Acta Mathematica Sinica, English Series
Jun., 2013, Vol. 29, No. 6, pp. 1055–1066
Published online: February 6, 2013
DOI: 10.1007/s10114-013-1443-6
Http://www.ActaMath.com

Acta Mathematica Sinica, English Series

Modular Filter Convergence Theorems for Urysohn Integral Operators and Applications

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Abstract We prove some versions of modular convergence theorems for nonlinear Urysohn-type integral operators with respect to filter convergence. We consider pointwise filter convergence of functions giving also some applications to linear and nonlinear Mellin operators. We show that our results are strict extensions of the classical ones.

Keywords Egorov filter, Lebesgue filter, filter convergence, filter exhaustiveness, filter singularity, integral operator, Mellin operator, modular convergence, modular space

MR(2010) Subject Classification 41A35, 46E30, 47G10

1 Introduction

We consider the problem of approximating a given real-valued function f, defined on a topological measure space, by means of a sequence of integral operators $(T_n f)_n$. Operators like Mellin convolution, moment and sampling operators play an important role in several branches of Mathematics, for instance in reconstruction of signals and images, in Fourier analysis and operator theory (see, e.g., [1–7] and the references contained therein). In particular, the Urysohn-type operators are generalizations of suitable convolution operators, like for example the Mellin operators.

In this paper, we deal with nonlinear Urysohn operators, generated by kernels which satisfy some singularity properties with respect to filter convergence introduced in [8], and we extend some modular convergence theorems proved in [9–11].

More precisely, we introduce the notion of exhaustiveness of a function sequence at a point with respect to a filter and we establish new filter convergence theorems under suitable conditions on the filters.

As applications, we obtain modular convergence theorems for nonlinear Mellin-type operators with respect to the statistical convergence introduced by Steinhaus (see [12]) and we consider also the moment kernels and the Mellin–Gauss–Weierstrass kernels (see [11]).

Received August 19, 2011, accepted March 9, 2012
Supported by Universities of Perugia and Athens

Finally, we give some non-trivial examples which show that our results are proper extensions of the corresponding classical ones, by constructing suitable kernels, satisfying singularity conditions with respect to the filter involved, but not the classical ones.

2 Preliminaries

We recall some properties of the filters of $\mathbb{N}$.

Definition 2.1 (a) *A nonempty family $\mathcal{F}$ of subsets of $\mathbb{N}$ is called a filter of $\mathbb{N}$ if and only if $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$, and for each $A \in \mathcal{F}$ and $B \supset A$, we get $B \in \mathcal{F}$.*

(b) *Let $\mathcal{F}$ be a filter of $\mathbb{N}$. A collection of subsets $\mathcal{H} \subset \mathcal{F}$ is called a base of $\mathcal{F}$ if and only if for every $A \in \mathcal{F}$, there is an element $B \in \mathcal{H}$ with $B \subset A$.*

(c) *A sequence $(x_n)_n$ in $\mathbb{R}$ is said to be $\mathcal{F}$-bounded if and only if there exists a positive real constant M such that the set $\{n \in \mathbb{N} : |x_n| \leq M\}$ belongs to $\mathcal{F}$.*

(d) *Let G be any nonempty set and $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of functions. We say that $(f_n)_n$ is $\mathcal{F}$-dominated if and only if there exists a non-negative function $h : G \to \mathbb{R}$ such that*

$$\{n \in \mathbb{N} : |f_n(t)| \leq h(t), \forall\, t \in G\} \in \mathcal{F}.$$

In this case, we also say that h $\mathcal{F}$-dominates $(f_n)_n$.

(e) *Let (X, d) be a metric space. A sequence $(x_n)_n$ in X is said to be $\mathcal{F}$-convergent to $x \in X$ (and we write $x = (\mathcal{F}) \lim_n x_n$) if and only if for every $\varepsilon > 0$, we get*

$$\{n \in \mathbb{N} : d(x_n, x) \leq \varepsilon\} \in \mathcal{F}.$$

We now introduce the following (see also [13]).

Definition 2.2 *Let (X, d) be a metric space and for every $x \in X$ and $\delta > 0$, set $B(x, \delta) := \{z \in X : d(z, x) < \delta\}$. A sequence of functions $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, is called $\mathcal{F}$-exhaustive at x_0 if and only if for every $\varepsilon > 0$, there exist $\delta > 0$ and $A \in \mathcal{F}$ such that $|f_n(z) - f_n(x_0)| \leq \varepsilon$, whenever $n \in A$ and $z \in B(x_0, \delta)$.*

We recall some examples of filters (see also [14]).

Example 2.3 (a) The filter $\mathcal{F}_{\text{cofin}}$ of all subsets of $\mathbb{N}$ whose complement is finite is called the *Fréchet filter*.

Note that a sequence $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, is exhaustive at $x_0 \in X$ if and only if it is $\mathcal{F}_{\text{cofin}}$-exhaustive at x_0.

(b) A filter $\mathcal{F}$ of $\mathbb{N}$ is said to be *free* if and only if it contains the Fréchet filter.

(c) We say that a free filter $\mathcal{F}$ is a *P-filter* if and only if for any sequence $(A_j)_j$ in $\mathcal{F}$, there are sets $B_j \subset \mathbb{N}$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcap_{j=1}^{\infty} B_j \in \mathcal{F}$.

An example of P-filter is the filter $\mathcal{F}_d$ associated with statistical convergence, that is the set of all subsets of $\mathbb{N}$ whose asymptotic density is 1. Here the asymptotic density of a set $A \subset \mathbb{N}$ is defined as

$$d(A) = \lim_n \frac{\text{card}(A \cap \{1, \ldots, n\})}{n}$$

(if this limit exists) and "card" denotes the cardinality of the set in brackets.

We now recall the Egorov and Lebesgue filters, that is the filters with respect to which the Egorov theorem and the Lebesgue dominated convergence theorem are always true respectively (see also [14]).

