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Two Periodic Solutions of Nonlinear Systems with Feedback Control

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Abstract

In this paper, by the well-known Deimling fixed point theorem in a cone, we consider the following nonlinear functional differential system with feedback control

$$\begin{cases} \dfrac{dx}{dt} = -r(t)x(t) + \lambda F(t, x(t-\tau(t)), u(t-\delta(t))), \\ \dfrac{du}{dt} = -h(t)u(t) + g(t)x(t-\sigma(t)), \end{cases}$$

where λ is a positive parameter. We obtain the results on the existence and multiplicity of positive periodic solutions.

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1. Introduction

The existence problem of periodic solutions has been an interesting problem for a long time. We can find many pretty results on the existence problem by using the fixed point

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theorems (see [1, 2, 6]). Wang [8] considered the following periodic equation

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(x(t - \tau(t))), \quad \lambda > 0. \tag{1.1}$$

They showed that the number of positive periodic solutions of (1.1) could be determined by the asymptotic behaviors of the quotient $\dfrac{f(u)}{u}$ at zero and infinity based on the well known fixed point theorem (Deimling [3], Guo and Laksmikantham [4] and Krasnoselskii [5]).

Motivated by the above excellent works, we study the following nonlinear nonautonomous functional differential system with feedback control:

$$\begin{cases} \dfrac{dx}{dt} = -r(t)x(t) + \lambda F(t, x(t - \tau(t)), u(t - \delta(t))), \\ \dfrac{du}{dt} = -h(t)u(t) + g(t)x(t - \sigma(t)), \end{cases} \tag{1.2}$$

where $\tau(t),\ \delta(t),\ \sigma(t) \in C(R, R_+),\ r(t),\ h(t),\ g(t) \in C(R, R_+)$, all of the above functions are ω-periodic functions and $\omega > 0$ is a constant.

When $\lambda = 1$, system (1.2) was studied by Liu and Li [7]. In [8], the author can directly impose conditions on $f(u)$ and obtain the results because the function f is only about u in (1.1). The techniques in [8] are difficulty to be used here. In this paper, we assume that

$$b_1(t)f_1(x)u \le F(t, x, u) \le b_2(t)f_2(x)u,$$

where $b_1(t), b_2(t)$, $f_1(x)$ and $f_2(x)$ are positive continuous functions and satisfy

$$b_i(t) = b_i(t + \omega), \quad i = 1, 2.$$

Then by considering the quotient $\dfrac{f_i(x)}{x}$ $(i = 1, 2)$, results on the existence and multiplicity of positive periodic solutions of (1.2) are established by the well-known fixed point theorem applied in [8] and [9]. Our conditions on F and sufficient conditions on the existence of periodic solutions are explicit and easy to be verified.

The rest of the paper consists of three sections. In Section 2, we give some notations and our main results. Section 3 contains preparations for the proofs of theorems. In the fourth section, the proofs of the main results are given.

2. Assumptions and main results

In this paper, we always assume that

(A1) there exist positive functions $b_1(t), b_2(t), f_1(x)$ and $f_2(x)$ such that

$$b_1(t)f_1(x)u \le F(t, x, u) \le b_2(t)f_2(x)u,$$

where $b_i(t) = b_i(t + \omega)$ and $f_i(u)$ $(i = 1, 2)$ are continuous;

(A2) for all $s \in R$, $F(s+\omega, x(s+\omega-\tau(s+\omega)), u(s+\omega-\delta(s+\omega))) = F(s, x(s-\tau(s)), u(s-\delta(s)))$;

(A3) $r(t)$, $h(t)$ and $g(t)$ are nonnegative ω-periodic functions and $\int_0^\omega r(s)ds > 0$, $\int_0^\omega h(s)ds > 0$, $\int_0^\omega g(s)ds > 0$;

(A4) $\tau(t)$, $\delta(t)$ and $\sigma(t)$ are positive ω-periodic functions.

