Share Equilibrium in Local Public Good Economies*

by

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Abstract We define a concept of share equilibrium for local public good (or club) economies where individual members of the population may have preferences over the membership of their jurisdiction. A share equilibrium specifies one share index for each individual. These indices determine each individual's cost shares in any jurisdiction that he might join. We demonstrate that the same axioms as those that characterize the Lindahl equilibrium, as discussed in his 1919 paper, also characterize the share equilibrium. The share equilibrium extends the notions of ratio equilibrium and cost share equilibrium (due to Kaneko 1977, Mas-Colell and Silvestre 1989) to economies with a local public good and possibly multiple jurisdictions.

1 An introduction to share equilibrium

This paper introduces the concept of a "share equilibrium" in the context of an economy with one local public good and one private good. The share equilibrium is an extension of the concept of the ratio equilibrium, defined for economies with pure public goods (see Kaneko, 1977). In contrast to the ratio equilibrium, share equilibrium is defined for economies where public goods are excludable and may be subject to congestion and, in an equilibrium outcome, the population may be partitioned into several jurisdictions, each producing the public good for the exclusive use of their own membership. The share equilibrium endogenously determines cost shares for any member of the population for any jurisdiction to which he may belong. In many situations involving clubs, local public goods, or group activities, shares of the costs of group activities are determined by some relative share indices of the individuals involved in the activity. Two examples that may fit very well are: property taxes that depend on property evaluations and thus on relative property values and; condominium homeowners fees that depend on relative sizes of condominium units. The main result of this paper is an axiomatization of share equilibrium.

To motivate the share equilibrium and our axiomatization, recall that a (pure) public good is a commodity that can be consumed in its entirety by all members of an economy. Various solutions to the problem of allocation of costs of pure public good provision have been proposed. The most well-known is perhaps the Lindahl equilibrium, as formalized in Samuelson (1954), with personalized prices for public goods.[1] According to our reading of Lindahl's (1919) paper, however, individuals pay shares of the total costs

[1]See also Mas-Colell (1980) where the concept of public goods was extended to public projects and Wooders (1997) which treats Lindahl prices (within njurisdictions) for economies with local public goods.

and, in equilibrium, these shares must satisfy the property that the amount of public good demanded is the same for all members of the population, given their cost shares. Other papers that have taken approaches in line with Lindahl (1919) include Kaneko (1977) and Mas-Colell and Silvestre (1989), which introdues the concept of cost share equilibrium for economies with multiple public goods. While with constant returns to scale in production of (pure) public goods, the Lindahl equilibrium yields the same outcomes as the equilibrium concept defined in Samuelson (1954) and Foley (1970) , the two concepts have different underlying philosophies and are fundamentally different. This becomes clear when considering properties of the concepts. In van den Nouweland, Tijs and Wooders (2002) we translate the ideas of Lindahl into axioms and find a set of axioms that are satisfied by the ratio equilibrium but not by the Samuelson equilibrium. In the current paper we demonstrate that the same axions that characterize the ratio equilibrium also characterize the share equilibrium.[2] Since a share equilibrium is a new concept, there are many open questions; in a concluding section, we report on research in progress dealing with some of these.

To illustrate ideas and to relate our general model to commonly-encountered models in the local public good literature, we provide some examples; these demonstrate that our model and results allow the possibility of congestion and of multiple jurisdictions.

[2]Axiomatic characterization of solution concepts is a well-established approach to their study. Recent contributions include Moulin (2000), Dhillon and Mertens (1999), and Maskin (1999).

2 Share equilibrium

2.1 Local public good economies

We consider economies with one public good and one private good.

Definition 1 A *local public good economy* is a list

$$E = \langle N, D; (w_i)_{i\in N}; (u_i)_{i\in N}; f\rangle,$$

where N denotes the non-empty finite population of the economy, $D \subseteq N$ denotes the set of decision makers, $w_i \in \mathbb{R}_+$ is the non-negative endowment of individual $i \in N$ of the private good, $u_i : \mathbb{R}_+ \times \mathbb{R}_+ \times 2^{N(i)} \to \mathbb{R}$ is the utility function of individual i, and $f : \mathbb{R}_+ \times 2^N\backslash\emptyset \to \mathbb{R}_+$ is the cost function for the production of local public good in jurisdictions.

