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The Onset of Stationary and Oscillatory Convection in a Horizontal Porous Layer Saturated with Viscoelastic Liquid Heated and Soluted From Below: Effect of Anisotropy

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Abstract

The onset of double diffusive stationary and oscillatory convection in a viscoelastic Oldroyd type fluid saturated in an anisotropic porous layer heated and soluted from below is studied. The flow is governed by the extended Darcy model for Oldroyd fluid. Stability analysis based on the method of perturbations of infinitesimal amplitude is performed using the normal mode technique. The analysis examines the effect of the Darcy Rayleigh number, the solutal Darcy the Rayleigh number, the relaxation time, the retardation time and the Lewis number. Important conclusions include the destabilizing effect of the relaxation time, the Darcy Rayleigh number and the Lewis number and the stabilizing effect of the solutal Darcy Rayleigh number, the retardation time and anisotropy parameter. Some of the results are generalization of the previous findings for isotropic porous medium.

Key words: Viscoelastic Liquid, Porous Layer, Anisotropy, Stationary/Oscillatory Convection

MSC 2010 No.: 74E10, 76A05, 76A10, 76E06

1. Introduction

Early studies based on the Rayleigh-Benard convection through porous media are mainly concerned with the problems of convective instability in a Newtonian fluid. The growing volume of work devoted to the Rayleigh-Benard convection in case of fluid saturated porous media has been well documented by Ingham and Pop (1998), Nield and Bejan (1999), Vafai (2000) and Vadasz (2008). The study of viscoelastic fluid flow in porous media is of considerable interest in various engineering fields such as enhanced oil recovery, paper and textile coating, composite manufacturing process and bioengineering on the one hand, the high viscosity in viscoelastic fluids reduces the chances of occurrence of instability, its elastic nature, on the other hand, increases the chances of oscillatory convection. This phenomenon in viscoelastic fluid apart from its rheological importance makes the study of the Rayleigh-Benard convection for viscoelastic fluids interesting and challenging to the researchers.

However, compared to the well documented works on theoretical and experimental investigations of the Rayleigh Benard convection of Newtonian fluids in porous media, only limited work on viscoelastic fluid flow in porous media has appeared till date. This may be due to the difficulties in solving analytically as well as numerically the complex nature of viscoelastic fluids and the nonexistence of simple models for their description and formulation.

Recently, some interesting studies related to a viscoelastic fluid saturated porous medium based on the Rayleigh Benard convection have been reported [Rudraiah et al.(1989); Kim et al. (2003); Yoon et al.(2004); Malashetty et al.(2006); Malashetty and Swamy (2007); Tan and Masuka (2007); Niu et al. (2010)]. Two diffusing components heat and solute work as two stratifying agents and if the gradients of these agents, having different diffusivities are simultaneously present in a fluid layer, a variety of interesting convective phenomena can occur which are not possible in a single component fluid. Sea water and atmosphere are the examples of double-diffusive convection. Thermosolutal convection problems in fluids in porous media arise in oceanography, limnology, geophysics, ground water hydrology, soil sciences and astrophysics.

Nield (1968) was the first to investigate double diffusive convection in a porous medium using linear stability theory for various thermal and solutal boundary conditions. Rudraiah et al. (1982) used non-linear stability theory to investigate the double diffusive convection in a horizontal porous layer. Nield et al. (1993) examined the effect of inclined temperature and solutal gradients and showed that both the thermal and solutal Rayleigh numbers contribute significantly to the onset of convective instability. Rachana et al. (1995), examined numerically the hydromagnetic stability of an unbounded electrically conducting couple stress binary fluid mixture having temperature and concentration gradients. They plotted the neutral stability curves and found the range of the wave numbers having non-oscillatory unstable, oscillatory unstable and stable modes. Goel and Agrawal (1999) examined double diffusive convection in couple stress binary fluid mixture and showed that though rotation and magnetic field both inhibit the onset of instability; they do not reinforce each other when acting jointly.