We set

$$f_{20} = \lim_{u\to 0^+} \frac{f_2(u)}{u}, \quad f_{1\infty} = \lim_{u\to\infty} \frac{f_1(u)}{u}.$$

We also define

$$M(r) = \max_{u\in[0,r]}\{f_2(u)\}, \quad m(r) = \min\{f_1(u) : \frac{p}{q}r \le u \le r\};$$

$$n = \frac{1}{e^{\int_0^\omega h(\theta)d\theta} - 1}, \quad m = \frac{e^{\int_0^\omega h(\theta)d\theta}}{e^{\int_0^\omega h(\theta)d\theta} - 1};$$

$$p = \frac{1}{e^{\int_0^\omega r(\theta)d\theta} - 1}, \quad q = \frac{e^{\int_0^\omega r(\theta)d\theta}}{e^{\int_0^\omega r(\theta)d\theta} - 1};$$

$$\lambda_0 = \frac{q}{m(1)p^2 n \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta};$$

$$\lambda_1 = \frac{1}{M(1)qm \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta}.$$

Our main results are as follows.

Theorem 2.1: Assume (A1)–(A4) hold, then

1. if one of $f_{20} = 0$ and $\lim_{x\to\infty} f_2(x) = 0$ holds, then (1.2) has at least one positive ω-periodic solution for $0 < \lambda_0 < \lambda$;
   if both $f_{20} = 0$ and $\lim_{x\to\infty} f_2(x) = 0$ hold, then (1.2) has at least two positive ω-periodic solutions for $0 < \lambda_0 < \lambda$.
2. if one of $f_{1\infty} = \infty$ and $\lim_{x\to 0^+} f_1(x) = \infty$ holds, then (1.2) has at least one positive ω-periodic solution for $\lambda_1 > \lambda > 0$; if both $f_{1\infty} = \infty$ and $\lim_{x\to 0^+} f_1(x) = \infty$ hold, then (1.2) has at least two positive ω-periodic solutions for $\lambda_1 > \lambda > 0$.

Theorem 2.2: Assume (A1)$-$(A4) hold, $f_{1\infty} > 0$ and $f_{20} < \infty$. If $f_{20} < f_{1\infty}$ and $\frac{q^2}{p^3 n \int_0^\omega b_1(s)ds} < \frac{1}{qm \int_0^\omega b_2(s)ds}$, then (1.2) has at least one positive ω-periodic solution for

$$\frac{q^2}{p^3 n \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta f_{1\infty}} < \lambda < \frac{1}{qm \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta f_{20}}.$$

3. Preparations

We first state the well-known fixed point theorem to be employed in the proofs of the main results.

Lemma 3.1 (Deimling [3], Guo and Lakshmikantham [4] and Krasnoselskii [5]): Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|u\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|u\| = r\}$.

1. If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.
2. If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

In order to employ Lemma 3.1 to verify our results, we first transform (1.2) into one equation. Integrating the second equation in (1.2) from t to $t + \omega$, we get

$$u(t) = \int_t^{t+\omega} k(t,s)g(s)x(s - \sigma(s))ds =: (\Phi x)(t), \tag{3.1}$$

where

$$k(t,s) = \frac{e^{\int_t^s h(\theta)d\theta}}{e^{\int_0^\omega h(\theta)d\theta} - 1}.$$

Note that $u(t+\omega) = u(t)$ and $k(t+\omega, s+\omega) = k(t,s)$ and

$$n \leq k(t,s) \leq m, \quad s \in [t, t+\omega].$$

Combining (3.1) with (1.2), we get the following equation

$$x'(t) = -r(t)x(t) + \lambda F(t, x(t - \tau(t)), (\Phi x)(t - \delta(t))). \tag{3.2}$$

Obviously, the existence problem of ω-periodic solution of (1.2) is equivalent to that of (3.2). Therefore, it is sufficient to consider the existence and multiplicity of positive ω-periodic solutions of the equation (3.2).

We set

$$E = \{u(t) \in C(R, R) : u(t+\omega) = u(t)\}.$$

Define $\|u\| = \max\limits_{t \in [0,\omega]} |u(t)|$ in E. Let K and K_r in E be given by

$$K = \{u \in E : u(t) \geq \frac{p}{q}\|u\|\}, \quad K_r = \{u \in K : \|u\| < r\}, \quad r > 0.$$

It is easy to prove that $(E, \|\cdot\|)$ is a Banach space and K is a cone in E. Note that $\partial K_r = \{u \in K : \|u\| = r\}$.

