An Estimate of the Tree-Width of a Planar Minor

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Abstract. We give a more simple than in [8] proof of the fact that if a finite graph has no minors isomorphic to the planar grid of the size of $r \times r$, then the tree-width of this graph is less than exp(poly(r)). In the case of planar graphs we prove a linear upper bound which improves the quadratic estimate from [5].

1. **Introduction.** Neil Robertson and P.D. Seymour in [6] proved that for any r there exists $m = f(r)$ such that every graph has tree-width $\leq m$ provided it has no planar grid of size $r \times r$ as its minor. A nonelementary upper bound of $f(r)$ follows from their proof. In [3] we presented a proof giving an elementary upper bound. The method from [3] allows to obtain the bound $m \leq \exp(\text{poly}(r))$, where $\exp(x)$ is function 2^x. N. Robertson, P.D. Seymour and R. Thomas [8] obtain a bound of less than 2^{9r^5}. When considering the case of planar graphs, N. Robertson and P.D. Seymour gave in [5] a proof with a quadratic upper bound of corresponding function $f(r)$. In Theorem 3 of the present paper we prove a linear upper bound for planar graphs. Incidentally (Theorem 2) we state in detail a shorter proof than in [8] for the bound exp(poly(r)) in general case. But let us remark that for this case a much simpler proof still, and with a better bound, can be found in [2].

The author does not know whether a polynomial upper bound is possible for the problem. If the answer to this question is affirmative, we will have the complete characterization of the graphs for which typical NP-problems (such as the problem of the existence of the Hamiltonian cycle) can be solved in polynomial time. This follows from the fact that such problems are solvable in polynomial time for any family of graphs with bounded tree-width, whereas for a family of graphs containing any plane grid they are NP-complete.

It is more convenient for us to use as in [3] the notion of n-divisibility instead of the notion of the tree-width. We prove in Theorem 1 that tree-width of a graph is related linearly (in both directions) with the minimal n for which the graph is n-divisible.

2. **Definitions and Theorems.** We will consider procedures of dividing of a finite graph into subgraphs: each subgraph arising in the process of dividing and

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having more than one vertex, at the next step is divided into two subgraphs, until all subgraphs have only one vertex (at the beginning of the process we have only one subgraph — the graph itself; here we mean by *subgraph* a subset of vertices of the graph together with all edges between them, but when we say that a subgraph is divided into two subgraphs we mean that the set of its vertices is partitioned into two parts). We'll say that a subgraph B of a graph G *is separable from its complement by no more than* n *vertices* iff there are no more than n vertices in the graph G such that any boundary for B (i.e. having only one end in B) edge is incident with at least one of these vertices. We'll call these "separating" vertices *marked for* B.

Definition. A graph is called m*-divisible*, if there exists a procedure of its dividing where each arising subgraph is separable from its complement by no more than m vertices. We say that a graph is m*-nondivisible* if it is not m-divisible.

We'll call *degree of nondivisibility* of a graph the minimal n such that the graph is n-divisible. Theorem 1 shows that the notions of tree-width and degree of nondivisibility are in fact equivalent. Let us recall the definition of tree-width from [4]. Let $V(G)$ denote the set of vertices of a graph G.

Definition. *A tree-decomposition* of a graph G is a family $(X_i|i \in I)$ of subsets of $V(G)$, together with a tree T with $V(T) = I$, with the following properties.

1. $\bigcup\limits_{i\in I} X_i = V(G)$.
2. Every edge of G has both its ends in some X_i $(i \in I)$.
3. For any $i, j, k \in I$, if j lies on the path of T from i to k, then $(X_i \bigcap X_k) \subseteq X_j$.

The *width of the tree-decomposition* is $\max\limits_{i\in I}(|X_i| - 1)$. The *tree-width* of G is the minimum $m \geq 0$ such that G has a tree-decomposition of width $\leq m$.

Theorem 1. a) *Any graph having tree-width* n *is* $(n + 1)$*-divisible.*
b) *Any* n*-divisible graph has tree-width no more than* $3n$*.*

Proof. Let us prove the item a). Consider the tree T of a tree-decomposition of a graph G. Let us describe a process of dividing of G. First, we separate from G the subgraph X_t corresponding to the root t of T. Remaining part is devided into parts that equal to $\bigcup\limits_{v\in T'} X_v \setminus X_t$ for each subtree T' with a root in a son of t (we separate these parts one by one; they are pairwise disjoint because by Property 3 their intersection would lie in X_t; by Property 2 there are not edges in G connecting these parts). In each part we again sepapate the subgraph situated in the root of corresponding subtree, then again divide the rest into parts and so on.

