t_0) = X_m(t_0)$, i.e. if

$$X(t_0) = \begin{pmatrix} x(t_0) \\ x^{[2]}(t_0) \\ \vdots \\ x^{[m]}(t_0) \end{pmatrix} \in I\!R^{b(n,m)}, \tag{35}$$

then the following identity relates the state $x(t)$ of (1) and the state $X(t)$ of (34) for $t \geq t_0$

$$x(t) = S_m X(t), \tag{36}$$

with

$$S_m = \begin{bmatrix} I_n & O_{n\times(n^2+\cdots+n^m)} \end{bmatrix}. \tag{37}$$

This happens because, thanks to Lemma III.1, with the initialization (35), it follows that $X(t) = X_m(t)$ for all $t \geq t_0$, so that the product $S_m X(t)$ in (36) simply selects the first n components of $X_m(t)$, that is the state $x(t)$ of system (1).

From what discussed, the following equations can be used to represent system (23) for $t \geq t_0$

$$\dot{X}(t) = \mathcal{A}_u(t)X(t) + \mathcal{B}_u(t), \quad X(t_0) = X_m(t_0) \tag{38}$$

$$\tilde{y}(t) = \mathcal{C}_y(t)X(t) \tag{39}$$

It must be stressed that the Kronecker powers of vectors contain redundant terms. It follows that redundant components are present in the extended state vector X_m, so that the extended state space results to be output-indistinguishable. Such redundancy can be eliminated by considering suitably defined reduction matrices. First of all note that $x^{[i]}$, the i-th Kronecker power of $x \in I\!R^n$, has n^i components, but only $\binom{n+i-1}{i}$ are distinct terms (the number of ways to choose i elements from a set of n, with repetitions allowed). Defining the following functions of pairs of integers

$$b(n,m) = n\frac{1-n^m}{1-n} = \sum_{i=1}^{m} n^i, \tag{40}$$

$$c(n,m) = \binom{n+m}{m} - 1 = \sum_{i=1}^{m}\binom{n+i-1}{i}, \tag{41}$$

it is easy to see that the vector X_m has $b(n,m)$ components, but only $c(n,m)$ are distinct (obviously $c(n,m) < b(n,m)$).

A block-diagonal reduction matrix $\overline{T}_{n,m} \in I\!R^{c(n,m)\times b(n,m)}$ can be suitably defined, as described in detail in [14], for the selection of a nonredundant subvector $\overline{X}_m \in I\!R^{c(n,m)}$ from $X_m \in I\!R^{b(n,m)}$. A block-diagonal matrix $T_{n,m} \in I\!R^{b(n,m)\times c(n,m)}$ allows to reconstruct the redundant vector X_m from $\overline{X}_m$. In formulas

$$\overline{X}_m(k) = \overline{T}_{n,m}X_m, \quad X_m = T_{n,m}\overline{X}_m(k). \tag{42}$$

Using Lemma III.1 and the reduction matrices (42), system can be embedded in the following system

$$\begin{aligned} \dot{\mathcal{X}}(t) &= \bar{\mathcal{A}}_u(t)\mathcal{X}(t) + \overline{\mathcal{B}}_u(t) \\ \tilde{y}(t) &= \bar{\mathcal{C}}_y\mathcal{X}(t), \end{aligned} \tag{43}$$

where $\mathcal{X}(t) \in I\!R^{c(n,m)}$ and

$$\begin{aligned} &\bar{\mathcal{A}}_u(t) = T_{n,m}\mathcal{A}_u(t)\overline{T}_{n,m}, \qquad \overline{\mathcal{B}}_u(t) = T_{n,m}\mathcal{B}_u(t), \\ &\bar{\mathcal{C}}_y(t) = \mathcal{C}_y(t)\overline{T}_{n,m}. \end{aligned} \tag{44}$$

Now the embedding of the system (23) into the extended system (43) should be intended as follows: if the initial value of the extended state of (43) is set to

$$\mathcal{X}(t_0) = T_{n,m}\begin{pmatrix} x(t_0) \\ x^{[2]}(t_0) \\ \vdots \\ x^{[m]}(t_0) \end{pmatrix} = T_{n,m}X_m(t_0), \tag{45}$$

then the outputs of the two systems is the same for any input, and the state $x(t)$ of system (23) is recovered by selecting the first n components of the extended state $\mathcal{X}(t)$:

$$\begin{aligned} x(t) &= \Sigma_m\mathcal{X}(t), \\ \text{where } \Sigma_m &= \begin{bmatrix} I_n & 0_{n\times(c(n,m)-n)} \end{bmatrix}. \end{aligned} \tag{46}$$