(d) A filter $\mathcal{F}$ of $\mathbb{N}$ is said to be an *Egorov filter* if and only if for every measure space $(G, \mathcal{B}, \mu)$, with μ finite and positive, for each sequence $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, pointwise $\mathcal{F}$-convergent to 0 and for every $\varepsilon > 0$, there exists a subset $B \in \mathcal{B}$ with $\mu(B) < \varepsilon$, such that

$$(\mathcal{F}) \lim_n \Big[\sup_{t \in G \setminus B} |f_n(t)| \Big] = 0.$$

(e) A filter $\mathcal{F}$ of $\mathbb{N}$ is said to be a *Lebesgue filter* if and only if for every measure space $(G, \mathcal{B}, \mu)$, with μ finite and positive, we have

$$(\mathcal{F}) \lim_n \int_G f_n d\mu = 0 \tag{2.1}$$

whenever $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence, pointwise $\mathcal{F}$-convergent to 0 and with the property that there exists a non-negative function $h \in L^1(G, \mathcal{B}, \mu)$, with $|f_n(t)| \le h(t)$ for all $t \in G$.

Note that every Egorov filter is a Lebesgue filter (see [14, Corollary 2.3]), but not vice versa: indeed, the above defined $\mathcal{F}_d$ is a Lebesgue filter, but not an Egorov filter (see [14, Proposition 3.1]).

Remark 2.4 It is easy to see that the definition of Lebesgue filter can be equivalently formulated if we require that (2.1) holds whenever $(f_n)_n$ is a sequence pointwise $\mathcal{F}$-convergent to 0 and $\mathcal{F}$-dominated by a function $h \in L^1(G, \mathcal{B}, \mu)$.

3 Main Results

We now introduce some notations and structural hypotheses, following [9–11].

Notations and Assumptions 3.1 (a) Let G be a locally compact Hausdorff topological space, $\mathcal{B}$ be the σ-algebra of all Borel subsets of G, and $\mu : \mathcal{B} \to \mathbb{R}$ be a positive σ-finite measure. We denote by $L^0(G, \mathcal{B}, \mu)$ the space of all real-valued μ-measurable functions with identification up to sets of measure μ zero, by $C(G)$ the space of all real-valued continuous bounded functions on G and by $C_c(G)$ the subspace of $C(G)$ of all continuous functions with compact support on G.

(b) Denote by $\mathcal{L}$ the set of all non-negative measurable functions $L : G \times G \to \mathbb{R}$ such that the sections $L(\cdot, t)$ and $L(s, \cdot)$ belong to $L^1(G, \mathcal{B}, \mu)$ for all t, $s \in G$ respectively.

(c) Let $\mathbb{R}_0^+$ be the set of all non-negative real numbers and Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that ψ is continuous, nondecreasing, $\psi(0) = 0$ and $\psi(u) > 0$ whenever $u > 0$, and let $\Xi = (\psi_n)_n \subset \Psi$ be a sequence of functions which are $\mathcal{F}$-exhaustive at 0 and for every $u > 0$, the sequence $(\psi_n(u))_n$ is $\mathcal{F}$-bounded. Denote by $\mathcal{K}_\Xi$ the class of all sequences of functions $K_n : G \times G \times \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, such that: $K_n(\cdot, \cdot, u)$ is measurable on $G \times G$ for all $u \in \mathbb{R}$ and $n \in \mathbb{N}$; $K_n(s, t, 0) = 0$ for every $n \in \mathbb{N}$ and s, $t \in G$; there are sequences $(L_n)_n \subset \mathcal{L}$ and $(\psi_n)_n \subset \Psi$, such that

$$|K_n(s, t, u) - K_n(s, t, v)| \le L_n(s, t)\psi_n(|u - v|) \tag{3.1}$$

for all $n \in \mathbb{N}$, s, $t \in G$ and u, $v \in \mathbb{R}$.

(d) Let $\mathbb{K} = (K_n)_n \in \mathcal{K}_\Xi$ and consider a sequence $\boldsymbol{T} = (T_n)_n$ of Urysohn-type operators defined as

$$(T_n f)(s) = \int_G K_n(s,t,f(t))d\mu(t) \quad \text{for all } s \in G, \tag{3.2}$$

where $f \in \mathrm{Dom}\,\boldsymbol{T} = \bigcap_{n=1}^{\infty} \mathrm{Dom}\,T_n$, where for each $n \in \mathbb{N}, \mathrm{Dom}\,T_n$ is the subset of $L^0(G,\mathcal{B},\mu)$ on which $T_n f$ is well defined as a μ-measurable function of $s \in G$.

The following extension of the notion of singularity given in [9] will be fundamental in proving our modular convergence theorems with respect to filter convergence.

Definition 3.2 *Let $\mathbb{K} \in \mathcal{K}_\Xi$ be as in 3.1. We say that $\mathbb{K}$ is $\mathcal{F}$-singular if and only if*

3.2.1) *there exists a positive real number D_1 with*

$$\Lambda = \left\{ n \in \mathbb{N} : \int_G L_n(s,t)d\mu(t) \le D_1 \text{ for all } s \in G \right\} \in \mathcal{F};$$

3.2.2) *for every $s \in G$ and for each neighborhood $U_s \subset G$ of s, we get*

$$(\mathcal{F}) \lim_n \int_{G \setminus U_s} L_n(s,t)d\mu(t) = 0;$$

3.2.3) *for every $s \in G$ and $u \in \mathbb{R}$, we have*

$$(\mathcal{F}) \lim_n \int_G K_n(s,t,u)d\mu(t) = u.$$

Let Φ be the set of all continuous non-decreasing functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\varphi(0) = 0$, $\varphi(u) > 0$ for all $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$ in the usual sense, and let $\widetilde{\Phi}$ the set of all elements of Φ which are convex functions. For all $\varphi \in \Phi$, let us consider the functional ρ^φ defined as

$$\rho^\varphi(f) = \int_G \varphi(|f(s)|)d\mu(s) \quad \text{for all } f \in L^0(G,\mathcal{B},\mu).$$

The subspace

$$L^\varphi(G) = \{ f \in L^0(G,\mathcal{B},\mu) : \rho^\varphi(\lambda f) < +\infty \text{ for some } \lambda > 0 \}$$

is the Orlicz space generated by φ (see also [3, 6]).