Let $e = (a, b)$ be an edge boundary for some subgraph G' arising in the process and corresponding to a subtree T' with root t'. It is easy to see that e has its external end (say, b) in X_f, where f is farther of t'. Indeed, by Property 2, there is r such that $a, b \in X_r$. We have $r \in T'$ since by construction (and by Property 3) for any $j \notin T'$ $G' \bigcap X_j = \emptyset$ and $a \in G'$. As $b \notin G'$, there exists an ancestor s of r such that $b \in X_s$. Then, by Property 3, $b \in X_f$. Thus, G' is separable from its complement by vertices of X_f, the number of which is $\leq n+1$.

At the end of the process G will be devided into subgraphs having $\leq n+1$ vertices. We split from them vertices one by one untill all subgraphs consist of only one vertex. The item a) is proved.

Let us prove the item b). First, we prove the following lemma.

Lemma 1. *Let a graph G be n-divisible and a corresponding process of its dividing be given. Then for each arising non-one-vertex subgraph P we can mark $\leq n$ vertices separating P from its complement, such that at the next partitioning P into P_1 and P_2 the following conditions hold:*
1. *If a vertex a does not belong to P and a is marked for at least one of P_1, P_2 then a is marked for P.*
2. *If $a \in P_i$ and a is marked for P then a is marked for P_i ($i = 1, 2$).*
3. *If $a \in P_i$ and a is marked for P_j where $j \neq i$ then a is marked for P_i ($i = 1, 2$).*

Proof. For each subgraph P arising in the process let $n_P \leq n$ be the minimal number such that there exists a set consisting of n_P vertices separating P from $G \setminus P$. Among all such separating sets of cardinality n_P we select (and mark for P) a set M_P with minimal number of external (i.e. not belonging to P) vertices.

Let us prove the item 1. Consider the set M of vertices from $G \setminus P$ marked for P_1, say, but not for P. Let, contrary to the statement, $M \neq \emptyset$. Let C be the set of vertices in P_1 joint by an edge to M and unmarked for P_1.

Let $|C| > |M|$. Evidently, all vertices in C belong to M_P. Any boundary for P edge, incident with a vertex in C, has another end in M_{P_1} hence it leads either to M or to M_P. Therefore if we replace in M_P the subset C with the set M, we obtain a set of vertices separating P from $G \setminus P$ and having less elements than M_P. This contradicts to minimality of M_P.

Now, let $|C| \leq |M|$. Every vertex in P_1 which is adjacent with a vertex in M either lies in C or is marked for P_1. Therefore if we replace in M_{P_1} the subset M with the set C, we obtain a new set separating P_1 from $G \setminus P_1$. It has no more vertices than the initial set, but the number of external for P_1 vertices is reduced. This contradiction proves the item 1.

Let us prove the item 2. Let, say, $i = 1$. Consider the set M of vertices in P_1 marked for P but not for P_1. Let $M \neq \emptyset$. Let C be the set of vertices in $G \setminus P$ joint by an edge to M and not belonging to M_P. Let $|M| \leq |C|$. Evidently, all C is marked for P_1. Let us replace in the set M_{P_1} the subset C with the set M. We obtain a new set of vertices separating P_1 from $G \setminus P_1$. It has no more vertices than the old set but the number of external vertices is reduced. This contradicts to the choice of M_{P_1}. Now, let $|M| > |C|$. Replace in M_P the subset M with the set C. We obtain a new set of vertices separating P from $G \setminus P$ and having less vertices than M_P. This contradiction proves the item 2.