This definition allows the possibility that not all individuals are involved in making decisions. For example, N may be the set of legal residents of the USA and D may be the set of registered voters. Residents who are not registered voters (the people in $N \setminus D$) are present in the economy, consume public goods and pay taxes toward paying for those goods, but they cannot vote and influence decisions regarding local public good provision. Or N may be the entire population and D may be the heads of households.[3] Non-empty subsets of the population N are *jurisdictions*. We denote the set of all possible jurisdictions by $2^N\backslash\emptyset$ and the set of all jurisdictions containing individual $i \in N$ by $2^{N(i)}$.

A utility function u_i specifies i's utility $u_i(x_i, y, J)$ of consuming an amount x_i of the private good and an amount y of the local public good while being a member of jurisdiction J ($i \in J$).

[3] Distinguishing between the sets N and D is essential for our axiomatization, as we will discuss.

A cost function f specifies, in terms of amounts of the private good, the cost $f(y, J)$ of producing the amount y of local public good in jurisdiction J that is borne by the decision makers in J.

The only assumptions we require are:

Assumptions

(1) For each $i \in N$, the utility function u_i is strictly increasing in both x_i and y.

(2) The cost function f is non-decreasing in y and $f(0, J) = 0$ for each jurisdiction J.

We denote the family of all public good economies satisfying these assumptions by $\mathcal{E}$. Whenever we need to distinguish between different economies, we may use the notations $N(E)$, $D(E)$, and f_E instead of just N, D, and f, respectively.

A specification of a jurisdiction structure of the population, levels of local public good provided in the jurisdictions, and private good consumption levels is called a configuration.

Definition 2 A *configuration* in economy $E \in \mathcal{E}$ with population N is a vector

$$(\mathbf{x}, \mathbf{y}, \mathbf{P}) = ((x_i)_{i \in N}, (y_P)_{P \in \mathbf{P}}, \mathbf{P}),$$

where, for each $i \in N$, $x_i \in \mathbb{R}_+$ is the consumption of the private good by individual i, $\mathbf{P}$ is a partition of N into jurisdictions and, for each $P \in \mathbf{P}$, $y_P \in \mathbb{R}_+$ is the level of local public good provided in jurisdiction $P \subseteq N$. The set of configurations in an economy with population N is denoted $C(N)$. A configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P}) \in C(N(E))$ is *feasible* for E if $f(y_P, P) \leq \sum_{i \in P \cap D} (w_i - x_i)$ for each $P \in \mathbf{P}$.

If $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ is a configuration in economy E, then we denote the jurisdiction containing individual $i \in N(E)$ by $P(i)$; thus $i \in P(i) \in \mathbf{P}$.

2.2 Share equilibrium defined

A share equilibrium for a local public good economy E consists of a vector of positive share indices $\mathbf{s} = (s_i)_{i \in N(E)} \in \mathbb{R}_{++}^{N(E)}$ and a configuration. The share indices determine the shares of the cost of the production of local public good in all jurisdictions that might possibly be formed in a relative manner: individual i with a share index s_i pays s_i/s_j times as much as individual j with share index s_j in any jurisdiction that contains both i and j. Thus, share indices create a budget constraint that limits the amounts of public good that individuals can afford in various jurisdictions. Each decision maker chooses utility-maximizing jurisdiction membership and consumption of the private and local public good within their budget constraint, taking their (relative) share indices as given. In equilibrium, these choices must be consistent. This requirement endogenously determines equilibrium share indices.[4]

We require the following notation. For local public good economy $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f \rangle$ and share indices $\mathbf{s} = (s_i)_{i \in N} \in \mathbb{R}_{++}^{N}$, the relative share of a decision-maker $i \in D$ as a member of a jurisdiction $J \subseteq N$ is $s_i^{J,D} := s_i / \left(\sum_{j \in J \cap D} s_j \right)$. Note that these relative shares are only as compared to the other decision makers in the jurisdiction. The reason for this is that only the individuals in D are involved in making decisions and they have to cover the costs $f(y, J)$ that are not borne by the people in $J \setminus D$.

Definition 3 A pair $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ consisting of share indices and a configuration is a *share equilibrium* in economy E if for each $i \in D$,[5]

1. $s_i^{P(i),D} f(y_{P(i)}, P(i)) + x_i = w_i$, and,

[4] Analogously, in a private goods exchange economy individuals take prices as given and equilibrium prices are determined endogenously.