We now define the modular convergence in the context of filters and some related notions.

Definition 3.3 (a) *A sequence $(f_n)_n$ of functions in $L^\varphi(G)$ is $\mathcal{F}$-modularly convergent to $f \in L^\varphi(G)$ if and only if there is a positive real number $\lambda > 0$ with*

$$(\mathcal{F}) \lim_n \rho^\varphi[\lambda(f_n - f)] = 0.$$

Note that the $\mathcal{F}_{\mathrm{cofin}}$-modular convergence coincides with the usual modular convergence.

(b) *The sequence $(f_n)_n$ in $L^\varphi(G)$ is $\mathcal{F}$-strongly convergent to $f \in L^\varphi(G)$ if and only if*

$$(\mathcal{F}) \lim_n \rho^\varphi[\lambda(f_n - f)] = 0$$

for every $\lambda > 0$. Observe that the $\mathcal{F}_{\mathrm{cofin}}$-strong convergence is equivalent to the usual strong convergence.

(c) *Given a subset $\mathcal{A} \subset L^\varphi(G)$ and $f \in L^\varphi(G)$, we say that $f \in \overline{\mathcal{A}}$ (that is, f is in the modular closure of $\mathcal{A}$) if and only if there is a sequence $(f_n)_n$ in $\mathcal{A}$ such that $(f_n)_n$ is modularly convergent to f in the usual sense.*

Finally, we introduce a technical concept, which relates the modular ρ^φ with the sequence $(\psi_n)_n$ introduced in 3.1 (c), and which will have importance in proving our main results. Given $\eta \in \Phi$, set

$$\rho^\eta(f) = \int_G \eta(|f(s)|)d\mu(s) \quad \text{for every } f \in L^0(G, \mathcal{B}, \mu).$$

Definition 3.4 *Given two functions φ, $\eta \in \Phi$ and a family $\Xi = (\psi_n)_n \subset \Psi$ as in (3.1), we say that the triple $(\rho^\varphi, \psi_n, \rho^\eta)$ is $\mathcal{F}$-properly directed if and only if there is a sequence $(c_n)_n$ in $\mathbb{R}$ with $(\mathcal{F})\lim_n c_n = 0$, such that for all $\lambda \in (0,1)$, there exists $C_\lambda \in (0,1)$ with $\rho^\varphi(C_\lambda(\psi_n \circ g)) \le \rho^\eta(\lambda g) + c_n$ whenever $n \in \mathbb{N}$ and $g \in L^0(G, \mathcal{B}, \mu)$, $g \ge 0$.*

We now extend [9, Theorem 1] to the context of filters. From now on, we suppose that our involved filter is a fixed free filter and we use the structural hypotheses and notations introduced above.

Theorem 3.5 *Let $f \in L^\infty(G, \mathcal{B}, \mu)$ and $\mathbb{K} = (K_n)_n \in \mathcal{K}_\Xi$ be $\mathcal{F}$-singular. Then for every continuity point $s \in G$ of f, we get $(\mathcal{F})\lim_n T_n f(s) = f(s)$.*

Proof Let $s \in G$ be a fixed continuity point of f. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}|(T_n f)(s) - f(s)| &\le \int_G |K_n(s,t,f(t)) - K_n(s,t,f(s))|d\mu(t) \\ &\quad + \left| \int_G K_n(s,t,f(s))d\mu(t) - f(s) \right| \\ &= I_1 + I_2.\end{aligned}$$

By virtue of the condition 3.2.3) of $\mathcal{F}$-singularity, we get $(\mathcal{F})\lim_n I_2 = 0$. So, in order to prove the theorem, it is enough to estimate the quantity I_1.

Fix arbitrarily $\varepsilon > 0$. By the $\mathcal{F}$-exhaustiveness at 0 of $(\psi_n)_n$, there is a $\sigma > 0$ and a set $\Pi \in \mathcal{F}$ with $\psi_n(u) \le \varepsilon$ whenever $|u| \le \sigma$ and $n \in \Pi$.

By continuity of f at the point s, there exists a neighborhood V_s of s such that $|f(t) - f(s)| < \sigma$ whenever $t \in V_s$. From (3.1), 3.2.1) and the $\mathcal{F}$-exhaustiveness at 0 of $(\psi_n)_n$, there exists a set $\Lambda \in \mathcal{F}$ such that for all $n \in \Lambda \cap \Pi$, we get

$$\begin{aligned}I_1 &\le \int_G L_n(s,t)\psi_n(|f(t) - f(s)|)d\mu(t) \\ &= \int_{V_s} L_n(s,t)\psi_n(|f(t) - f(s)|)d\mu(t) + \int_{G\setminus V_s} L_n(s,t)\psi_n(|f(t) - f(s)|)d\mu(t) \\ &\le \psi_n(\sigma) \cdot D_1 + \psi_n(2\|f\|_\infty) \cdot \int_{G\setminus V_s} L_n(s,t)d\mu(t) \\ &\le \varepsilon D_1 + \psi_n(2\|f\|_\infty) \cdot \int_{G\setminus V_s} L_n(s,t)d\mu(t).\end{aligned}$$

By 3.2.2), there is a $\Pi_s \in \mathcal{F}$ (depending on s) such that for every $n \in \Pi_s$, we get

$$\int_{G\setminus V_s} L_n(s,t)d\mu(t) \le \varepsilon.$$

Thus $\Pi_s \cap \Pi \in \mathcal{F}$. Let $\Lambda_s = \Pi_s \cap \Pi \cap \Lambda$. We have obtained

$$I_1 \le \varepsilon D_1 + \varepsilon \psi_n(2\|f\|_\infty)$$

for all $n \in \Lambda_s$. Since, by hypothesis, the sequence $(\psi_n(2\|f\|_\infty))_n$ is $\mathcal{F}$-bounded, there exist a positive real number D' and a set $Q \in \mathcal{F}$, depending only on f, such that $\psi_n(2\|f\|_\infty) \le D'$ whenever $n \in Q$. Hence $(\mathcal{F}) \lim_n I_1 = 0$. So $(\mathcal{F}) \lim_n T_n f(s) = f(s)$, and thus we obtain the assertion. □