Sharma and Rana (2002) examined the stationary convection in the case of thermosolutal instability of Walters (model B) a visco-elastic rotating fluid permeated with suspended particles and variable gravity field in a porous medium and showed that the solute gradient and rotation have stabilizing effects while the suspended particles are found to have a destabilizing effect on the system and the medium permeability has the dual effect on the system under certain conditions. The stability of a Maxwell fluid in the Bénard problem based on the Darcy Maxwell model for a double diffusive mixture in a porous medium heated and salted from below has been examined by Wang and Tan (2008).

Due to geographical and pedagogical processes like sedimentation, compactation, frost action and reorientation of the solid matrix, inhomogeneity and anisotropicity are characteristics of most of the natural porous materials. It is to be noted that early studies on convection in a porous medium have usually ignored these aspects of porous materials. There are artificial porous media encountered in numerous systems in industries as well like pelleting used in chemical engineering process, fiber material used for insulating purpose and many more. Despite the practical importance of the topic, very few studies are reported on the Rayleigh Benard convection for anisotropic porous material. Epherre (1975) performed the first study of the onset of convection in a horizontal layer with an anisotropic permeability. Tyvand (1980) studied the problem of thermohaline instability in anisotropic porous media.

The problem of natural convection in both isotropic and anisotropic porous channels has been studied by Nilsen and Storesletten (1990). Malashetty (1993) investigated the effect of anisotropy on the onset of convection in a double-diffusive flow. More recently, Malashetty and Swamy (2007) investigated the stability of Oldroyd fluid for anisotropic porous layer heated from below and cooled from above. In fact their investigation was an extension of the problem discussed by Yoon et al. (2004) for anisotropic medium.

Yoon et al. (2004) analyzed the onset of thermal convection in a horizontal porous layer saturated with viscoelastic liquid using the simplified constitutive model to examine the effects of relaxation times. They also examined the effect of rotation and anisotropy on the onset of convection in a horizontal porous layer by using a linear and a weak nonlinear theory. Saravanan and Arunkumar (2010) examined the effect of gravity modulation on the onset of convection in a horizontally saturated and transversely anisotropic porous fluid layer in which the applied temperature gradient is opposite to that of gravity and based on the Darcy-Brinkman boundary layer correction. Bhadauria and Srivastava (2010) investigated the thermal instability in an electrically conducting two component fluid saturated, porous medium confined between two horizontal surfaces subjected to a vertical magnetic field and considering temperature modulation of the boundaries, characterized by the Brinkman–Darcy model. The thermal instability in a rotating anisotropic porous medium, saturated with viscoelastic fluid based upon linear and non-linear theory has been investigated by Kumar and Bhadauria (2011).

As far as the double diffusive convection in anisotropic porous media is concerned, very few studies are available so far. Malashetty and Basavaraja (2004) studied the effect of time-periodic boundary temperatures on the onset of double diffusive convection in a fluid saturated anisotropic porous medium by making a linear stability analysis. The linear stability of a viscoelastic liquid saturated horizontal anisotropic porous layer heated from below and cooled from above is investigated for Oldroyd type model by Malashetty and Swamy (2007). The double diffusive convection in a horizontal anisotropic porous layer saturated with a binary fluid, heated and soluted from below with Soret effect has been studied analytically using both linear and non-linear stability analysis by Gaikwad et al. (2009). Malashetty et al. (2009) also examined the onset of double diffusive convection in a binary viscoelastic Oldroyd type fluid saturated anisotropic porous layer using a linear and weak non-linear stability analysis.

Recently, Shiina and Hishida (2010) obtained the critical Rayleigh number Ra_c at the onset of natural convection by linear stability analysis for high porosity anisotropic horizontal porous layers. Malashetty et al. (2011) examined the onset of double diffusive convection in a binary viscoelastic Oldroyd type fluid saturated anisotropic rotating porous layer by using a linear and a weakly non-linear stability analysis. Capone et al. (2011) analyzed the double-diffusive convection in an anisotropic porous layer with a constant through flow, with penetrative convection being simulated via an internal heat source. Chen et al. (2011) examined the stability analysis of thermosolutal convection in a horizontal porous layer when the solid and fluid phases are not in a local thermal equilibrium, and the solubility of the dissolved component depends on temperature for double-diffusive convection.