[5] The equality in condition 1 could be "$\leq$ ", but using "=" is without loss of generality since condition 2 cannot be satisfied if $s_i^{P(i),D} f(y_{P(i)}, P(i)) + x_i < w_i$.

2. for all $(\overline{x}_i, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ satisfying $i \in \overline{J}$ and $s_i^{\overline{J},D} f(\overline{y}, \overline{J}) + \overline{x}_i \leq w_i$, it holds that $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\overline{x}_i, \overline{y}, \overline{J})$.

The set of share equilibria of an economy E is denoted $SE(E)$.

Note that, if $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is a share equilibrium in an economy E and $\alpha > 0$, then $(\alpha\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is also a share equilibrium in economy E.

Share equilibrium is an extension of ratio equilibrium, which is defined for pure public good economies only, to local public good economies. Pure public good economies can be interpreted as a special case of local public good economies where the only possible jurisdiction structure is that which consists if one jurisdiction which contains the entire population of the economy (i.e., $\mathbf{P} = \{N\}$). Using the notation in the current paper, a *ratio equilibrium* is a pair consisting of share indices $\mathbf{s}$ and a configuration $(\mathbf{x}, \mathbf{y}, \{N\})$ such that for each $i \in D$, (1) $s_i^{N,D} f(y_N, \{N\}) + x_i = w_i$, and (2) for all $(\overline{x}_i, \overline{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$ satisfying $s_i^{N,D} f(\overline{y}, \{N\}) + \overline{x}_i \leq w_i$, it holds that $u_i(x_i, y_N, \{N\}) \geq u_i(\overline{x}_i, \overline{y}, \{N\})$.[6]

Remark 1 Let $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f \rangle \in \mathcal{E}$ be a local public good economy and let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ be a share equilibrium in economy E. The following hold.

1. If $\mathbf{P} = \{N\}$, then $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \{N\}))$ is a ratio equilibrium of the pure public good economy that is obtained from E by restricting the set of possible jurisdiction structures to $\{N\}$.

2. Let $J \in \mathbf{P}$ be a jurisdiction formed as part of the share equilibrium. Then, the restriction of the share equilibrium $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ to J, i.e.,

[6] Note that this definition of ratio equilibrium also is applicable to pure public good economies in which not all people are involved in the decision-making process, but setting $N = D$ gives the definition of ratio equilibrium as in van den Nouweland, Tijs, and Wooders (2002).

$((s_i)_{i\in J}, ((x_i)_{i\in J}, y_J, \{J\}))$, is a ratio equilibrium of the pure public good economy that is obtained from E by restricting the population to J and the set of possible jurisdiction structures to $\{J\}$.

The following example illustrates that share equilibrium allows for non-trivial equilibrium jurisdiction structures and that our model allows local public goods of the sort modeled in Allouch and Wooders (2008) and its antecedents.

Example 1 *Share equilibria.* Consider an economy E in which all people are decision makers ($N(E) = D(E)$) who all are identical in terms of their endowments, preferences, and effects on others. Each individual has an endowment w and utility function $u_i(x_i, y, J) = x_i^\alpha y^{1-\alpha} - v(J)$, where $\alpha \in (0,1)$ and v is a congestion or crowding function defined by

$$v(J) = \begin{cases} 0 & \text{if } k = 2 \text{ or } k = 3 \\ 1 & \text{otherwise} \end{cases}$$

The cost function is given by $f(y, J) = |J|\, y$; thus *per capita* cost of public goods are constant across levels of local public good and jurisdictions.[7]

Assign each individual member of the population i the share index $s_i = 1$. Then, in any jurisdiction J, an individual pays the share $\frac{1}{|J|}$ of the cost of local public good production and thus the optimal level of local public good is the level that maximizes $u_i(w - \frac{1}{|J|}f(y, J), y, J) = (w - y)^\alpha y^{1-\alpha} - v(J)$, which is $y^* = (1-\alpha)w$. Thus, in any jurisdiction J, an individual achieves the maximum possible utility of $u_i^*(x_i^*, y^*, J) = \alpha^\alpha(1-\alpha)^{1-\alpha}w - v(J)$ by consuming $y^* = (1-\alpha)w$ of the local public good and $x_i^* = \alpha w$ of the private good. Because the first component in this expression is constant

[7] This special case often appears in the literature and makes the good a *local public service*.

across jurisdictions, the second component becomes decisive for the choice of jurisdiction, and only jurisdictions of size 2 or 3 are formed in equilibrium.