Theorem 3.6 *Let $\mathcal{F}$ be a Lebesgue filter of $\mathbb{N}$, $\varphi \in \widetilde{\Phi}$, $\Xi = (\psi_n)_n \subset \Psi$ and $\mathbb{K} = (K_n)_n \in \mathcal{K}_\Xi$ be $\mathcal{F}$-singular. Then*

$$(\mathcal{F}) \lim_n \rho^\varphi[\lambda(T_n f - f)\chi_S] = 0$$

for every $f \in C(G)$, $\lambda > 0$ and $S \in \mathcal{B}$ with $\mu(S) < +\infty$.

Proof Choose arbitrarily $f \in C(G)$. By hypothesis, Theorem 3.5 and the continuity of φ, we have

$$(\mathcal{F}) \lim_n \varphi(\lambda|(T_n f)(s) - f(s)|) = 0 \quad \text{for all } s \in G \text{ and } \lambda > 0.$$

Indeed, it is enough to observe that a function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous if and only if for any free filter $\mathcal{F}$ and for all sequences $(x_n)_n$ in $\mathbb{R}_0^+$, $\mathcal{F}$-converging to $x \in \mathbb{R}_0^+$, we have $(\mathcal{F}) \lim_n \varphi(x_n) = \varphi(x)$, taking into account that $\varphi(0) = 0$.

Let now $D_1 > 0$ and $\Lambda \in \mathcal{F}$ be according to $\mathcal{F}$-singularity, and fix arbitrarily $\lambda > 0$. For all $n \in \Lambda$ and $s \in G$, we have

$$2\lambda|(T_n f)(s)| \le 2\lambda \int_G L_n(s,t)\psi_n(|f(t)|)d\mu(t) \le 2\lambda D_1 \psi_n(\|f\|_\infty). \tag{3.3}$$

Let $S \in \mathcal{B}$, $\mu(S) < +\infty$. From (3.3), taking into account that $\varphi(0) = 0$ and since φ maps bounded sets into bounded sets, there exist $M^* > 0$ and $\Lambda^* \in \mathcal{F}$, without loss of generality, we assume $\Lambda^* \subset \Lambda$, such that for all $n \in \Lambda^*$ and $s \in G$, we get

$$\varphi(2\lambda|(T_n f)(s)|\chi_S(s)) \le \varphi(2\,\lambda\, D_1\, \psi_n(\|f\|_\infty)\, \chi_S(s)) \le M^* \chi_S(s). \tag{3.4}$$

By the convexity of φ, for all $s \in G$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \varphi(\lambda|(T_n f)(s) - f(s)|\chi_S(s)) &\le \varphi(2\,\lambda|(T_n f)(s)|\chi_S(s)) + \varphi(2\,\lambda|f(s)|\chi_S(s)) \\ &\le M^* \chi_S(s) + \varphi(2\lambda\|f\|_\infty)\chi_S(s). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain that the functions $s \mapsto \varphi(\lambda|(T_n f)(s) - f(s)|\chi_S(s))$, $n \in \mathbb{N}$, are $\mathcal{F}$-dominated by a suitable function h, which belongs to $L^1(G, \mathcal{B}, \mu)$. From this, since

$$(\mathcal{F}) \lim_n \varphi(\lambda|(T_n f)(s) - f(s)|\chi_S(s)) = 0$$

and $\mathcal{F}$ is a Lebesgue filter, we get

$$(\mathcal{F}) \lim_n \rho^\varphi[\lambda(T_n f - f)\chi_S] = 0,$$

that is the assertion. □

We now state the following technical lemma, analogous to [10, Theorem 5].

Lemma 3.7 *Let $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$ and $\Xi = (\psi_n)_n \subset \Psi$ be such that $(\rho^\varphi, \psi_n, \rho^\eta)$ is $\mathcal{F}$-properly directed, and assume that $\mathbb{K} = (K_n)_n$ is $\mathcal{F}$-singular.*

Then there is a sequence $(c_n)_n$ in $\mathbb{R}$ with $(\mathcal{F}) \lim_n c_n = 0$ and such that for every $\lambda > 0$, there exists a constant $a = a_\lambda > 0$ (depending only on λ) with

$$\rho^\varphi[a(T_n f - T_n g)] \le \rho^\eta[\lambda(f - g)] + c_n,$$

for all $n \in \mathbb{N}$ and $f,\, g \in L^0(G, \mathcal{B}, \mu) \cap \mathrm{Dom}\, \boldsymbol{T}$.

Proof Let $\lambda > 0$ be fixed, and let $(c_n)_n$, C_λ be according to the property of $\mathcal{F}$-properly directed triple. Let $a > 0$ be such that $a\, D_1 \leq C_\lambda$, where D_1 is as in 3.2.1). Proceeding analogously as in [10, Theorem 5] and using the Jensen inequality and the Fubini theorem, for each $n \in \mathbb{N}$ and $f,\, g \in L^0(G, \mathcal{B}, \mu) \cap \mathrm{Dom}\, \boldsymbol{T}$, we have

$$\rho^\varphi[a(T_n f - T_n g)] \leq \int_G \eta(\lambda |f(t) - g(t)|) d\mu(t) + c_n = \rho^\eta[\lambda(f-g)] + c_n,$$

and $(\mathcal{F}) \lim_n c_n = 0$. Thus we get the assertion. □

We now prove the following

Theorem 3.8 *Let $\mathcal{F}$ be a Lebesgue filter of $\mathbb{N}$, $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$ and $\Xi = (\psi_n)_n \subset \Psi$ be such that $(\rho^\varphi, \psi_n, \rho^\eta)$ is $\mathcal{F}$-properly directed. Let $\mathbb{K} = (K_n)_n$ be $\mathcal{F}$-singular.*

Then for every $f \in L^{\varphi+\eta}(G) \cap \mathrm{Dom}\, \boldsymbol{T}$, there exists a positive real number a such that

$$(\mathcal{F}) \lim_n \rho^\varphi[a(T_n f - f)\chi_S] = 0$$

whenever $S \in \mathcal{B}$ with $\mu(S) < +\infty$.