Kumar (2011) examined the combined effect of magnetic field and dust particles on the stability of a stratified couple-stress fluid through a porous medium in the presence of magnetic field. The double-diffusive convection in an anisotropic porous layer heated and salted from below with internal heat source using linear and non-linear stability analyses has been investigated by Bhadauria (2012). Some of the important results obtained by him for binary fluid mixture are, the effects of mechanical anisotropy and internal Rayleigh number destabilized the system while the concentration Rayleigh number are sustain the stability of the system, the magnitude of stream functions increases as the thermal Rayleigh number increases.

The aim of the present work is to extend the study of Yoon et al. (2004) to examine the effect of anisotropy parameter in double diffusive convection at the onset of oscillatory convection in a horizontal porous layer saturated with viscoelastic (Oldroyd) fluid. The modified Darcy equation incorporating the viscoelastic effects and two relaxation times for Oldroyd fluid suggested by Alisaev and Mirzadyanzade (1975) and Akhatov and Chembrisova (1993) respectively have been considered.

2. Physical Problem and Its Formulation

We consider an infinite horizontal anisotropic porous layer of thickness '*d*' saturated with Oldroydian viscoelastic fluid confined between two rigid boundaries as shown in Figure 1. The system is heated and soluted from below such that the two rigid boundaries are maintained at different temperatures and concentrations. The assumptions used in the present paper are

(i) The porous medium is anisotropic and homogeneous.
(ii) The saturating fluid is incompressible and non-Newtonian (Oldroydian).
(iii) The onset of thermal and solutal convection is under the Boussinesq approximation.
(iv) The bottom boundary is kept at temperature T_1, concentration C_1 and the upper boundary is kept at a lower temperature T_2, lower concentration C_2 with fixed $\Delta T = T_1 - T_2$ (> 0) and $\Delta C = C_1 - C_2$ (> 0).

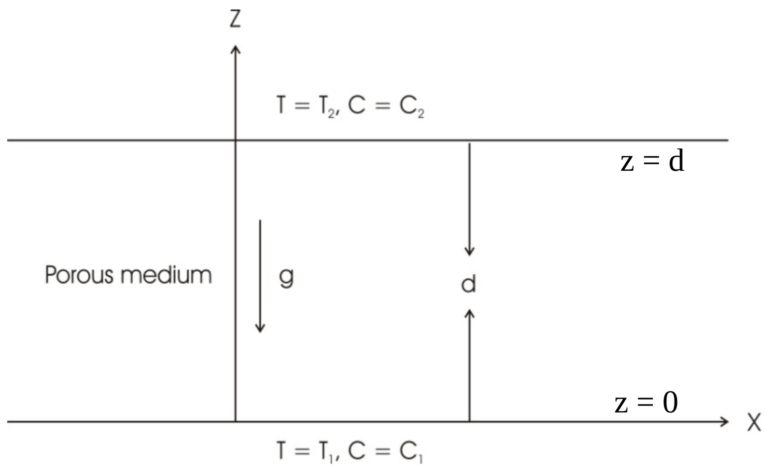


Figure 1. Physical Configuration

In view of these assumptions and following Yoon et al. (2004), the governing equations for anisotropic porous medium are written as

$$\nabla.\boldsymbol{q} = 0\,, \tag{1}$$

$$\left(1+\bar{\varepsilon}\frac{\partial}{\partial t}\right)\boldsymbol{q} = -\frac{\boldsymbol{K}}{\mu}\left(1+\bar{\lambda}\frac{\partial}{\partial t}\right)\left[\nabla p + \boldsymbol{k_1}\rho g\right], \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{q}.\nabla\right)T = \kappa\nabla^2 T\,, \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{q}.\nabla\right)C = \kappa'\nabla^2 C \tag{4}$$