IV. AN ASYMPTOTIC STATE OBSERVER FOR *BDRO* SYSTEMS

In this section, following the same approach of [15], an asymptotic observer for nonlinear systems of the type (1) is presented and a sufficient condition for the asymptotic convergence to zero of the state observation error is given. The observer is constructed on the basis of the extended system (43), that has the simpler structure of a time-varying linear system.

Theorem IV.1. *Consider the BDRO system* (1) *and the construction that has led to the extended system* (43). *Assume that for a given pair* $(x(t_0), u)$ *the pair* $(\bar{\mathcal{A}}_u(t), \bar{\mathcal{C}}_y(t))$ *defined in* (44) *is such that there exist positive scalars* α, β, δ, *with* $\alpha < \beta$, *such that for all* $t \geq t_0$

$$\alpha I_{c(n,m)} \leq \int_t^{t+\delta} e^{\bar{\mathcal{A}}_u^T(\tau)}\bar{\mathcal{C}}_y^T(t)\bar{\mathcal{C}}_y(t)e^{\bar{\mathcal{A}}_u(\tau)}d\tau \leq \beta I_{c(n,m)}. \tag{47}$$

Then, for any $\widehat{\mathcal{X}}(t_0) \in I\!R^{c(n,m)}$, *the system*

$$\begin{aligned} \dot{\widehat{\mathcal{X}}}(t) &= \bar{\mathcal{A}}_u(t)\widehat{\mathcal{X}}(t) + \overline{\mathcal{B}}_u(t) \\ &\quad + P(t)\bar{\mathcal{C}}_y^T\left(\tilde{y}(t) - \bar{\mathcal{C}}_y(t)\widehat{\mathcal{X}}(t)\right), \end{aligned} \tag{48}$$

$$\begin{aligned} \dot{P}(t) &= \left(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}_y^T\bar{\mathcal{C}}_y\right)P(t) \\ &\quad + P(t)\left(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}_y^T\bar{\mathcal{C}}_y\right)^T + Q(t), \end{aligned} \tag{49}$$

$$\hat{x}(t) = \Sigma_m\widehat{\mathcal{X}}(t), \tag{50}$$

defined for $t \geq t_0$, *with* $Q(t)$ *and* $P(t_0)$ *symmetric positive definite,* $Q(t) \geq q_m I_{c(n,m)}$ *for some positive* q_m, *is an asymptotic observer for the system* (1), *i.e.*

$$\lim_{t\to\infty}\|x(t) - \hat{x}(t)\| = 0. \tag{51}$$

Proof: The proof follows the same passages of Theorem IV.1 of [15]. It is sufficient to show that equations (48)–(50) are an asymptotic observer for the extended system (43), i.e.

$$\lim_{t\to\infty}\|\mathcal{X}(t) - \widehat{\mathcal{X}}(t)\| = 0. \tag{52}$$

First, note that the assumption (47) coincides with the *uniform observability* of a linear time-varying system, and implies that $P(t)$ admits upper and lower bounds (see [16], [17]), i.e. there exist positive scalars p_m and p_M such that

$$p_m I_{c(n,m)} \le P(t) \le p_M I_{c(n,m)}. \tag{53}$$

Let $\varepsilon(t) = \mathcal{X}(t) - \widehat{\mathcal{X}}(t)$ be the observation error of the extended system. Subtracting equation (43) from (48), the error dynamics is obtained

$$\dot{\varepsilon}(t) = \big(\bar{\mathcal{A}}_u(t) - K(t)\bar{\mathcal{C}}_y\big)\,\varepsilon(t), \quad t \ge t_0. \tag{54}$$

Consider the following function of the observation error

$$V(\varepsilon, t) = \varepsilon^T P^{-1}(t)\varepsilon, \tag{55}$$

positive definite for all $t \ge t_0$ because (53) implies

$$\frac{1}{p_M} I_{c(n,m)} \le P^{-1}(t) \le \frac{1}{p_m} I_{c(n,m)}. \tag{56}$$