Proof Let $f \in L^{\varphi+\eta}(G) \cap \mathrm{Dom}\, \boldsymbol{T}$. By [15, Proposition 1], $L^{\varphi+\eta}(G)$ is the modular closure of $C_c(G)$ with respect to the modular topology in $L^{\varphi+\eta}$ associated with the usual modular convergence. So there exist a constant $\lambda' \in (0,1)$ and a sequence $(f_k)_k$ of elements of $C_c(G)$ such that for every $S \in \mathcal{B}$ with $\mu(S) < +\infty$, and for any $\varepsilon > 0$, there is $\overline{k} = \overline{k}(\varepsilon) \in \mathbb{N}$ with

$$\rho^{\varphi+\eta}[\lambda'(f_k - f)\chi_S] \leq \frac{\varepsilon}{3} \tag{3.6}$$

for every $k \geq \overline{k}$.

In correspondence with λ', let $a_{\lambda'} > 0$ be a constant according to Lemma 3.7. Applying Theorem 3.6 to λ' and f_k, $k \in \mathbb{N}$, for all $\varepsilon > 0$, $S \in \mathcal{B}$ with $\mu(S) < +\infty$ and $k \geq \overline{k}$, we have

$$A_k := \left\{ n \in \mathbb{N} : \rho^\varphi[\lambda'(T_n f_k - f_k)\chi_S] \leq \frac{\varepsilon}{3} \right\} \in \mathcal{F}.$$

Let $a > 0$ be such that $3a \leq \min(\lambda', a_{\lambda'})$. By the convexity of φ, for all $n \in \mathbb{N}$, we get

$$\begin{aligned}
&\int_G \varphi(a|T_n f(s) - f(s)|\chi_S(s)) d\mu(s) \\
&\quad \leq \int_G \varphi(3a|T_n f(s) - T_n f_{\overline{k}}(s)|\chi_S(s)) d\mu(s) + \int_G \varphi(3a|T_n f_{\overline{k}}(s) - f_{\overline{k}}(s)|\chi_S(s)) d\mu(s) \\
&\qquad + \int_G \varphi(3a|f_{\overline{k}}(s) - f(s)|\chi_S(s)) d\mu(s) \\
&\quad = \rho^\varphi[3a(T_n f - T_n f_{\overline{k}})\chi_S] + \rho^\varphi[3a(T_n f_{\overline{k}} - f_{\overline{k}})\chi_S] + \rho^\varphi[3a(f_{\overline{k}} - f)\chi_S] \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

Observe that, by Lemma 3.7, for all $n \in \mathbb{N}$, we get $I_1 \leq \rho^\eta[\lambda'(f - f_{\overline{k}})\chi_S] + c_n$, where $(\mathcal{F}) \lim_n c_n = 0$; moreover, by Theorem 3.6, we have $I_2 \leq \frac{\varepsilon}{3}$ for every $n \in A_{\overline{k}}$. We obtain

$$\begin{aligned}
I_1 + I_3 &\leq \rho^\eta[\lambda'(f - f_{\overline{k}})\chi_S] + c_n + \rho^\varphi[\lambda'(f - f_{\overline{k}})\chi_S] \\
&= \rho^{\varphi+\eta}[\lambda'(f - f_{\overline{k}})\chi_S] + c_n \leq \frac{\varepsilon}{3} + c_n
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, for all $n \in A_{\overline{k}}$, we get

$$\int_G \varphi(a|T_n f(s) - f(s)|\chi_S(s))d\mu(s) \leq I_1 + I_2 + I_3 \leq \frac{2}{3}\varepsilon + c_n. \tag{3.7}$$

Since $(\mathcal{F})\lim_n c_n = 0$, for every $\varepsilon > 0$, the set

$$F_\varepsilon = \left\{ n \in \mathbb{N} : |c_n| \leq \frac{\varepsilon}{3} \right\} \in \mathcal{F}. \tag{3.8}$$

Let $\overline{k}$ be as above, and set $E = F_\varepsilon \cap A_{\overline{k}}$. Note that $E \in \mathcal{F}$. From (3.7) and (3.8), for every $n \in E$, we have

$$0 \leq \rho^\varphi[a(T_n f - f)\chi_S] \leq \varepsilon.$$

This means

$$(\mathcal{F})\lim_n \rho^\varphi[a(T_n f - f)\chi_S] = 0,$$

that is the assertion. □

4 Applications

As an application, we consider nonlinear Mellin operators and in particular moment operators and Mellin–Gauss–Weierstrass-type operators (see also [11]). We recall the following structural assumptions according to [11].

Assumptions 4.1 (a) Let $G = \mathbb{R}^+$, and for any measurable set $S \subset \mathbb{R}^+$, put $\mu(S) = \int_S \frac{dt}{t}$. Let $\widetilde{\mathcal{L}}$ be the set of all sequences of measurable functions $\widetilde{L_n} : \mathbb{R}^+ \to \mathbb{R}_0^+$, $n \in \mathbb{N}$, such that $\widetilde{L_n} \in L^1(\mu)$.