and

$$\rho = \rho_0\left[1-\alpha\left(T-T_1\right)+\alpha'\left(C-C_1\right)\right], \tag{5}$$

where $\boldsymbol{q}$ =(u, v, w) is the velocity vector and $\boldsymbol{K}$ is the permeability tensor $k_x\left(\hat{i}\hat{i}+\hat{j}\hat{j}\right)+k_z\left(\hat{k}\hat{k}\right)$ of the porous medium. In a plane parallel to horizontal boundary, the permeability is same in both x and y directions, however it changes with z, μ is the effective viscosity of the fluid, $\bar{\lambda}$ is the relaxation time, $\bar{\varepsilon}$ is the retardation time, ρ is the density, g is the magnitude of gravitational acceleration, κ is thermal diffusivity, κ' is solutal diffusivity, α is thermal expansion coefficient and α' is solutal expansion coefficient. Equation (2) represents the modified Darcy equation suggested by Alisaev and Mirzadjanzade (1975) taking into account the Oldroyd's linear model with anisotropic effect.

3. Basic State

The basic state of the system given by

$$\boldsymbol{q} = \left(0,0,0\right),\; p = p\left(z\right),\;\; \rho = \rho\left(z\right),\; T = T\left(z\right) \text{ and } C = C\left(z\right) \tag{6}$$

yields,

$$\frac{dp}{dz}+\rho g=0\,, \tag{7}$$

$$T-T_1=-\beta z\,, \tag{8}$$

and

$$C-C_1=-\beta' z\,, \tag{9}$$

where $\beta=\left(\frac{\Delta T}{d}\right)$ and $\beta'=\left(\frac{\Delta C}{d}\right)$, both are positive.

4. Mathematical Analysis and Dispersion Relation

To examine the stability conditions by employing linear stability theory, equations (1) to (5) for disturbances (also called perturbations) of velocity, temperature and concentration are written in linear form as

$$\nabla.\boldsymbol{q}'=0\,, \tag{10}$$

$$\left(1+\bar{\varepsilon}\frac{\partial}{\partial t}\right)\boldsymbol{q}'=-\frac{\boldsymbol{K}}{\mu}\left(1+\bar{\lambda}\frac{\partial}{\partial t}\right)\left[\nabla p'+\boldsymbol{k_1}\rho g\right], \tag{11}$$

$$\frac{\partial\theta'}{\partial t}-\beta w'=\kappa\nabla^2\theta'\,, \tag{12}$$

$$\frac{\partial\gamma'}{\partial t}-\beta' w'=\kappa'\nabla^2\gamma' \tag{13}$$

and

$$\rho'=-\rho_0\left(\alpha\theta'-\alpha'\gamma'\right), \tag{14}$$

where $\boldsymbol{q}'=\left(u',v',w'\right)$, p', θ', γ' and ρ' are disturbances in velocity, pressure, temperature, concentration and density.

Under the normal mode analysis, we assume the time-dependent periodic disturbances in a horizontal plane of the form

$$\left[w',p',\theta',\gamma'\right]=\left[w(z),p(z),\theta(z),\gamma(z)\right]e^{\left(ia_x x+ia_y y+nt\right)}, \tag{15}$$

where a_x and a_y are the real wave numbers in the *x* and *y* directions respectively and *n*, in general, is complex such that $n\left(=n_r+in_i\right)$. Infinitesimal perturbations of the rest state may either damp or grow depending on the value of the parameter n. $n_r<0$ means that the system is stable and

$n_r > 0$, even for a single perturbation, indicates that the system is unstable. When $n_r = 0$ and $n_i = 0$ system is marginally stable under the principle of exchange of stabilities while $n_r = 0$ and $n_i \neq 0$ represents over stability of periodic oscillatory motion.