Let the map $T_\lambda : K \to E$ be defined by

$$T_\lambda u(t) = \lambda \int_t^{t+\omega} G(t,s)F(s, u(s - \tau(s)), (\Phi u)(s - \delta(s)))ds,$$

where

$$G(t,s) = \frac{e^{\int_t^s r(\theta)d\theta}}{e^{\int_0^\omega r(\theta)d\theta} - 1}.$$

It is easy to see that $G(t+\omega, s+\omega) = G(t,s)$ for all $(t,s) \in R^2$ and

$$p \le G(t,s) = \frac{e^{\int_t^s r(\theta)d\theta}}{e^{\int_0^\omega r(\theta)d\theta} - 1} \le q, \quad s \in [t, t+\omega].$$

Next, we show that T_λ defined above satisfies the conditions in Lemma 3.1 and give our priori estimations.

Lemma 3.2: Assume (A1)–(A4) hold. Then $T_\lambda(K) \subset K$ and $T_\lambda : K \to K$ is completely continuous.

Proof: For $u \in K$, we have

$$\begin{aligned}
&T_\lambda u(t+\omega)\\
&\quad= \lambda \int_{t+\omega}^{t+2\omega} G(t+\omega, s)F(s, u(s-\tau(s)), (\Phi u)(s-\delta(s)))ds\\
&\quad= \int_t^{t+\omega} \lambda G(t+\omega, \theta+\omega)F(\theta+\omega, u(\theta+\omega-\tau(\theta+\omega)),\\
&\qquad\quad (\Phi u)(\theta+\omega-\delta(\theta+\omega)))d\theta\\
&\quad= \lambda \int_t^{t+\omega} G(t,\theta)F(\theta, u(\theta-\tau(\theta)), (\Phi u)(\theta-\delta(\theta)))d\theta = T_\lambda u(t).
\end{aligned}$$

For $u \in K$ and $t \in [0,\omega]$, we can show that

$$\begin{aligned}
T_\lambda u(t) &\ge \lambda p \int_0^\omega F(s, u(s-\tau(s)), (\Phi u)(s-\delta(s)))ds\\
&\ge \frac{\lambda p}{q} q \int_0^\omega F(s, u(s-\tau(s)), (\Phi u)(s-\delta(s)))ds\\
&\ge \frac{\lambda p}{q} \max_{t\in[0,\omega]} \{\int_t^{t+\omega} G(t,s)F(s, u(s-\tau(s)), (\Phi u)(s-\delta(s)))ds\}\\
&= \frac{p}{q}\|T_\lambda u\|.
\end{aligned}$$

So $T_\lambda(K) \subset K$. By the assumptions (A1)$-$(A4), it is easy to show that T_λ is completely continuous by Arzela-Ascoli theorem. ■

Lemma 3.3: Assume (A1)–(A4) hold. The existence of positive ω-periodic solutions of (3.2) is equivalent to the existence of the positive fixed point of T_λ in K.

This lemma is obvious and its proof is omitted.

Lemma 3.4: Assume (A1)$-$(A4) hold and let $\eta > 0$. If $f_1(u(t)) \geq \eta$ for any $u \in K$ and for any $t \in [0, \omega]$, then

$$\|T_\lambda u\| \geq \frac{\lambda p^2 n\eta}{q} \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta \|u\|.$$

Proof: From the proof of Lemma 3.2, we have

$$T_\lambda u(t) \geq \lambda p \int_0^\omega F(s, u(s-\tau(s)), (\Phi u)(s-\delta(s)))ds.$$

Then for $u \in K$ and $t \in [0, \omega]$, we have

$$\begin{aligned} T_\lambda u(t) &\geq \lambda p \int_0^\omega \{b_1(s) f_1(u(s-\tau(s))) \int_{s-\delta(s)}^{s-\delta(s)+\omega} k(s,\theta) g(\theta) u(\theta - \sigma(\theta)) d\theta\} ds \\ &\geq \lambda pn \int_0^\omega \{b_1(s) f_1(u(s-\tau(s))) \int_0^\omega g(\theta) u(\theta-\sigma(\theta))d\theta\}ds \\ &\geq \lambda pn\eta \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)u(\theta-\delta(\theta))d\theta \\ &\geq \frac{\lambda p^2 n\eta}{q} \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta\|u\|, \end{aligned}$$

since $f_1(u(t)) \geq \eta$ for $u \in K$. ■

Lemma 3.5: Assume (A1)–(A4) hold and let $\eta > 0$. If $f_1(u(t)) \geq \eta u(t)$ for any $u \in K$ and for any $t \in [0, \omega]$, then

$$\|T_\lambda u\| \geq \frac{\lambda p^3 n\eta}{q^2} \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta \|u\|^2.$$