Thus, we find share equilibria in which $s_i = 1$ for all i, the jurisdiction structure is a partition of the population into jurisdictions of size 2 or 3, $y_J = (1-\alpha)w$ in each of those jurisdictions, and $x_i = \alpha w$ for all i. Note that we are identifying multiple share equilibrium configurations. For example, if the population size equals 6, there are 10 possible partitions into 3 jurisdictions of size 2 and another 10 possible partitions into 2 jurisdictions of size 3.

Note that none of the share equilibria that we found are a ratio equilibrium of the economy E. Thus, this example illustrates that some reverse of Remark 1 does not hold. ■

3 An axiomatization of the share equilibrium

In this section, we extend the axiomatization in van den Nouweland, Tijs, and Wooders (2002) to share equilibrium and demonstrate that the same axioms as those that characterize ratio equilibrium also characterize share equilibrium.

3.1 The axiomatization

We define a *solution* on $\mathcal{E}$ as a mapping ϕ that assigns to each public good economy $E \in \mathcal{E}$ a set of pairs each consisting of a vector of numbers and a configuration, that is,

$$\phi(E) \subseteq \mathbb{R}^{N(E)} \times C(N(E)).$$

Share equilibrium is an example of a solution on $\mathcal{E}$ and we characterize it using properties of solutions. The axiomatizing properties are consistency, converse consistency, and one-person rationality. The axioms consistency and converse consistency both involve the appropriate restriction of economies to subsets of individuals.

Reduced economies Take as given a local public good economy $E = \langle N, D; (w_i)_{i\in N}; (u_i)_{i\in N}; f\rangle$. Let $R \subseteq D$ be a subset of decision makers, $R \neq \emptyset$,[8] and let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \mathbb{R}^N \times C(N)$. The *reduced economy* of E with respect to R and $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is the economy

$$E^{R,\mathbf{s}} = \langle N, R; (w_i)_{i\in N}; (u_i)_{i\in N}, f_{R,\mathbf{s}}\rangle,$$

in which the set of decision-makers is R and the cost function $f_{R,\mathbf{s}}$ is given by

$$f_{R,\mathbf{s}}(\overline{y}, J) = \left[\sum_{i\in J\cap R} s_i^{J,D}\right] f(\overline{y}, J)$$

for all $\overline{y} \in \mathbb{R}_+$ and $J \subseteq N$.

It is important to note that a reduced economy of an economy $E \in \mathcal{E}$ is itself an element of $\mathcal{E}$, so that we can apply a solution ϕ to it.

Consistency A solution ϕ on $\mathcal{E}$ is *consistent* (CONS) if it satisfies the following condition.

> If $E \in \mathcal{E}$, $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E)$, and $R \subset D(E)$, $R \neq \emptyset$,
> then $E^{R,\mathbf{s}} \in \mathcal{E}$ and $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E^{R,\mathbf{s}})$.

A solution is consistent if, once agreement on relative cost shares has been reached, the withdrawal of some decision makers from the decision-making process will not change the final outcome of the process.

Converse Consistency A solution ϕ on $\mathcal{E}$ is *converse consistent* (COCONS) if, for every $E \in \mathcal{E}$ with a population of at least two members ($|N(E)| \geq 2$),

[8] Think of R as standing for "reduced-economy decision-makers".

for every set of share indices $\mathbf{s} = (s_i)_{i \in N(E)} \in \mathbb{R}^{N(E)}$ and for every configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P}) \in C(N(E))$, the following condition is satisfied.

> If, for every $R \subset D(E)$ with $R \notin \{\emptyset, D(E)\}$ it holds that
> $E^{R,\mathbf{s}} \in \mathcal{E}$ and $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E^{R,\mathbf{s}}),$ then $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E)$.

Whereas consistency states that agreements that are acceptable to the group of all decision makers should be acceptable to all smaller groups as well, converse consistency states that if a pair of share indices and a configuration constitutes an acceptable solution to all proper subgroups of the set of decision makers, then the pair also constitutes an acceptable solution to the group as a whole.