Let us prove the item 3. Let, say, $i = 1$, $j = 2$. Consider the set M of vertices in P_1 marked for P_2 but not for P_1. Let $M \neq \emptyset$. Let C be the set of vertices in P_2 joint by an edge to M and unmarked for P_2. Let $|M| \leq |C|$. Evidently, $C \subseteq M_{P_1}$. Replace in the set M_{P_1} the subset C with M. We obtain a new set of vertices separating P_1 from $G \setminus P_1$ with smaller number of external vertices. This contradicts to the choice of M_{P_1}. Now, let $|M| > |C|$. Replace in M_{P_2} M with

C. We obtain a new set of vertices separating P_2 from $G \setminus P_2$ and having less vertices than M_{P_2}. This contradiction proves the item 3. Lemma 1 is proved. □

So, let a graph G be n-divisible. Let us take the process of its dividing and mark for each arising subgraph P the set M_P separating P from $G \setminus P$ as in the proof of Lemma 1. We'll represent this process in the form of a binary dividing tree D with subgraphs placing in its vertices (in a natural manner). A tree-decomposition tree T is obtained from D by ascribing to each vertex p the set X_p equal to union of all marked vertices for three subgraphs: the subgraph P and two its sons into which it is partitioned (if they exist).

It remaines to prove three properties from the definition of tree-decomposition. Property 1 is obvious, since by definition of M_P any vertex is marked for corresponding one-vertex subgraph. Let us prove Property 2. Let e be an edge of graph G. Consider a moment of dividing process when the ends of e turn out to be in different subgraphs and let A be that of them for which marked end of e is external (if there is no such subgraphs, Property 2 for e, clearly, holds). Consider the sequence of such descendants of subgraph A that have e as a boundary edge. Since for one-vertex subgraphs only one internal vertex is marked, there are two neighboring subgraphs in this sequence — a farther and a son such that for the farther an external end of e is marked and for the son — an internal end. This implies satisfaction of Property 2 for e.

Let us prove Property 3. Let a vertex a is marked for two subgraphs G_1 and G_2. It is sufficient to show that a is marked for all subgraphs on the path in dividing tree connecting G_1 and G_2 exept, maybe, their common ancestor. Let A be a subgraph on the path between G_1 and G_2. Consider two cases.

Case 1. One of G_1, G_2 is ancestor of another, say, G_1 is ancestor of G_2. Let the vertex a not belong to G_1. Then it follows from item 1 of Lemma 1 that if a is unmarked for A then a is unmarked for all descendants of A. Hence, as a is marked for G_2 then a is marked for A. Now, let $a \in G_2$. Then by item 2 of Lemma 1 as a is marked for G_1 then a is marked for any descendant of G_1 which contains a, including A. Now, let $a \in G_1$, $a \notin G_2$. Let B be the nearest to G_1 descendant of G_1 on the path to G_2 such that $a \notin B$. Then if A lies between B and G_2 (or if $A = B$), it is easy to see that from item 1 and the fact that a is marked for G_2 it follows that a is marked for A. And if A lies between G_1 and B ($A \neq B$) then from item 2 and the fact that a is marked for G_1 it follows that a is marked for A.

Case 2. None of subgraphs G_1, G_2 is ancestor of another. Let P be the nearest to them their common ancestor. It is easy to see that for both sons of P (as well as for all subgraphs between them and G_1, G_2) a is marked: if a son does not contain a this follows from item 1, if a son contains a then from item 3 of Lemma 1 and from the fact that another son does not contain a. Further, all considerations, evidently, are reduced to the case 1. Theorem 1 is proved. □

N. Robertson and P.D. Seymour in [4] nonconstructively proved the existense of a polynomial algorithm to test if a graph has tree-width $\leq m$ for fixed m. We briefly describe a polynomial algorithm which for any fixed n decides if an input graph is n-divisible and if so, constructs a process of its n-dividing.

We'll mean by *n-divisibility of a subgraph* its n-divisibility as a graph but we take its boundary edges into consideration (in particular, the subgraph itself must be separable from its complement by no more than n vertices). We call a vertex g belonging to a subgraph B *saturated for* B if there are more than n boundary edges incident with g (multiple edges are considered only once). An external vertex which is incident with a boundary edge will be called *external boundary vertex.*

Lemma 2. *If a graph G is n-divisible, there exists a process of its n-dividing such that for every arising subgraph B we at first separate from it one by one no more than n vertices so that remaining subgraph B' either becomes one-vertex or has no saturated vertices and has no more than n^2 external boundary vertices. Then we divide B' into connected components and only after this we divide this components.*

Proof. Let a non-one-vertex subgraph B be n-divisible and let no more than n vertices separating B from its complement be marked. Let us separate out of B a saturated (and, hence, marked) vertex b. It is easy to see that the rest B_1 is n-divisible because a process of n-dividing of B induces a process of n-dividing of B_1. (Indeed, a subgraph C arising in the induced process and corresponding to the subgraph $C' = C + \{b\}$ in the main process is separable from its complement by $\leq n$ vertices — these vertices are the same as for C' including b.) We show that any saturated for B_1 vertex b_1 is marked for B. Eash incident with b_1 and boundary for B_1 edge either is boundary for B or leads to b. If b_1 is not marked for B then at least n adjacent with b_1 vertices out of B must be marked for B. Besides, b is marked for B, and we have a contradiction.