(b) Let $\Xi = (\psi_n)_n \subset \Psi$ be as in 3.1 (c), and denote by $\widetilde{\mathcal{K}}_\Xi$ the set of all sequences of functions $\widetilde{K_n} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, such that

i) $\widetilde{K_n}(\cdot, u)$ is measurable for all $u \in \mathbb{R}$ and $n \in \mathbb{N}$, and $\widetilde{K_n}(t, 0) = 0$ for every $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$;

ii) there are sequences $(\widetilde{L_n})_n \subset \widetilde{\mathcal{L}}$ and $(\psi_n)_n \subset \Psi$, such that

$$|\widetilde{K_n}(t, u) - \widetilde{K_n}(t, v)| \leq \widetilde{L_n}(t)\psi_n(|u - v|) \tag{4.1}$$

for all $n \in \mathbb{N}$, $t \in \mathbb{R}^+$ and $u, v \in \mathbb{R}$.

(c) Let $\widetilde{\mathbb{K}} = (\widetilde{K_n})_n \in \widetilde{\mathcal{K}}_\Xi$ and let us consider a sequence $\widetilde{\boldsymbol{T}} = (\widetilde{T_n})_n$ of nonlinear Mellin operators defined as

$$(\widetilde{T_n} f)(s) = \int_0^{+\infty} \widetilde{K_n}\left(\frac{t}{s}, f(t)\right)\frac{dt}{t} \quad \text{for all } s \in \mathbb{R}^+, \tag{4.2}$$

where $f \in \mathrm{Dom}\,\widetilde{\boldsymbol{T}} = \bigcap_{n=1}^\infty \mathrm{Dom}\,\widetilde{T_n}$ is the subset of $L^0(\mathbb{R}^+, \mathcal{B}, \mu)$ on which $\widetilde{T_n} f$ is well defined.

Observe that, if we put

$$L_n(s, t) = \widetilde{L_n}\left(\frac{t}{s}\right), \quad K_n(s, t, u) = \widetilde{K_n}\left(\frac{t}{s}, u\right)$$

for all s, $t \in \mathbb{R}^+$ and $u \in \mathbb{R}$, then it is not difficult to check that, if $\widetilde{K_n}$, $\widetilde{L_n}$, $n \in \mathbb{N}$, satisfy Assumptions 4.1, then K_n, L_n, $n \in \mathbb{N}$, fulfil Assumptions 3.1.

We now introduce (filter) singularity in the context of Mellin operators.

Definition 4.2 *We say that $\widetilde{\mathbb{K}} = (\widetilde{K}_n)_n$ is $\mathcal{F}$-singular if and only if*

4.2.1) *there exists a $D_1 > 0$ with*

$$\Pi = \left\{ n \in \mathbb{N} : \int_0^{+\infty} \widetilde{L_n}(t)\, \frac{dt}{t} \leq D_1 \right\} \in \mathcal{F};$$

4.2.2) *for all $\delta > 1$, setting $U_\delta = \left[\frac{1}{\delta}, \delta\right]$, we get*

$$(\mathcal{F}) \lim_n \int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_n}(t)\, \frac{dt}{t} = 0;$$

4.2.3) *for every $u \in \mathbb{R}$, we have*

$$(\mathcal{F}) \lim_n \int_0^{+\infty} \widetilde{K_n}(t,u)\, \frac{dt}{t} = u.$$

Remark 4.3 Observe that

$$\int_0^{+\infty} \widetilde{L_n}\left(\frac{t}{s}\right) \frac{dt}{t} = \int_0^{+\infty} \widetilde{L_n}(z)\, \frac{s\,dz}{sz} = \int_0^{+\infty} \widetilde{L_n}(z)\, \frac{dz}{z},$$
$$\int_0^{+\infty} \widetilde{K_n}\left(\frac{t}{s}, u\right) \frac{dt}{t} = \int_0^{+\infty} \widetilde{K_n}(z,u) \frac{s\,dz}{sz} = \int_0^{+\infty} \widetilde{K_n}(z,u) \frac{dz}{z}$$

for all $n \in \mathbb{N}$, $s \in \mathbb{R}^+$ and $u \in \mathbb{R}$.

It is possible to adapt to the context of Mellin operators the convergence theorems proved in the previous section, and we can obtain, as particular cases, convergence theorems for these kinds of operators with respect to statistical convergence, since the filter $\mathcal{F}_d$ is a Lebesgue filter. We get the following consequence of Theorem 3.8.

Corollary 4.4 *Let $\mathcal{F}$ be a Lebesgue filter of $\mathbb{N}$, $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$ and $\Xi = (\psi_n)_n \subset \Psi$ be such that $(\rho^\varphi, \psi_n, \rho^\eta)$ is $\mathcal{F}$-properly directed. Let $\widetilde{\mathbb{K}} = (\widetilde{K_n})_n$ be $\mathcal{F}$-singular.*

Then for every $f \in L^{\varphi+\eta}(G) \cap \operatorname{Dom} \boldsymbol{T}$, there exists a positive real number a such that

$$(\mathcal{F}) \lim_n \rho^\varphi[a(\widetilde{T_n} f - f)\chi_S] = 0$$

for any $S \in \mathcal{B}$ with $\mu(S) < +\infty$.

In the linear frame, a particular case of Mellin-type kernels is the *moment kernel*, defined as

$$M_n(t) = n\, t^n \chi_{(0,1)}(t), \quad t \in \mathbb{R}^+. \tag{4.3}$$

For every $n \in \mathbb{N}$, $t \in \mathbb{R}^+$ and $u \in \mathbb{R}$, set

$$\widetilde{L_n}(t) = M_n(t), \quad \widetilde{K_n}(t,u) = \widetilde{L_n}(t) \cdot u. \tag{4.4}$$

Observe that, for any $\delta > 1$ and $n \in \mathbb{N}$, we get

$$\int_0^{+\infty} \widetilde{L_n}(t)\, \frac{dt}{t} = n \int_0^1 t^{n-1}\, dt = 1,$$
$$\int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_n}(t)\, \frac{dt}{t} = n \int_0^{1/\delta} t^{n-1}\, dt = \left(\frac{1}{\delta}\right)^n. \tag{4.5}$$

From (4.3)–(4.5), it follows that all $\mathcal{F}$-singularity conditions in Definition 4.2 are satisfied for every free filter $\mathcal{F}$.