Let $v(t) = V(\varepsilon(t), t)$. The time derivative of $v(t)$ along the error trajectories is

$$\begin{aligned} \dot{v}(t) &= \dot{\varepsilon}^T(t)P^{-1}(t)\varepsilon(t) + \varepsilon^T(t)\dot{P}^{-1}(t)\varepsilon(t) \\ &\quad + \varepsilon^T(t)P^{-1}(t)\dot{\varepsilon}(t) \\ &= \varepsilon^T(t)\Big[\big(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}_y^T\bar{\mathcal{C}}_y\big)^T P^{-1}(t) + \dot{P}^{-1}(t) \\ &\quad + P^{-1}(t)\left(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}_y^T\bar{\mathcal{C}}_y\right)\Big]\varepsilon(t). \end{aligned} \tag{57}$$

Recalling that $\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$, it follows

$$\begin{aligned} \dot{v}(t) &= \varepsilon^T(t)P^{-1}(t)\Big[P(t)\big(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}_y^T\bar{\mathcal{C}}_y\big)^T - \dot{P}(t) \\ &\quad + \left(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}_y^T\bar{\mathcal{C}}_y\right)P(t)\Big]P^{-1}(t)\varepsilon(t). \end{aligned} \tag{58}$$

From this, recalling (49), for $t \ge t_0$

$$\dot{v}(t) = -\varepsilon^T(t)P^{-1}(t)Q(t)P^{-1}(t)\varepsilon(t). \tag{59}$$

Since, by (56), it is

$$\dot{v}(t) \le -\frac{1}{p_M^2} q_m \|\varepsilon(t)\|^2, \tag{60}$$

$$v(t) \ge \frac{1}{p_M}\|\varepsilon(t)\|^2. \tag{61}$$

It follows

$$\dot{v}(t) \le -\frac{q_m}{p_M} v(t), \tag{62}$$

from which

$$v(t) \le e^{-\frac{q_m}{p_M}(t-t_0)} v(t_0) = \frac{1}{p_m} e^{-\frac{q_m}{p_M}(t-t_0)}\|\varepsilon(t_0)\|^2, \tag{63}$$

and finally

$$\|\varepsilon(t)\|^2 \le \frac{p_M}{p_m} e^{-\frac{q_m}{p_M}(t-t_0)}\|\varepsilon(t_0)\|^2. \tag{64}$$

This proves the convergence (51), and therefore (52).

Remark IV.2. Note that the condition (47) can only be tested *on-line* because matrix $\bar{\mathcal{C}}_y(t)$ depends on $y(t)$, the output of the original system (1), that is not available before time t. Moreover, also the input $u(t)$ applied to the system in general is available at time t (on-line). In practice, the observer (48)–(50) should be applied without a preliminary check of the condition (47). A positive definite initial value $P(t_0)$ ensures that for t sufficiently close to t_0 the matrix $P(t)$ remains nonsingular and bounded. However, it is convenient to monitor the minimum and maximum eigenvalues of $P(t)$ during its evolution. The divergence of $\lambda_{\text{Max}}\big(P(t)\big)$ or the approach to zero of $\lambda_{\text{min}}\big(P(t)\big)$ are caused by a loss of observability of the extended system (43), possibly due to a *bad input*, in the current time interval. In these cases it may be necessary to reset $P(t)$ to some well conditioned P_0, waiting for the system to recover observability thanks to a *good input*.

V. SIMULATION RESULTS

Simulations results are here reported in order to show the effectiveness of the proposed observer. Consider the following system:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x \in I\!R^3, \tag{65}$$

$$y(t) = \frac{-1 + x_1 + 2x_2x_3 - 2x_3^2}{1 + x_2^2}, \quad y \in I\!R. \tag{66}$$

where:

$$A = \begin{bmatrix} -0.5 & 1 & 2 \\ 0 & -1.5 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}. \tag{67}$$

The output in (66) can be put in the form (9) with:

$$\begin{aligned} n_0 &= -1, \qquad n_1^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ n_2^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 \end{bmatrix}, \\ d_0 &= 1, \qquad d_1^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \\ d_2^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{68}$$