Separating b_1 out of B_1, we obtain B_2 and so on until $B_i = B'$ has no saturated vertices. Evidently, we have to separate $\leq n$ vertices. The fact that B' has $\leq n^2$ external boundary vertices is obvious enough. Lemma 2 is proved. □

We'll call the process of dividing described in Lemma 2 *canonical process.* Now, we describe an algorithm. We consider the following totalities: either a one-vertex subgraph K or a pair $\langle K, P\rangle$ where P is a set of $\leq n^2$ vertices of an input graph G and K is a connected component of the subgraph $G \setminus P$. We will form step by step a list of all the totalities where K is n-divisible. Before the first step we put all one-vertex subgraphs down on the list. After the m-th step there will be all such pairs in our list that K is n-divisible by $\leq m$ partitioning (and, maybe, some other pairs with n-divisible K).

At the $(m+1)$-th step we look over all pairs $\langle K, P\rangle$ and for every pair which is not contained in our list we do the following. First, we verify that K is separable from its complement by $\leq n$ vertices. Let it be so. Then we suppose that K can be partitioned into two (unknown) parts K_1 and K_2 being n-divisible by $\leq m$ dividing. Look over all quadruples of sets of vertices $\langle O_1, O_2, P_1, P_2\rangle$ where $|O_1| \leq n$, $|O_2| \leq n$, $|P_1| \leq n^2$, $|P_2| \leq n^2$. The meaning is: O_i — the set of those marked for K_i vertices which by Lemma 2 can be separated so that the subgraph $K_i \setminus O_i$ has the properties stated in Lemma 2; P_i — the set of all external boundary for $K_i \setminus O_i$ vertices. For a quadruple corresponding to a canonical process, the subgraphs $K_1 \setminus O_1$ and $K_2 \setminus O_2$ which we try to find must be a union of some connected components of the subgraphs $G \setminus P_1$ and $G \setminus P_2$ respectively.

Let $K' = K \setminus (O_1 \bigcup O_2)$. We call a path *clear* if all its vertices except, maybe, ends lie in $K' \setminus (P_1 \bigcup P_2)$. For each vertex a in K' consider two the following conditions.
1. Either $a \in P_1$ or there exists a clear path leading from a to some vertex $b \in P_1 \bigcap K'$.
2. $a \notin P_1 \bigcup P_2$ and there exists a clear path leading from a to $P_2 \setminus K'$.
We put a in K_2 if at least one of the conditions holds, otherwise we put a in K_1. (Note, that if both conditions are not satisfied then the conponent of $G \setminus P_1$ containing a either does not belong to K' or coincides with a component of $G \setminus P_2$.)

After this partitioning of K we verify that P_1 and P_2 really are the sets of all external boundary vertices for $K_1 \setminus O_1$ and $K_2 \setminus O_2$ respectievly. It is easy to see that if it is not the case then the chosen quadruples of sets does not correspond to a canonical process of dividing. Finally, we verify that K_1 and K_2 are separable from their complements by a sets of $\leq n$ vertices including respectively O_1 and O_2.

We put $\langle K, P \rangle$ down on our list if and only if all the connected components of $G \setminus P_1$ and $G \setminus P_2$ contained in K' already present in the list. It is not difficult to see that the described algorithm is required.

Remark. There is also another notion being studied in literature — the *branchwidth* of a graph G. It is equal to the minimal t for which there exists a process of dividing of edges of G (like our process for vertices) such that for any arising set of edges E' it holds $|\mathrm{coup}(E')| \leq t$ where $\mathrm{coup}(E')$ is the set of vertices incident both with an edge in E' and with an edge not in E'. N. Robertson and P.D. Seymour in [7] proved linear equivalence of branchwidth and tree-width. Hans L. Bodlaender and Dimitrios M. Thilikos in [1] constructed a linear algorithm for recognition of the relation branchwidth $< T$ (for arbitrary fixed T).