Proof: From the proof of Lemma 3.4, for $u \in K$ and $t \in [0, \omega]$, we have

$$\begin{aligned} T_\lambda u(t) &\geq \lambda pn \int_0^\omega \{b_1(s) f_1(u(s-\tau(s))) \int_0^\omega g(\theta) u(\theta-\sigma(\theta))d\theta\}ds \\ &\geq \lambda pn\eta \int_0^\omega b_1(s)u(s-\tau(s))ds \int_0^\omega g(\theta)u(\theta-\delta(\theta))d\theta \\ &\geq \frac{\lambda p^3 n\eta}{q^2} \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta\|u\|^2, \end{aligned}$$

since $f_1(u(t)) \geq \eta u(t)$ for $u \in K$. ■

Lemma 3.6: Assume (A1)–(A4) hold and $\varepsilon > 0$. Let $r > 0$. If $f_2(u(t)) \leq \varepsilon$ for $u \in \partial K_r$ and for any $t \in [0, \omega]$, then

$$\|T_\lambda u\| \leq \lambda q m \varepsilon \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta \|u\|.$$

Proof: From the definition of T_λ and the assumptions (A1)–(A4), for $u \in \partial K_r$, we have

$$\begin{aligned}
\|T_\lambda u\| &\leq \lambda q \int_t^{t+\omega} F(s, u(s-\tau(s)), (\Phi u)(s-\delta(s)))ds \\
&= \lambda q \int_0^{\omega} F(s, u(s-\tau(s)), (\Phi u)(s-\delta(s)))ds \\
&\leq \lambda q \int_0^{\omega} \{b_2(s) f_2(u(s-\tau(s))) \int_{s-\delta(s)}^{s-\delta(s)+\omega} k(s,\theta) g(\theta) u(\theta-\sigma(\theta))d\theta\}ds \\
&\leq \lambda q m \varepsilon \int_0^{\omega} \{b_2(s) \int_0^{\omega} g(\theta) u(\theta-\sigma(\theta))d\theta\}ds \\
&\leq \lambda q m \varepsilon \int_0^{\omega} b_2(s)ds \int_0^{\omega} g(\theta)d\theta \|u\|.
\end{aligned}$$

Then $\|T_\lambda u\| \leq \lambda q m \varepsilon \int_0^{\omega} b_2(s)ds \int_0^{\omega} g(\theta)d\theta \|u\|$ for $u \in \partial K_r$ and $t \in [0, \omega]$. ■

Lemma 3.7: Assume (A1)–(A4) hold and $\varepsilon > 0$. Let $r > 0$. If $f_2(u(t)) \leq \varepsilon u(t)$ for $u \in \partial K_r$ and for any $t \in [0, \omega]$, then

$$\|T_\lambda u\| \leq \lambda q m \varepsilon \int_0^{\omega} b_2(s)ds \int_0^{\omega} g(\theta)d\theta \|u\|^2.$$

Proof: From Lemma 3.6, for $u \in \partial K_r$, we have

$$\begin{aligned}
\|T_\lambda u\| &\leq \lambda q m \int_0^{\omega} \{b_2(s) f_2(u(s-\tau(s))) \int_0^{\omega} g(\theta) u(\theta-\sigma(\theta))d\theta\}ds \\
&\leq \lambda q m \varepsilon \int_0^{\omega} \{b_2(s) u(s-\tau(s)) \int_0^{\omega} g(\theta) u(\theta-\sigma(\theta))d\theta\}ds \\
&\leq \lambda q m \varepsilon \int_0^{\omega} b_2(s)ds \int_0^{\omega} g(\theta)d\theta \|u\|^2.
\end{aligned}$$