One-Person Rationality: A solution ϕ on $\mathcal{E}$ satisfies *one-person rationality* (OPR) if for every economy $E = \langle N, \{i\}; (w_j)_{j \in N}; (u_j)_{j \in N}; f \rangle \in \mathcal{E}$, with one decision-maker i, it holds that

$$\begin{aligned}
&\phi(E) = \{(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \mid s_i > 0,\ f(y_{P(i)}, P(i)) + x_i = w_i \\
&\text{and } u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\overline{x}_i, \overline{y}, \overline{J}) \text{ for all } (\overline{x}_i, \overline{y}, \overline{J}) \\
&\text{satisfying } i \in \overline{J} \text{ and } f(\overline{y}, \overline{J}) + \overline{x}_i \leq w_i\}.
\end{aligned}$$

The one-person rationality axiom is a standard rationality axiom that dictates that in a one-decision-maker economy, the single decision maker maximizes his utility given his endowment of the private good and the cost of producing certain amounts of local public good in various jurisdictions.

We can now state our main result.

Theorem 1 The share equilibrium is the unique solution on $\mathcal{E}$ that satisfies one-person rationality, consistency, and converse consistency.

The proof of Theorem 1 is included in the Appendix

3.2 Discussion of the axioms

At the heart of our axiomatization is the notion of consistency, which relates a solution reached in an economy to solutions reached by subgroups of the population. To express this idea we needed to first consider how to reduce an economy to a population subgroup. The introduction of a set of decision makers in the definition of an economy enables us to create a reduced economy (itself satisfying our definition of a local public good economy) whose solution can be studied.

The idea behind the definition of the reduced economy is as follows. Suppose all decision makers agree on share indices $\mathbf{s}$ and configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$. This implies that they agree on cost-sharing schemes corresponding to the share indices $\mathbf{s}$, formation of local jurisdictions $\mathbf{P}$, and levels of local public good production for each of those jurisdictions. Then if, in addition to the people in $N \backslash D$, the people in $D \backslash R$ also withdraw from the decision-making process, the decision makers in R can reconsider the jurisdictions to be formed, the levels of local public good to be produced in those jurisdictions, and their relative shares of the residual cost of producing the local public good. When they reconsider, the members of R take into account that the members in $D \backslash R$ have agreed to their relative share indices and will pay their corresponding shares for the cost of local public good production.[9] Hence, when reconsidering the cost-sharing scheme, the decision makers in R face the residual cost

$$f_{R,\mathbf{s}}(\overline{y}, J) = \left[1 - \sum_{i \in J \cap (D \backslash R)} s_i^{J,D} \right] f(\overline{y}, J) = \left[\sum_{i \in J \cap R} s_i^{J,D} \right] f(\overline{y}, J)$$

[9]The reader will perceive that the set R in the reduced economy plays the same role that the set D plays in the economy that is reduced. Thus, the reduced economy has the same structure as the original economy and is itself a local public good economy for which solutions are defined.

for producing a level $\overline{y}$ of local public good in a jurisdiction J.

Notice that the people in $N \backslash R$ do not leave the economy, but only the decision-making process. In the reduced economy, share indices of the members of $N \backslash R$, which determine their cost shares of local public good production in jurisdictions, are taken as given.[10]

We illustrate reduced economies and the role of the share indices in consistency with the following example.

Example 2 *Decision makers cannot change cost shares of non-decision-makers.* Consider an economy with $N = D = \{1, 2, 3\}$. Suppose the members of D agree on share indices $s_1 = s_2 = 1$ and $s_3 = 2$. Then, in jurisdiction $J_1 = \{1, 2\}$ individual 1's cost share is $s_1^{J_1,D} = \frac{1}{2}$, and 1 would pay $\frac{1}{2} f(y, J_1)$ if a amount y of public good were produced in this jurisdiction. In jurisdiction $J_2 = \{1, 3\}$, individual 1's cost share is $s_1^{J_2,D} = \frac{1}{3}$ and 1 would pay $\frac{1}{3} f(y, J_2)$, and in jurisdiction $J_3 = N$, individual 1 would pay $\frac{1}{4} f(y, J_3)$.