Let us turn to our main result. Recall that a graph A is a *minor* of a graph B if we can map every vertex of the graph A to a nonempty connected subgraph of the graph B (moreover, different vertices correspond to disjoint subgraphs) and map every edge of the graph A to an edge of the graph B joining those two subgraphs which correspond to the ends of the edge in A.

Theorem 2. *For any natural $r \geq 2$ there exists $m \leq r^2 \exp(r^{20})$ such that if a finite graph G has no minors isomorphic to the planar grid of the size of $r \times r$, then this graph is m-divisible.*

Proof. We say that two subgraphs P_1 and P_2 of a graph G are *n-separable through a subgraph C* of the graph G if we can select $\leq n$ vertices in C with the following property: any path between P_1 and P_2 which has all interior vertices in C and contains at least two edges, passes through at least one of the selected vertices.

Lemma 3. *For any n, k in any (nk)-nondivisible graph there exist a connected subgraph C and k connected subgraphs, pairwise disjoint and disjoint from C, such that any two of these k subgraphs are n-nonseparable through C.*

Proof. Let $m = nk$. Let a graph G be m-nondivisible. We will carry out some procedure on G described below. Before the beginning of every stage of this procedure the conditions described in the following paragraph will be satisfied.

Some pairwise disjoint connected subgraphs are selected in the graph G. One of them is m-nondivisible. We'll call this subgraph "*central subgraph*" and denote it by C. The selected subgraphs joined by an edge to C will be called "*boundary subgraphs*". There are not more than k boundary subgraphs. Any edge boundary for C has external end in one of boundary subgraphs. For each boundary subgraph P we can select $\leq n$ vertices in $C \bigcup P$ such that any edge which joins P to C is incident with at least one of the selected vertices.

It follows from the conditions above that C is separable from its complement by $\leq m$ vertices. So, since C is m-nondivisible, for any partition of C into two subgraphs, at least one of them is m-nondivisible. Before the beginning of our procedure the subgraph C is a connected m-nondivisible component of the graph G. The boundary subgraphs are absent.

Before the beginning of every stage, the number of the boundary subgraphs is either strictly less than k or equal to k. In the first case let c be an arbitrary vertex in C. Then the subgraph $C_1 = C \setminus \{c\}$ is m-nondivisible and is separable from $G \setminus C_1$ by $\leq m$ vertices. Let C_0 be a m-nondivisible component of the subgraph C_1. C_0 becomes the new central subgraph, and $\{c\}$ becomes the new boundary subgraph. Clearly, the inductive conditions are satisfied.

In the second case if there is no pair of boundary subgraphs being n-separable through C then our procedure is completed, and we have found the required subgraphs. Otherwise let P_1 and P_2 be such a pair. Consider the set M of vertices in C which are joined by an edge to $P_1 \bigcup P_2$. If M consists of only one vertex c then we separate c in the same way as in the first case. In this case we exclude P_1 and P_2 from the set of the selected subgraphs. Clearly, inductive conditions are satisfied. If $|M| > 1$ then we mark $\leq n$ vertices in C separating P_1 and P_2. Let us prove the following fact:

there exists a partitioning of C into nonempty parts C_1 and C_2 such that the graphs $P_1 \bigcup C_1$ and $P_2 \bigcup C_2$ are connected and each edge connecting C_1 with $P_2 \bigcup C_2$ or C_2 with $P_1 \bigcup C_1$ is incident with one of the marked vertices.

Choose in C two different vertices c_1 and c_2 such that c_i is joined to P_i by an edge. Ascribe c_i to C_i. Ascribe to C_i the remaining vertices in C which can be joined to P_i by a path with all interior vertices unmarked and lying in C. (If both 1 and 2 can serve as i, we act arbitrary). Consider the subgraph C' in C consisting of vertices which were not ascribed neither to C_1 nor to C_2. Since C is connected, for each connected component K of the graph C' there is a vertex in $C \setminus C'$ which is joined to K by an edge. Fix such a vertex a. Ascribe K to C_i which contains a. Now, the stated fact became obvious enough.