Then $\|T_\lambda u\| \leq \lambda q m \varepsilon \int_0^{\omega} b_2(s)ds \int_0^{\omega} g(\theta)d\theta \|u\|^2$ for $u \in \partial K_r$ and $t \in [0, \omega]$. ■

Lemma 3.8: Assume (A1)–(A4) hold and let $r > 0$, then for $u \in \partial K_r$

$$\|T_\lambda u\| \geq \lambda \frac{p^2}{q} n m(r) \int_0^{\omega} b_1(s)ds \int_0^{\omega} g(\theta)d\theta \|u\|.$$

Proof: For $u \in \partial K_r$, we have $f_1(u(t)) \geq m(r)$ for $t \in [0, \omega]$. Therefore

$$T_\lambda u(t) \geq \lambda p n \int_0^{\omega} b_1(s) f_1(u(s-\tau(s)))ds \int_0^{\omega} g(\theta) u(\theta-\sigma(\theta))d\theta$$

$$\geq \lambda \frac{p^2}{q} nm(r) \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta \|u\|.$$

■

Lemma 3.9: Assume (A1)–(A4) hold and let $r > 0$, then for $u \in \partial K_r$

$$\|T_\lambda u\| \leq \lambda qmM(r) \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta \|u\|.$$

Proof: For $u \in \partial K_r$, we have $f_2(u(t)) \leq M(r)$ for $t \in [0, \omega]$. Then, from the proof of Lemma 3.6, we get

$$\begin{aligned} T_\lambda u(t) &\leq \lambda qm \int_0^\omega b_2(s) f_2(u(s-\tau(s)))ds \int_0^\omega g(\theta)u(\theta - \sigma(\theta))d\theta \\ &\leq \lambda qmM(r) \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta \|u\|. \end{aligned}$$

■

4. Proofs of main results

4.1. Proof of Theorem 2.1

(i) Take $r_1 = 1$. By Lemma 3.8, for $\lambda > \lambda_0$, we have

$$\|T_\lambda u\| \geq \lambda \frac{p^2}{q} nm(r_1) \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta \|u\| > \|u\|, \ \ u \in \partial K_{r_1}.$$

If $f_{20} = 0$, then for any $\varepsilon > 0$, we can find a positive constant r_2 with $0 < r_2 < r_1$ such that $f_2(u) \leq \varepsilon u$ for $0 \leq u \leq r_2$. Take ε such that

$$\lambda qm\varepsilon \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta < 1.$$

Thus by Lemma 3.7, for $u \in \partial K_{r_2}$ and $t \in [0, \omega]$, we have

$$\begin{aligned} \|T_\lambda u\| &\leq \lambda qm\varepsilon \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta \|u\|^2 \\ &= \lambda qm\varepsilon r_2 \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta \|u\| \\ &< \lambda qm\varepsilon \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta \|u\| < \|u\|. \end{aligned}$$

It follows from Lemma 3.1 that

$$i(T_\lambda, K_{r_1}, K) = 0, \ \ i(T_\lambda, K_{r_2}, K) = 1, \ \ i(T_\lambda, K_{r_1} \backslash \overline{K}_{r_2}, K) = -1.$$

Then T_λ has at least a positive fixed point in $K_{r_1}\backslash\overline{K}_{r_2}$, which is a positive ω-periodic solution of (3.2).

If $\lim\limits_{x\to\infty} f_2(x)=0$, then for any $\varepsilon>0$, there must be $H>0$ such that $f_2(u)\le\varepsilon$ for $u\ge H$ and $t\in[0,\omega]$. We take ε such that

$$\lambda qm\varepsilon\int_0^\omega b_2(s)ds\int_0^\omega g(\theta)d\theta<1.$$

Take $r_3=\max\left\{3r_1,\dfrac{Hq}{p}\right\}$. Then it follows that $u(t)\ge\dfrac{p}{q}\|u\|\ge H$ for $u\in\partial K_{r_3}$ and $t\in[0,\omega]$. Thus in view of Lemma 3.6, we have

$$\|T_\lambda u\|\le\lambda qm\varepsilon\int_0^\omega b_2(s)ds\int_0^\omega g(\theta)d\theta\|u\|<\|u\|,\ \ u\in\partial K_{r_3},\ \ t\in[0,\omega].$$

Lemma 3.1 implies

$$i(T_\lambda,K_{r_1},K)=0,\ \ i(T_\lambda,K_{r_3},K)=1,\ \ i(T_\lambda,K_{r_3}\backslash\overline{K}_{r_1},K)=1.$$

Then T_λ has at least a positive fixed point in $K_{r_3}\backslash\overline{K}_{r_1}$, which is also a positive ω-periodic solution of (3.2).