Now, suppose that individual 1 leaves the decision-making process, agreeing to the share index $s_1 = 1$. In the reduced economy, the set of decision-makers is $R = \{2, 3\}$. The cost function $f_{R,\mathbf{s}}(\cdot, \cdot)$ for the reduced economy is different from the original cost function $f(\cdot, \cdot)$ only for jurisdictions that include decidion-maker 1. Specifically, $f_{R,\mathbf{s}}(y, J_1) = \frac{1}{2} f(y, J_1)$, $f_{R,\mathbf{s}}(y, J_2) = \frac{2}{3} f(y, J_2)$, and $f_{R,\mathbf{s}}(y, J_3) = \frac{3}{4} f(y, J_3)$. The remaining decision makers in R can now reconsider not only the jurisdictions that they want to form and the levels of local public good for those jurisdictions, but also their share indices.

[10] It is interesting to note that an issue to be resolved in defining consistency of a solution concept is to ask what elements are individual-specific and remain fixed for the members of $N \backslash R$. In their study of the Walrasian equilibrium for a private goods economy, van den Nouweland et al. (1996) keep fixed the allocations of those individuals who are not members of the reduced economy and axiomatize Walrasian allocations. In the current paper we are axiomatizing equilibrium cost shares and thus, analogously, the cost shares of those non-decision makers are fixed.

Suppose that 3 changes his share index to $\widetilde{s}_3 = 1$. This changes 3's relative standing vis-a-vis 2 and shifts costs from individual 3 to individual 2. For example, in jurisdiction $J_4 = \{2, 3\}$, decision maker 3 will now have to pay the cost $\frac{1}{2} f_{R,\mathbf{s}}(y, J_4) = \frac{1}{2} f(y, J_4)$, whereas before he changed his share index, he would have had to pay $\frac{2}{3} f(y, J_4)$. However, note that changing his share index does not allow decision maker 3 to put a larger share of the (cost) burden of producing local public good on the non-decision-maker 1, because 1's share of the cost in various jurisdictions was agreed upon before he left the decision-making process. For example, if decision maker 3 now wants to form jurisdiction J_2 with individual 1, then 3 will still have to shoulder the cost $f_{R,\mathbf{s}}(y, J_2) = \frac{2}{3} f(y, J_2)$. In jurisdiction J_3 with all three members of the population, individuals 2 and 3 pay $\frac{1}{2} f_{R,\mathbf{s}}(y, J_3) = \frac{3}{8} f(y, J_3)$ each. ■

Once we know how to reduce economies, and thus can formulate consistency and converse consistency, the axiomatization follows straightforwardly. Consistency and converse consistency link solutions in larger economies to those in smaller economies and vice versa, whereas the axiom one-person rationality considers solutions for economies with one decision-maker only. Therefore, the one-person rationality axiom is the one that anchors the solution concept and is arguably the most important axiom in the characterization because it sets share equilibrium apart from other solution concepts that satisfy consistency properties. Thus, it is important to note that one-person rationality is a rationality assumption much like those that prevail throughout economics.

4 Conclusions

There are many situations in which people are obliged to pay their share of the cost of a public good according to some sharing formula, even when they do not participate in the decision-making process. One example is given by condominium home owners association agreements in which owners of units within the condominium contract to pay shares, sometimes based on relative sizes of units owned, of costs of goods and services paid for by the association.[11] If the method of reaching agreements is consistent, condominium owners need not be concerned if they are unable to attend a meeting of the home owners association, because when reaching agreements in meetings, attendees will have no incentive to overturn agreements that may have been reached in informal discussions.[12]

To summarize, in this paper, we introduced the concept of a share equilibrium for local public good economies where relative share indices determine shares of costs of local public good provision for each level of public good provision and each jurisdiction. The relative cost shares, and hence the relative share indices of individuals who end up in the same jurisdiction, can be determined just by considering the differences between their endowment and their consumption of private good. Note, however, that the share indices

[11] One of the authors recently considered purchasing a townhouse in a complex where the commons and exterior of the units are maintained by a home owners association. The association covenants stipulate that during any meeting of the members at which at least 50% of all the votes in the association are represented, a majority vote is binding on all members. Among other things, the members can decide to take on new projects and to increase members' contributions to the home owners association to cover the projects' costs. The covenants also stipulate that any such contributions have to be paid within 30 days; otherwise financial penalties are impossed. The covenants governing the home owners association to which the other author belongs are similar.