Another example of Mellin-type kernels is the Mellin–Gauss–Weierstrass kernel, defined by putting

$$\widetilde{L_n}(t) = \frac{n}{2\sqrt{\pi}} \mathrm{e}^{-\frac{n^2}{4}\log^2 t}, \quad \widetilde{K_n}(t,u) = \widetilde{L_n}(t) \cdot u, \quad n \in \mathbb{N}, t \in \mathbb{R}^+, u \in \mathbb{R}. \tag{4.6}$$

We have

$$\int_0^{+\infty} \widetilde{L_n}(t)\, \frac{dt}{t} = \frac{n}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{n^2}{4}w^2}\, dw = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-v^2}\, dv = 1. \tag{4.7}$$

Moreover, for any fixed $\delta > 1$ and for n large enough (depending on δ), we get

$$\begin{aligned} \int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_n}(t)\, \frac{dt}{t} &= \frac{n}{2\sqrt{\pi}} \int_0^{1/\delta} \mathrm{e}^{-\frac{n^2}{4}\log^2 t} \frac{dt}{t} + \frac{n}{2\sqrt{\pi}} \int_\delta^{+\infty} \mathrm{e}^{-\frac{n^2}{4}\log^2 t} \frac{dt}{t} \\ &= \frac{n}{2\sqrt{\pi}} \int_{-\infty}^{-\log\delta} \mathrm{e}^{-\frac{n^2}{4}w^2}\, dw + \frac{n}{2\sqrt{\pi}} \int_{\log\delta}^{+\infty} \mathrm{e}^{-\frac{n^2}{4}w^2}\, dw \\ &= \frac{n}{\sqrt{\pi}} \int_{\log\delta}^{+\infty} \mathrm{e}^{-\frac{n^2}{4}w^2}\, dw = \frac{2}{\sqrt{\pi}} \int_{\frac{n\log\delta}{2}}^{+\infty} \mathrm{e}^{-v^2}\, dv \\ &\le \frac{2}{\sqrt{\pi}} \int_{\frac{n\log\delta}{2}}^{+\infty} \mathrm{e}^{-v}\, dv = \frac{2}{\sqrt{\pi}} \mathrm{e}^{-\frac{n\log\delta}{2}}. \end{aligned} \tag{4.8}$$

Hence, for all $\delta > 1$, we have

$$(\mathcal{F}) \lim_n \int_{\mathbb{R}^+ \setminus U_\delta} \widetilde{L_n}(t)\, \frac{dt}{t} = 0 \tag{4.9}$$

for every free filter $\mathcal{F}$ of $\mathbb{N}$. From (4.7) and (4.9), it follows that all conditions of $\mathcal{F}$-singularity are fulfilled, even in the classical case, that is when $\mathcal{F} = \mathcal{F}_{\text{cofin}}$.

In a similar way, we consider nonlinear moment kernels or nonlinear Mellin–Gauss–Weierstrass kernels, by setting

$$\widetilde{K_n}(t,u) = \widetilde{L_n}(t)\, G_n(u), \quad n \in \mathbb{N}, t \in \mathbb{R}^+, u \in \mathbb{R},$$

where $(G_n)_n$ is a sequence of functions satisfying a Lipschitz condition of the type

$$|G_n(u) - G_n(v)| \le \psi_n(|u-v|) \quad \text{for all } n \in \mathbb{N} \text{ and } u, v \in \mathbb{R},$$

where $(\psi_n)_n \subset \Psi$ and Ψ is as in Assumptions 4.1 (b), and $(\mathcal{F}) \lim_n G_n(u) = u$ (see also [11]).

We now give some examples in which our results hold for filter convergence with respect to any fixed Lebesgue filter $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$, in particular for the statistical convergence (that is in the case $\mathcal{F} = \mathcal{F}_d$), but not for ordinary convergence, showing that our results are proper extensions of the corresponding classical ones (see also [16], where a similar argument is used for Korovkin-type theorems).

Example 4.5 Let $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$ be any Lebesgue filter, and H be an infinite set, such that $\mathbb{N} \setminus H \in \mathcal{F}$: since $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$, then H does exist. We consider both moment-type and Mellin–Gauss–Weierstrass-type kernels. For every $t > 0$ and $n \in \mathbb{N}$, set

$$L_n^*(t) = \begin{cases} \widetilde{L_n}(t), & \text{if } n \in \mathbb{N} \setminus H, \\ \mathrm{e}^{3n^2} \widetilde{L_n}(t), & \text{if } n \in H, \end{cases} \tag{4.10}$$

where $\widetilde{L_n}(t)$ is as in (4.4) or as in (4.6).

By (4.5), (4.7) and (4.8), it follows that the $\mathcal{F}$-singularity conditions are satisfied. Set $\varphi(u) = \eta(u) = u^p$, where $u \in \mathbb{R}_0^+$ and $p \geq 1$ is arbitrarily chosen. For each $n \in \mathbb{N}$, define $\psi_n : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by putting $\psi_n(u) = u$, $u \in \mathbb{R}_0^+$. It is easy to see that the triple $(\rho^\varphi, \psi_n, \rho^\eta)$ is $\mathcal{F}$-properly directed with $c_n = 0$ for all $n \in \mathbb{N}$. Thus, in this case, the hypotheses of Corollary 4.4 are fulfilled. Therefore, the kernels L_n^*, $n \in \mathbb{N}$, satisfy our main results with respect to $\mathcal{F}$-convergence.

We now prove that the kernels L_n^*, $n \in \mathbb{N}$, do not fulfil the classical versions of theorems analogous to Theorems 3.6 and 3.8.