If $f_{20}=0$ and $\lim\limits_{x\to\infty} f_2(x)=0$, from the above argument, it is easy to conclude that T_λ has at least two positive fixed points in $K_{r_1}\backslash\overline{K}_{r_2}$ and $K_{r_3}\backslash\overline{K}_{r_1}$, that is, (3.2) has at least two positive ω-periodic solutions x_1 and x_2 in $K_{r_1}\backslash\overline{K}_{r_2}$ and $K_{r_3}\backslash\overline{K}_{r_1}$ for $\lambda>\lambda_0$, which satisfy

$$r_2<\|x_1\|<r_1<\|x_2\|<r_3.$$

(ii) We take $r_1=1$. Then by Lemma 3.9, for $0<\lambda<\lambda_1$, we have

$$\|T_\lambda u\|\le\lambda qm\int_0^\omega b_2(s)ds\int_0^\omega g(\theta)d\theta M(r_1)\|u\|<\|u\|,\ \ u\in\partial K_{r_1}.$$

If $\lim\limits_{x\to0^+} f_1(x)=\infty$, then for any $\eta>0$ there must be a positive number $r_2<r_1$ such that $f_1(u)\ge\eta$ for $u\in[0,r_2]$ and $t\in[0,\omega]$. We choose η such that

$$\frac{\lambda p^2n\eta}{q}\int_0^\omega b_1(s)ds\int_0^\omega g(\theta)d\theta>1.$$

Therefore by Lemma 3.4, we get

$$\|T_\lambda u\|\ge\frac{\lambda p^2n\eta}{q}\int_0^\omega b_1(s)ds\int_0^\omega g(\theta)d\theta\|u\|>\|u\|,\ \ u\in\partial K_{r_2}.$$

Then it follows from Lemma 3.1 that

$$i(T_\lambda,K_{r_1},K)=1,\ \ i(T_\lambda,K_{r_2},K)=0$$

and $i(T_\lambda, K_{r_1}\backslash\overline{K}_{r_2}, K) = 1$. So T_λ has at least one positive fixed point in $K_{r_1}\backslash\overline{K}_{r_2}$ for $0 < \lambda < \lambda_1$.

If $f_{1\infty} = \infty$, then there is $H > 0$ such that $f_1(u) \geq \eta u$ for $u \in [H, \infty)$, where $\eta > 0$ satisfies

$$\frac{\lambda p^3 n\eta}{q^2}\int_0^\omega b_1(s)ds\int_0^\omega g(\theta)d\theta > 1.$$

Take $r_3 = \max\{3r_1, \dfrac{Hq}{p}\}$. Then for $u \in \partial K_{r_3}$ and $t \in [0,\omega]$, there must be $u(t) \geq \dfrac{p}{q}\|u\| \geq H$. Again, it follows from Lemma 3.5 that

$$\begin{aligned}\|T_\lambda u\| &\geq \frac{\lambda p^3 n\eta}{q^2}\int_0^\omega b_1(s)ds\int_0^\omega g(\theta)d\theta\|u\|^2\\ &= \frac{\lambda p^3 n\eta}{q^2}\int_0^\omega b_1(s)ds\int_0^\omega g(\theta)d\theta\|u\|r_3 > \|u\|,\ \ u \in \partial K_{r_3}.\end{aligned}$$

Lemma 3.1 tells that

$$i(T_\lambda, K_{r_1}, K) = 1,\ \ i(T_\lambda, K_{r_3}, K) = 0.$$

Thus $i(T_\lambda, K_{r_3}\backslash\overline{K}_{r_1}, K) = -1$. So T_λ has at least one positive fixed point in $K_{r_3}\backslash\overline{K}_{r_1}$ for $0 < \lambda < \lambda_1$.