[12] Then why do association members go to home owners meetings? This question was posed at a presentation of this research at the University of Warwick. Legal requirements (in the states of Oregon and Tennessee, at least) stipulate that a quorum must be present at the meeting to make binding decisions on the association members. We conjecture that it is this stipulation that motivates meetings in the evenings with refreshments.

also contain information on relative cost shares of individuals in *hypothetical* jurisdictions, that is, ones that do not appear in the equilibrium jurisdiction structure. In pure public good economies, where all individuals are by definition in the same jurisdiction, there is no need to consider hypothetical jurisdictions and the cost shares in such jurisdictions. Hence, for pure public good economies, it suffices to specify shares for each member of the population for the jurisdiction consisting of the total population. In this manner, we obtain the ratio equilibrium for pure public good economies as a special case of the share equilibrium. Our main result is the axiomatic characterization of the share equilibrium. This gives us insight into the properties of this equilibrium concept and shows that it is in the same spirit as the ratio equilibrium.

Since a share equilibrium is a new concept, there are many open questions. In further research we consider existence of share equilibrium and the core. We have results that show that the competitive nature of share equilibrium results in it generating configurations that are in the core, and also that share equilibrium exists for reasonably large classes of economies.We have results that show that the competitive nature of share equilibrium results in it generating configurations that are in the core, and also that share equilibrium exists for interesting classes of economies. We plan to investigate the relationship of share equilibrium to the partnered core and also noncooperative foundations of share equilibrium.[13] An important extension would be to consider economies with multiple public goods, as in Mas-Colell and Silvestre (1989).

[13]See Reny and Wooders (1996) for a definition of the partnered core.

5 Appendix

Proof of Theorem 1. For a clearer exposition, we break the proof of Theorem 1 up into 5 distinct steps.

Step 1. The share equilibrium on the family $\mathcal{E}$ of local public good economies is consistent.

Proof of Step 1. Let $E = \langle N, D; (w_i)_{i\in N}; (u_i)_{i\in N}; f\rangle \in \mathcal{E}$ be a local public good economy, let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E)$, and let $R \subseteq D$, $R \neq \emptyset$. Let $f_{R,\mathbf{s}}$ be the cost function of the reduced economy $E^{R,\mathbf{s}}$. Obviously, $E^{R,\mathbf{s}} \in \mathcal{E}$.

For all $i \in R$ and $J \subseteq N$ with $i \in J$ it holds that

$$\begin{aligned} s_i^{J,D} f(\overline{y}, J) &= \frac{s_i}{\sum_{j\in J\cap R} s_j} \frac{\sum_{j\in J\cap R} s_j}{\sum_{j\in J\cap D} s_j} f(\overline{y}, J) \\ &= \frac{s_i}{\sum_{j\in J\cap R} s_j} \left[\sum_{j\in J\cap R} s_j^{J,D}\right] f(\overline{y}, J) \\ &= s_i^{J,R} f_{R,\mathbf{s}}(\overline{y}, J) \end{aligned}$$

for all $\overline{y} \in \mathbb{R}_+$. We now derive

$$s_i^{P(i),R} f_{R,\mathbf{s}}(y_{P(i)}, P(i)) + x_i = s_i^{P(i),D} f(y_{P(i)}, P(i)) + x_i = w_i.$$

Let $(\overline{x}_i, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ be such that $i \in \overline{J}$ and $s_i^{\overline{J},R} f_{R,\mathbf{s}}(\overline{y}, \overline{J}) + \overline{x}_i \leq w_i$. Then $s_i^{\overline{J},D} f(\overline{y}, \overline{J}) + \overline{x}_i \leq w_i$ and, because $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E)$, we know that $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\overline{x}_i, \overline{y}, \overline{J})$. This proves that $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E^{R,\mathbf{s}})$.

Step 2. The share equilibrium on the family $\mathcal{E}$ of local public good economies satisfies converse consistency.