One of C_i is m-nondivisible, let it be C_1. From the proven fact it follows that C_1 is separable from $G \setminus C_1$ by $\leq m$ vertices. Let C_0 be an m-nondivisible component of the subgraph C_1. It will be the new central subgraph. The subgraph $P_2 \bigcup C_2$ will be the new boundary subgraph replacing P_2. It is easy to verify that the inductive conditions are satisfied.

Our procedure will end in a construction of the required subgraphs. This completes the proof of Lemma 3. □

Let us take $n = \exp(r^{20})$, $k = r^2$ in Lemma 3. We will use the following theorem of Menger.

Menger's Theorem. *Two given nonadjacent vertices a and b of a graph cannot be separated by deleting n vertices (different from a, b) if and only if there exist* $n + 1$ *pairwise vertex-disjoint paths between a and b.*

It follows from this theorem that for each pair of boundary subgraphs in G there exist $n + 1$ pairwise vertex-disjoint (except ends) paths between these subgraphs having all interior vertices in C. For all these pairs we fix n corresponding paths. Let us order the formed families of paths and denote them by $S_1, S_2, \dots S_{\frac{k(k-1)}{2}}$. We will reconstruct these families as follows.

At the next stage we take the next family S_i in this ordering. By S_i we mean the family which was formed from the original S_i by the reconstruction made up to the current moment. We assume as an inductive condition that for each $j < i$ the family S_j consists of only one path and this path does not cross any path of any other family. For each $j > i$ we take for the new S_j some subfamily of the old S_j of cardinality $l = |S_i|/\exp(r^{10})$. Consider the graph $S_i \bigcup S_j \subseteq C$ which consists of all vertices and edges belonging to S_i or to S_j except for the end vertices and edges. Let us draw in $S_i \bigcup S_j$ a new family S_j of the cardinality l so that it joins the same boundary subgraphs as the old S_j and the number of edges in $S_i \bigcup S_j$ belonging to S_j but not to S_i is minimal. One of the two following cases holds.

Case 1. There is a path q in S_i which does not cross $\geq |S_j|/\exp(r^{10})$ paths in each S_j when $j > i$. In this case we take $\{q\}$ for the new S_i, and for each $j > i$ we take for the new S_j the subfamily of the old S_j which consists of all paths not crossed by q. Evidently, the inductive condition is satisfied.

Case 2. There is no path described in the case 1. In this case we stop our procedure.

If we have the case 1 at every stage than at the end of the procedure we will have the complete graph with k vertices (and, hence, the $r \times r$ grid) as a minor of our graph.

Assume that we have the case 2 at i-th stage. Then there exists $j > i$ such that not less than $|S_i|/k^2$ paths in S_i cross $\geq |S_j| - |S_j|/\exp(r^{10})$ paths in S_j. Fix such j and denote the set of $|S_i|/k^2$ described paths in S_i by S_i^1. We will find the $r \times r$ grid in $S_i^1 \bigcup S_j$.

We order paths in S_j in the order of the decrease the number of paths in S_i^1 crossed by the paths in S_j. Let $V = \{q_1, q_2, \dots, q_k\}$ be the set of the initial k paths in this ordering.

Lemma 4. *There exist at least* $|S_j|\exp(r^9)$ *paths in* S_i^1 *crossing each path in* V*.*

Proof. Denote $b = |S_j|$. Let us show that the path q_k (and, hence, each path in V) crosses at least $N = |S_i^1| - b - \exp(r^{10})$ paths in S_i^1. Indeed, in all there exist at least $P = \left(1 - \frac{1}{\exp(r^{10})}\right) |S_i^1| b$ pairs of crossing paths. Even if the paths $q_1, \ldots, q_{k-1}$ cross all paths in S_i^1, it remains $E = P - (k-1)|S_i^1|$ such pairs for the other paths in S_j. Evidently, q_k must cross at least
$\frac{E}{b} = |S_i^1| - \frac{|S_i^1|}{\exp(r^{10})} - (k-1)\frac{|S_i^1|}{b} \geq |S_i^1| - b - (k-1)\exp(r^{10}) \geq N$
path in S_i^1 as we wanted. Hence, there exist
$|S_i^1| - kN' = \frac{b\exp(r^{10})}{k^2} - kb - k\exp(r^{10}) \geq$
$\geq b\left(\frac{\exp(r^{10})}{r^4} - r^2 - \frac{r^2\exp(r^{10})}{\exp(r^{19})}\right) \geq b\exp(r^9)$
paths in S_i^1, crossing each path in V. The set of such paths we denote by U. Lemma 4 is proved. □