In this example $n = 3$, $m = 2$ and therefore the dimension $c(n,m)$ of the extended system is 9. The simulations reported in this section are performed with the following input

$$u(t) = 0.5\big(1 + \sin(2\pi t/T)\big), \quad \text{with } T = 4. \tag{69}$$

The matrix $Q(t)$ and the initial value for the matrix $P(t)$ in the Riccati equation (49) have been chosen of the type

$$P(t_0) = \alpha\overline{S}, \qquad Q(t) = \overline{S}, \tag{70}$$

$$\text{where} \quad \overline{S} = \begin{bmatrix} \beta I_3 & O_{3\times 6} \\ O_{6\times 3} & \beta^2 I_6 \end{bmatrix}, \quad \text{with } \beta = 20. \tag{71}$$

The graphics of this section present the simulation results for different values of the parameter α in (70). The values used are

$$\alpha_1 = 10, \quad \alpha_2 = 100, \quad \alpha_3 = 1.000. \tag{72}$$

Figures 1-3 show the true and the observed state components. It is evident that the convergence speed increases by increasing α. Figure 3 reports the output $y(t)$ and the *observed* output $\hat{y}(t)$, given by the output function in (66) computed at the observed state $\hat{x}(t)$.

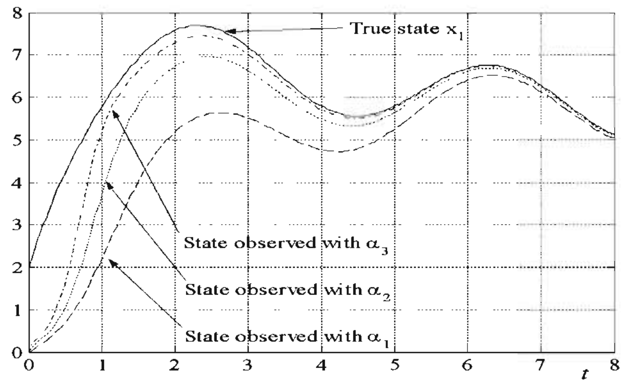

Fig. 1. **True and estimated state: the first component.**

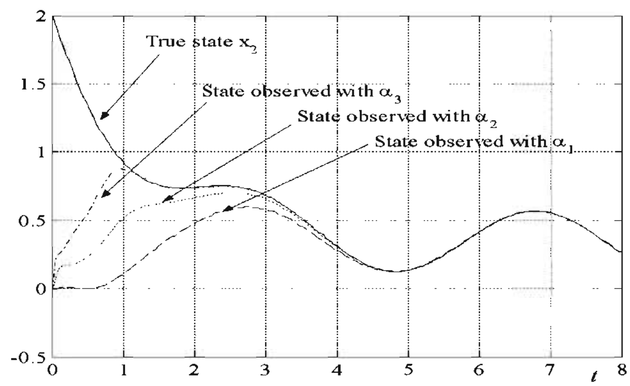

Fig. 2. **True and estimated state: the second component.**

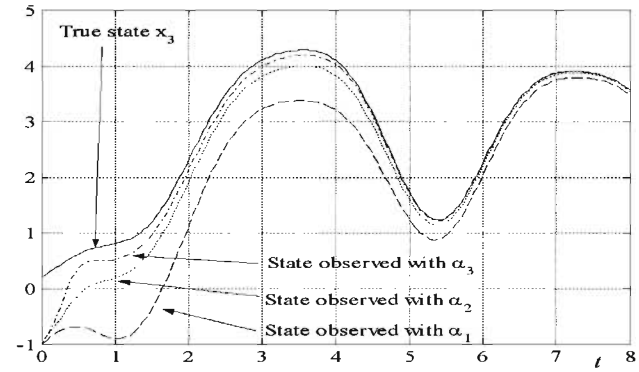

Fig. 3. **True and estimated state: the third component.**

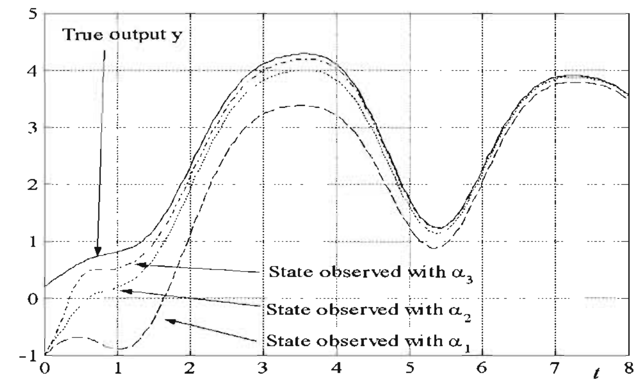

Fig. 4. **True and estimated output.**

VI. CONCLUSIONS

The problem of the state observation for the class of systems with bilinear input-state dynamics and state-output functions that are ratios of polynomial is investigated in this paper, and an asymptotic observer is presented. The observer gain is time varying and is obtained as the solution of a differential Riccati equation. The observer behavior has been numerically tested on some examples and has always given good results.

References

[1] R.R. Mohler, "Natural bilinear control processes," *IEEE Trans. Syst., Sci. & Cybern.*, SSC-6, pp. 192–197, 1970.

[2] C. Bruni, G. Di Pillo, G. Koch, "Bilinear systems: An appealing class of "nearly linear" systems in theory and applications," *IEEE Trans. A. C.*, Vol. 19, No. 4, pp. 334–348, 1974.