Substituting equation (15) in equations (10)-(14), the stability or instability governing equations of the system are given by

$$\left(1+\bar{\varepsilon}n\right)\left(D^2 - K_1 a^2\right)w = -\frac{k_x \rho_0 g}{\mu} a^2 \left(1+\bar{\lambda}n\right)\left(\alpha\theta - \alpha'\gamma\right), \tag{16}$$

$$\kappa\left(D^2 - a^2\right)\theta = n\theta - \beta w, \tag{17}$$

and

$$\kappa'\left(D^2 - a^2\right)\gamma = n\gamma - \beta' w, \tag{18}$$

where

$$D = \frac{d}{dz}, \ a = \sqrt{a_x^2 + a_y^2} \text{ and } K_1 = \frac{k_x}{k_z}.$$

Employing the following non-dimensional parameters

$$\left(x^*, y^*, z^*\right) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right), \ a^* = \frac{a}{1/d}, \ \sigma = \frac{nd^2}{\kappa}, D^* = \frac{D}{1/d}, \ k_x^* = \frac{k_x}{d^2}, \ k_z^* = \frac{k_z}{d^2}$$

$$w^* = \frac{w}{U_0}, \theta^* = \frac{\theta\kappa}{\beta d^2 U_0} \text{ and } \gamma^* = \frac{\gamma\kappa'}{\beta' d^2 U_0}, \text{ (} U_0 \text{ is the dimension of velocity).} \tag{19}$$

and removing asterisks, we get the non-dimensional form of equations (16) to (18) as

$$\left(1+\varepsilon\sigma\right)\left(D^2 - K_1 a^2\right)w = -\left(1+\lambda\sigma\right)\left(\theta - N\gamma Le\right)a^2 Ra_{Dx}, \tag{20}$$

$$\left(D^2 - a^2 - \sigma\right)\theta = -w \tag{21}$$

and

$$\left(D^2 - a^2 - \sigma Le\right)\gamma = -w, \tag{22}$$

where

$Le = \dfrac{\kappa}{\kappa'}$ (Lewis number),

$Ra_{Dx} = \dfrac{k_x g \alpha \beta d^4}{\kappa \nu}$ (Darcy Rayleigh number),

$\lambda = \dfrac{\kappa}{d^2}\bar{\lambda}$ (non-dimensional relaxation time),

$\varepsilon = \dfrac{\kappa}{d^2}\bar{\varepsilon}$ (non-dimensional retardation time),

$$N=\frac{\alpha'\beta'}{\alpha\beta} \text{ (buoyancy ratio),}$$

and

$$\nu=\frac{\mu}{\rho_0} \text{ (kinematic viscosity).}$$

The combined stability governing equation is obtained as

$$\begin{aligned}&(1+\varepsilon\sigma)\left(D^2-K_1a^2\right)\left[\left(D^2-a^2\right)^2-\sigma(1+Le)\left(D^2-a^2\right)+\sigma^2 Le\right]w\\&=a^2(1+\lambda\sigma)\left[\left(Ra_{Dx}-Rs_{Dx}\right)\left(D^2-a^2\right)w-\sigma\left(Ra_{Dx}Le-Rs_{Dx}\right)w\right],\end{aligned} \tag{23}$$

where

$$Rs_{Dx}=N\,Le\,Ra_{Dx}=\frac{k_x g\alpha'\beta' d^4}{\kappa'\nu} \text{ (solutal Darcy Rayleigh number) .}$$