Proof of Step 2. Let $E = \langle N, D; (w_i)_{i\in N}; (u_i)_{i\in N}; f\rangle \in \mathcal{E}$ be a local public good economy with $|N| \geq 2$ and let share indices $\mathbf{s} = (s_i)_{i\in N} \in \mathbb{R}^N$ and configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P}) = ((x_i)_{i\in N}, (y_P)_{P\in\mathbf{P}}, \mathbf{P}) \in C(N)$ be such that, for every $R \subseteq D$ with $R \notin \{\emptyset, D\}$, it holds that $E^{R,\mathbf{s}} \in \mathcal{E}$ and $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E^{R,\mathbf{s}})$. Then $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E^{\{i\},\mathbf{s}})$ for each $i \in D$.

Let $i \in D$ and consider the reduced economy $E^{\{i\},\mathbf{s}}$. Using that $\mathbf{s}_i^{J,\{i\}} = 1$ for all $J \subseteq N$ with $i \in J$, and $(\mathbf{s},(\mathbf{x},\mathbf{y},\mathbf{P})) \in SE(E^{\{i\},\mathbf{s}})$, we find that $f_{\{i\},\mathbf{s}}(y_{P(i)}, P(i)) + x_i = w_i$ and $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\overline{x}_i, \overline{y}, \overline{J})$ for all $(\overline{x}_i, \overline{y}, \overline{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ satisfying $i \in \overline{J}$ and $f_{\{i\},\mathbf{s}}(\overline{y}, \overline{J}) + \overline{x}_i = w_i$. Using $f_{\{i\},\mathbf{s}}(\overline{y}, J) = s_i^{J,D}\ f(\overline{y}, J)$ for all $\overline{y} \in \mathbb{R}_+$ and $J \subseteq N$ with $i \in J$, we derive that conditions 1 and 2 of share equilibrium are satisfied when applied to $(\mathbf{s},(\mathbf{x},\mathbf{y},\mathbf{P}))$ and decision maker i in the economy E. Since we derived this for an arbitrary decision maker $i \in D$, we can conclude that $(\mathbf{s},(\mathbf{x},\mathbf{y},\mathbf{P})) \in SE(E)$.

Step 3. The share equilibrium on the family $\mathcal{E}$ of local public good economies satisfies one-person rationality.

Proof of Step 3. Let $E = \langle N, \{i\}; (w_j)_{j\in N}; (u_j)_{j\in N}; f\rangle \in \mathcal{E}$ be a local public good economy with one decision maker, i. Then, in every jurisdiction J containing i, individual i pays the share $s_i^{J,\{i\}} = 1$ of the cost $f(y, J)$ of producing any amount of local public good y. Thus, the conditions for share equilibrium reduce to those in the axiom OPR.

Step 4. Let ϕ and ψ be two solutions on $\mathcal{E}$ that both satisfy one-person rationality. If ϕ is consistent and ψ is converse consistent, then it holds that $\phi(E) \subseteq \psi(E)$ for all $E \in \mathcal{E}$.

Proof of Step 4. This step is generic for axiomatizations using consistency and converse consistency. For example, it appears as Lemma 4 in van den Nouweland, Tijs, and Wooders (2002). We refer the reader to that paper for a proof.

Step 5. The share equilibrium is the unique solution on $\mathcal{E}$ that satisfies one-person rationality, consistency, and converse consistency.

Proof of Step 5. In Steps 1, 2, and 3 we proved that the share equilibrium satisfies CONS, COCONS, and OPR. To prove uniqueness, assume that ϕ is a solution on $\mathcal{E}$ that also satisfies the three foregoing axioms. Let $E \in \mathcal{E}$ be arbitrary. Then, Step 4 shows that $\phi(E) \subseteq SE(E)$ by CONS of ϕ and

COCONS of the share equilibrium, and that $SE(E) \subseteq \phi(E)$ by CONS of the share equilibrium and COCONS of ϕ. Hence, $\phi(E) = SE(E)$. ■

Careful examination of the proof of Theorem 1 reveals that consistency, converse consistency, and one-person rationality also characterize share equilibrium as a solution on a set of local public good economies that consist of a specific economy $E \in \mathcal{E}$ and all its reduced economies with respect to a fixed set of share indices $\mathbf{s} \in \mathbb{R}^{N(E)}$, i.e., the class of local public good economies $\{E^{R,\mathbf{s}} \mid R \subseteq D(E)\}$.

We conclude this appendix with the remark that the three axioms used to characterize the share equilibrium in Theorem 1 are logically independent. Examples demonstrating this are available from the authors upon request.

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