We'll call paths in U *vertical* and in V — *horizontal*. Consider a horizontal path q. Clearly, there is an edge $e \notin U$ on q such that the path q crosses equal (to within 1) number of different vertical paths on each side from e. It follows from the minimality of the number of edges in S_j that after removal of the edge e there will be no b (recall, $b = |S_j|$) pairwise vertex-disjoint paths between the boundary subgraphs joined by S_j. By Menger's theorem they are $(b-1)$-separable. Fix $(b-1)$ vertices separating these subgraphs. Clearly, on each path in S_j except q there is just one fixed vertex. There are no more than $(b-1)$ vertical paths passing through the fixed vertices. It is easy to see that any other path in U does not cross q on both sides from e, otherwise we could go from "the left" boundary subgraph to "the right" one not passing both through e and through the fixed vertices. Since on each side from e the path q crosses half of vertical paths, there are two large subfamilies U_l and U_r in U such that U_l crosses q only on "the left" side from e and U_r only on "the right" side. It is easy to see that on any horizontal path q' there is an edge e' such that U_l crosses q' only on "the left" side from e' and U_r — only on "the right" side. (Indeed, it is sufficient to show that for any $q_1 \in U_l$, $q_2 \in U_r$ there are not vertices a_l, a_r on q' such that $a_l \in q_2$, $a_r \in q_1$ and a_l lies on the left of a_r on q'. But if it is not the case we could easily bypass both e and all fixed vertices going from the left to the right.)

Similarly, we divide each of two "halves" of the path q (before e and after e) in two equal parts with respect to the corresponding part of vertical paths. We continue this procedure until the path q (and, hence, all horizontal paths) is divided into $r^2\exp(r^4)$ segments. At the end of the procedure we have subfamily $U_1 \subseteq U$, $|U_1| = r^2\exp(r^4)$ and the partition of each horizontal path into segments such that each path in U_1 crosses any horizontal path on only one segment, and different paths on different segments. All horizontal paths cross paths in U_1 in the same order. (Of course, at each step of dividing of a subset of vertical paths into two parts, we throw out $\leq b$ "bad" vertical paths. But b is small in comparison with $|U|$ which ensures realizability of the procedure.)

We'll say that a path q crosses a path p *only once* if their common vertices and edges constitute exactly one (maybe, one-vertex) path (thus, this path is a subpath of both p and q). We will use the following trivial fact. *Let S_1 and S_2 be* *families of n pairwise vertex-disjoint paths such that any two paths in different families cross only once and all paths in S_1 cross paths in S_2 in the same order and all paths in S_2 cross paths in S_1 in the same order. Then the graph $S_1 \bigcup S_2$ has as a minor the grid of size of $n \times n$.*

For each $\alpha \in U_1$ we consider the following graph. Its vertices are horizontal paths. Vertices x and y are joined by an edge if there is a segment of the path α such that its end vertices are on the paths x and y and all its interior vertices are not in V. Clearly, the constructed graph is connected. Consider the subfamily $U_2 \subseteq U_1$ of r^2 paths such that all paths in U_2 correspond to the same graph. Let us take a frame tree in this graph. Evidently, a tree with r^2 vertices has either the height $\geq r$ or the number of leaves $\geq r$. In the first case, clearly, we have the $r \times r$ grid as a minor of our graph. In the second case consider the linear ordering of U_2 in which paths in U_2 are crossed by horizontal paths. Let us divide U_2 into r groups of neighboring paths with respect to this ordering. We use every group for the passing a path which in some fixed order crosses only once horizontal paths corresponding to leaves of the tree. We use non-leaf vertices of the tree for a moving from a leaf to another leaf vertex. Each such moving takes place in individual tree. Thus we have the $r \times r$ grid as a minor. This completes the proof of Theorem 2. □

Remark. It is shown in [2] that the degree 20 in the bound $r^2 \exp(r^{20})$ can be improved substantially while making the proof even simpler.

As we can see from the following theorem, for planar graphs there is a linear upper bound of the value of m.