If $\lim\limits_{x\to 0^+} f_1(x) = f_{1\infty} = \infty$, from the above discussion, we can see that T_λ has at least two fixed points in $K_{r_1}\backslash\overline{K}_{r_2}$ and $K_{r_3}\backslash\overline{K}_{r_1}$, that is, (3.2) has at least two positive ω-periodic solutions x_1 and x_2 in $K_{r_1}\backslash\overline{K}_{r_2}$ and $K_{r_3}\backslash\overline{K}_{r_1}$ for $0 < \lambda < \lambda_1$, which satisfy

$$r_2 < \|x_1\| < r_1 < \|x_2\| < r_3.$$

4.2. Proof of Theorem 2.2

If $f_{1\infty} > f_{20}$ and $\dfrac{q^2}{p^3 n\int_0^\omega b_1(s)ds} < \dfrac{1}{qm\int_0^\omega b_2(s)ds}$, then for

$$\frac{q^2}{p^3 n\int_0^\omega b_1(s)ds\int_0^\omega g(\theta)d\theta f_{1\infty}} < \lambda < \frac{1}{qm\int_0^\omega b_2(s)ds\int_0^\omega g(\theta)d\theta f_{20}},$$

there must exist a positive number $\varepsilon \in (0, f_{20})$ such that

$$\frac{q^2}{p^3 n\int_0^\omega b_1(s)ds\int_0^\omega g(\theta)d\theta(f_{1\infty}-\varepsilon)} < \lambda < \frac{1}{qm\int_0^\omega b_2(s)ds\int_0^\omega g(\theta)d\theta(f_{20}+\varepsilon)}.$$

If $f_{20} < \infty$, then there is a number $1 > r_1 > 0$ such that $f_2(u) \leq (f_{20}+\varepsilon)u$ for $u \in [0, r_1]$. Thus $f_2(u) \leq (f_{20}+\varepsilon)u$ for $u \in \partial K_{r_1}$. By Lemma 3.7, we have

$$\|T_\lambda u\| \leq \lambda qm(f_{20}+\varepsilon)\int_0^\omega b_2(s)ds\int_0^\omega g(\theta)d\theta\|u\|^2$$

$$= \lambda qm(f_{20} + \varepsilon) \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta \|u\| r_1 < \|u\|, \ \ u \in \partial K_{r_1}.$$

If $f_{1\infty} > 0$, we can find a number $H > 1 > r_1$ such that $f_1(u) \geq (f_{1\infty} - \varepsilon)u$ for $u \in [H, \infty)$. Let $r_2 = \max\{3r_1, \dfrac{Hq}{p}\}$. It follows that $u(t) \geq \dfrac{p}{q}\|u\| \geq H$ for $u \in \partial K_{r_2}$. Thus $f_1(u) \geq (f_{1\infty} - \varepsilon)u$ for $u \in \partial K_{r_2}$ and $t \in [0, \omega]$. By Lemma 3.5, we get

$$\|T_\lambda u\| \geq \frac{\lambda p^3 n(f_{1\infty} - \varepsilon)}{q^2} \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta \|u\|^2$$

$$= \frac{\lambda p^3 n(f_{1\infty} - \varepsilon)}{q^2} \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta \|u\| r_2 > \|u\|, \ \ u \in \partial K_{r_2}.$$

From Lemma 3.1, we can obtain

$$i(T_\lambda, K_{r_1}, K) = 1, \ \ i(T_\lambda, K_{r_2}, K) = 0.$$

So $i(T_\lambda, K_{r_2} \backslash \overline{K}_{r_1}, K) = -1$ and T_λ has at least one fixed point in $K_{r_2} \backslash \overline{K}_{r_1}$ for

$$\frac{q^2}{p^3 n \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta f_{1\infty}} < \lambda < \frac{1}{qm \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta f_{20}}.$$

That is, (3.2) has at least one positive ω-periodic solution in $K_{r_2} \backslash \overline{K}_{r_1}$ for

$$\frac{q^2}{p^3 n \int_0^\omega b_1(s)ds \int_0^\omega g(\theta)d\theta f_{1\infty}} < \lambda < \frac{1}{qm \int_0^\omega b_2(s)ds \int_0^\omega g(\theta)d\theta f_{20}}.$$

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