[3] P. D'Alessandro, A. Isidori, A. Ruberti. "Realization and structure theory of bilinear dynamical systems," SIAM J. Control, Vol. 12, No. 3, pp. 517–535, 1974.

[4] D. Williamson, "Observation of bilinear systems with application to biological control," Automatica, Vol. 13, pp. 243–25, May 1977.

[5] C. Bruni, A. Gandolfi, A. Germani, "Observability of linear-in-the-state systems: a functional approach," *IEEE Trans. A. C.*, Vol. AC–25, No. 3, pp. 566–567, June 1980.

[6] O. Grasselli, A. Isidori, "An existence theorem for observers of bilinear systems," *IEEE Trans. A. C.*, Vol. 26, No. 6, pp. 1299–1300, 1981.

[7] I.A. Derese, E.J. Noldus, "Existence of bilinear state observers for bilinear systems," *IEEE Trans. A. C.*, Vol. 26, No. 2, pp. 590–592, 1981.

[8] P. Sen, "On the choice of input for observability in bilinear systems," *IEEE Trans. A. C.*, Vol. 26, No. 2, pp. 451–454, 1981.

[9] R. Martinez-Guerra, J. De Leon-Morales, "Observer synthesis for a class of bilinear systems: a differential algebraic approach," Proc. of the 33rd IEEE Conf. on Decision and Control, CDC'94, Vol. 2, pp. 1209–1210, 14-16 Dec. 1994 .

[10] Sze-Kui Ng, "Optimal linear nonanticipative control of partially observable bilinear systems," *IEEE Trans. A. C.*, Vol. 29, No. 3, pp. 271–274, 1984.

[11] A. Germani, C. Manes, P. Palumbo, "Linear filtering for bilinear stochastic differential systems with unknown inputs," *IEEE Trans. A. C.*, Vol. 47, No. 10, pp. 1726–1730, 2002.

[12] F. Carravetta, A. Germani, M. Raimondi, "Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise," *IEEE Trans. A. C.*, Vol. 42, NO. 8, pp. 1106–1126, 1997.

[13] A. Graham, *Kronecker Products and Matrix Calculus with Applications,* Ellis Horwood ltd. Publishers, Chichester, 1981.

[14] F. Carravetta, A. Germani, M. Raimondi, "Polynomial filtering for linear discrete-time non-Gaussian systems," *SIAM J. Contr. Optim.*, Vol. 34, No. 5, pp. 1666–1690, 1996.

[15] D. Di Martino, A. Germani, C. Manes, P. Palumbo, "State observation for systems with linear dynamics and polynomial output," proc. of 43rd IEEE Conf. on Decision and Control (CDC04), Paradise Island, Bahamas, pp. 3886–3891, Dec. 2004.

[16] R.S. Bucy, "Global theory of the Riccati equation," *J. Comput. Syst. Sci.*, Vol. 1, pp. 349–361, 1967.

[17] A.H. Jazwinski, *Stochastic Processes and Filtering Theory.* New York: Academic, 1970.