Theorem 3. *For any r there exists $m \leq cr$ where $c = 2^{16}$ such that if a finite planar graph has no minors isomorphic to the planar $r \times r$ grid, then this graph is m-divisible.*

Proof. Let us take $k = 5$, $n = cr$ in Lemma 3, where $c = 2^{16}$. We construct the families $S_1, S_2, \ldots, S_{10}$ in the same way as in the proof of Theorem 2. We will carry out the same procedure with families of paths as in the proof of Theorem 2. At i-th stage we take the family S_i being a subfamily of the original S_i. There are two possible cases. In the first case there exists a path $q \in S_i$ which crosses less than half of paths in each S_j when $j > i$. Then for each $j > i$ we take for the new S_j the subfamily consisting of the paths of the old S_j which are not crossed by q. After that we proceed to the next stage.

Since the complete graph with five vertices can not be a minor of a planar graph, we will have at some i-th stage ($i \leq 9$) the second case, that is, there is no path described in the first case. Then there exists $j > i$ such that $\geq \frac{cr}{10 \cdot 2^{10}} \geq 4r$ paths in S_i are crossed by $\geq |S_j|/2$ paths in S_j. Let us fix such j and denote the set of $\geq 4r$ described paths corresponding to j by S_i^1.

Clearly, we can consider the connected graphs A and B joined by S_i to be trees. Then it is easy to see that paths in S_j together with A, B divides the plane into $|S_j|$ parts called *faces* and every face has exactly two paths on its boundary. Let us number paths in S_j by numbers $1, \ldots, |S_j|$ so that the pairs of paths $(i, i+1)$ where $i < |S_j|$ and $(|S_j|, 1)$ are neighboring i.e. some face has in its boundary both paths. This numbering gives a cycle order on S_j.

It is easy to see that to pass from some path in S_j to another path in S_j we must cross all paths of one of two sets between them. Therefore, each path in S_i^1 crosses $|S_j|/2$ paths in S_j which form a segment in the cycle order. Let us divide S_j into four equal segments in the order. Evidently, there exists a quarter such that $\geq |S_i^1|/4$ paths in S_i^1 contain a subpath crossing all paths of this quarter and having its ends on the two exterior paths q_1, q_2 of the quarter and having all its interior vertices out of q_1, q_2. Denote the set of such subpaths on paths in S_i^1 by S_i^2 and denote the considered quarter of paths in S_j by S_j^1. Clearly, $|S_i^2| \geq r$, $|S_j^1| \geq r$.

Let us draw in the graph $S_i^2 \bigcup S_j^1$ a family U of $|S_i^2|$ pairwise vertex-disjoint paths between q_1 and q_2 and a family V of $|S_j^1|$ pairwise vertex-disjoint paths between A and B, such that the number of edges of the graph $U \bigcup V$ is minimal. We'll call paths in U *vertical* and in V — *horizontal*. Clearly, each vertical path crosses each horizontal path. It is evident also that vertical paths divide the part of the plane bounded by q_1, q_2, A, B into parts and the set U (as well as the parts of the plane) are ordered in a natural way so that to pass from some vertical path to another vertical path we must cross all the paths between them. The same is true for horizontal paths. Therefore, for the proof of the existence of the grid it is sufficient to show that each vertical path crosses each horizontal path only once. Suppose that it is not true. Let α be the nearest to q_1 horizontal path which crosses some vertical path β in vertices a_1 and a_2 not connected by a path in $\alpha \cap \beta$.

We will show that the subpath $[a_1, a_2]$ of the path β does not pass through the part of the plane lying between q_1 and α. Assume that it is not true. Then either this subpath crosses the path α' neighboring to α from the side of q_1 or there exists a subpath l of the path β with the ends lying on α and the interior vertices lying out of V. The first case contradicts the condition of the choice of the path α, since β crosses α' not only in $[a_1, a_2]$. In the second case we can pass α along l and reduce the number of edges in $U \bigcup V$. This contradicts the minimality of this number.

If there are no vertices of vertical paths on the segment $r = [a_1, a_2]$ of the path α except the vertices of β, then we can pass β along r, which contradicts the minimality of the number of edges. Otherwise, assume that there is a vertex b on r belonging some vertical path β'. The subpath of β' from b to q_2 can not lie entirely between α and q_2 because it does not cross β. But this subpath can not pass through the part of the plane between α and q_1, because by the same argument as for β we obtain from this assumption a contradiction either with the condition of the choice of α or with the minimality of the number of edges in $U \bigcup V$. This contradiction completes the proof of Theorem 3. □

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