First of all, observe that in the case of the moment kernel, for every compact interval $[a, b] \subset \mathbb{R}^+$, for all $n \in H$ and $s \in \mathbb{R}^+$, we get

$$\int_a^b L_n^*\left(\frac{t}{s}\right)\frac{dt}{t} = n\,\mathrm{e}^{3n^2}\int_a^b \left(\frac{t}{s}\right)^n \chi_{(0,1)}\left(\frac{t}{s}\right)\frac{dt}{t} = n\,\mathrm{e}^{3n^2}\int_a^b \left(\frac{t}{s}\right)^n \chi_{(0,s)}(t)\frac{dt}{t}$$

$$= \frac{n\mathrm{e}^{3n^2}}{s^n}\int_a^b t^{n-1}\chi_{(0,s)}(t)\,dt = \begin{cases} \mathrm{e}^{3n^2}\frac{b^n - a^n}{s^n}, & \text{if } s \geq b, \\ \mathrm{e}^{3n^2}\frac{s^n - a^n}{s^n}, & \text{if } a \leq s < b, \\ 0, & \text{if } 0 < s < a. \end{cases} \tag{4.11}$$

In the case of the Mellin–Gauss–Weierstrass kernel, for every $[a, b] \subset \mathbb{R}^+$, $n \in H$ and $s > 0$, we have

$$\int_a^b L_n^*\left(\frac{t}{s}\right)\frac{dt}{t} = \frac{n\,\mathrm{e}^{3n^2}}{2\sqrt{\pi}}\int_{a/s}^{b/s} \mathrm{e}^{-\left(\frac{n}{2}\log t\right)^2}\frac{dt}{t} = \frac{\mathrm{e}^{3n^2}}{\sqrt{\pi}}\int_{\frac{n}{2}\log\left(\frac{a}{s}\right)}^{\frac{n}{2}\log\left(\frac{b}{s}\right)} \mathrm{e}^{-w^2}\,dw. \tag{4.12}$$

Let now $S = [\mathrm{e}^{-1/4}, \mathrm{e}^{1/4}]$, $f \in C_c(\mathbb{R}^+)$, $f \geq 0$ be such that $f(t) = 1$ for all $t \in [\mathrm{e}^{-3}, \mathrm{e}^{-2}]$ and the support of f is contained in $[\mathrm{e}^{-4}, \mathrm{e}^{-1}]$. For every $n \in \mathbb{N}$ and $s > 0$, set

$$(T_n^* f)(s) = \int_0^{+\infty} L_n^*\left(\frac{t}{s}\right) f(t)\frac{dt}{t}.$$

For all $\lambda > 0$ and $n \in H$, we get

$$\rho^\varphi[\lambda(T_n^* f - f)\chi_S] = \lambda^p \int_{\mathrm{e}^{-1/4}}^{\mathrm{e}^{1/4}} |(T_n^* f)(s) - f(s)|^p \frac{ds}{s} = \lambda^p \int_{\mathrm{e}^{-1/4}}^{\mathrm{e}^{1/4}} |(T_n^* f)(s)|^p \frac{ds}{s}.$$

We now claim that

$$\lim_{n\in H}(T_n^* f)(s) = +\infty \quad \text{for all } s \in S. \tag{4.13}$$

As a consequence, we will obtain that $\lim_{n\in H}\rho^\varphi[\lambda(T_n^* f - f)\chi_S] = +\infty$, and hence the sequence $(\rho^\varphi[\lambda(T_n^* f - f)\chi_S])_n$ does not converge in the usual sense, though it is $\mathcal{F}$-convergent to 0. In the case of the moment-type kernel, from (4.11), it follows that

$$(T_n^* f)(s) = \int_0^{+\infty} L_n^*\left(\frac{t}{s}\right) f(t)\frac{dt}{t} \geq \int_{\mathrm{e}^{-3}}^{\mathrm{e}^{-2}} L_n^*\left(\frac{t}{s}\right)\frac{dt}{t} = \mathrm{e}^{3n^2}\left[\frac{\mathrm{e}^{-2n} - \mathrm{e}^{-3n}}{s^n}\right]$$

for each $n \in H$ and $s \in S$. Since $S = [\mathrm{e}^{-1/4}, \mathrm{e}^{1/4}]$, then $\frac{1}{s^n} \geq \mathrm{e}^{-n/4}$, and therefore,

$$(T_n^* f)(s) \geq \mathrm{e}^{3n^2}(\mathrm{e}^{-2n} - \mathrm{e}^{-3n})\mathrm{e}^{-n/4} > \mathrm{e}^{3n}(\mathrm{e}^{-2n} - \mathrm{e}^{-3n})\mathrm{e}^{-n/4} = \mathrm{e}^{3n/4} - \mathrm{e}^{-n/4}$$

for all $n \in H$ and $s \in S$. Thus, we get (4.13).

For the Mellin–Gauss–Weierstrass-type kernel, from (4.12), for all $n \in H$ and $s \in S$, we have

$$\begin{aligned}(T_n^* f)(s) &= \int_0^{+\infty} L_n^*\left(\frac{t}{s}\right) f(t) \frac{dt}{t} \geq \int_{\mathrm{e}^{-3}}^{\mathrm{e}^{-2}} L_n^*\left(\frac{t}{s}\right) \frac{dt}{t} \\ &= \frac{\mathrm{e}^{3n^2}}{\sqrt{\pi}} \int_{\frac{n}{2}\log\left(\frac{\mathrm{e}^{-3}}{s}\right)}^{\frac{n}{2}\log\left(\frac{\mathrm{e}^{-2}}{s}\right)} \mathrm{e}^{-w^2}\, dw \geq \frac{\mathrm{e}^{3n^2}}{\sqrt{\pi}} \int_{-\frac{11}{8}n}^{-\frac{9}{8}n} \mathrm{e}^{-w^2}\, dw \\ &\geq \frac{n}{4} \frac{\mathrm{e}^{3n^2}}{\sqrt{\pi}} \mathrm{e}^{-\frac{121}{64}n^2} = \frac{n}{4\sqrt{\pi}} \mathrm{e}^{\frac{71}{64}n^2}.\end{aligned}$$

So we obtain (4.13) even in this case. Thus we proved that our results are proper extensions of the corresponding classical ones.

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