The boundaries are considered to be impermeable and rigid, therefore the appropriate boundary conditions are

$$w=D^2w=0 \qquad \text{at } z=0,1. \tag{24}$$

5. Stability Analysis

The eigenvalue problem given by (23) and (24) involving Ra_{Dx}, Rs_{Dx}, a, ε, λ, Le, K_1 and σ as parameters, is solved upon assuming that amplitude $w(z)$ is small enough and can be expressed as

$$w=w_0\sin(m\pi z) \text{ for } m=1,2,3,..., \tag{25}$$

where w_0 denotes the amplitude. Substituting equation (25) into equation (23), we obtain

$$A_1\sigma^3+A_2\sigma^2+A_3\sigma+A_4=0, \tag{26}$$

where

$$A_1=\varepsilon Le\left(m^2\pi^2+K_1a^2\right), \tag{27}$$

$$A_2=\lambda a^2 Le\left[X_1-Ra_{Dx}\right], \tag{28}$$

$$A_3=a^2\left[Le+\lambda\left(m^2\pi^2+a^2\right)\right]\left[X_2-Ra_{Dx}\right], \tag{29}$$

and

$$A_4=a^2\left(m^2\pi^2+a^2\right)\left[X_3-Ra_{Dx}\right], \tag{30}$$

with

$$X_1 = \frac{1}{Le}\left[Rs_{Dx} + \frac{\left(m^2\pi^2 + K_1 a^2\right)\left\{Le + \varepsilon\left(1+Le\right)\left(m^2\pi^2 + a^2\right)\right\}}{\lambda a^2}\right], \tag{31}$$

$$X_2 = \left[\frac{1+\lambda\left(m^2\pi^2 + a^2\right)}{Le+\lambda\left(m^2\pi^2 + a^2\right)} Rs_{Dx} + \frac{\left(m^2\pi^2 + K_1 a^2\right)\left(m^2\pi^2 + a^2\right)\left\{\left(1+Le\right) + \varepsilon\left(m^2\pi^2 + a^2\right)\right\}}{a^2\left\{Le+\lambda\left(m^2\pi^2 + a^2\right)\right\}}\right], \tag{32}$$

and

$$X_3 = Rs_{Dx} + \frac{\left(m^2\pi^2 + K_1 a^2\right)\left(m^2\pi^2 + a^2\right)}{a^2}. \tag{33}$$

Observe that A_1 is positive definite whereas A_2, A_3 and A_4 may be positive or negative real numbers.

If σ_1, σ_2 and σ_3 are the roots of equation (26), then

$$\sigma_1 + \sigma_2 + \sigma_3 = -\frac{A_2}{A_1}, \tag{34}$$

$$\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \frac{A_3}{A_1} \tag{35}$$

and

$$\sigma_1\sigma_2\sigma_3 = -\frac{A_4}{A_1}. \tag{36}$$

It is apparent that modes are unstable under the condition

$$Ra_{Dx} > \max\left\{X_1, X_2, X_3\right\}, \tag{37}$$

as in that case A_2, A_3 and A_4 all will be negative ensuring the existence of positive root, i.e., unstable mode.

Further dividing equation (26) by σ^2 and equating the imaginary part, we have

$$\sigma_i\left[A_1 - \frac{A_3}{|\sigma|^2} - \frac{2\sigma_r A_4}{|\sigma^2|^2}\right] = 0, \tag{38}$$

which clearly shows that $\sigma_i = 0$ necessarily under condition (35). Therefore, unstable modes will grow periodically and not through oscillation.

A close observation of the coefficient A_i in equation (26) predicts that the system is again unstable if the conditions $A_2 < 0$, $A_3 > 0$ and $A_4 < 0$ hold, because in that case by Descartes' rule of sign at least one root is real and positive.

Dividing equation (26) by σ and equating the imaginary part of the resulting equation to zero, we get

$$\sigma_i\left[\left(2A_1\sigma_r + A_2\right)|\sigma|^2 - A_4\right] = 0 .$$

It has already been proved that if $A_2 < 0$ and $A_4 < 0$, then the system is unstable $\left(\sigma_r > 0\right)$ and the modes are oscillatory $(\sigma_i \neq 0)$. In that case, we rewrite the above equation as

$$\sigma_i\left[\left(2A_1\sigma_r - |A_2|\right)|\sigma|^2 + |A_4|\right] = 0 , \tag{39}$$

and for the consistency of equation (39), we must necessarily have

$$0 < \sigma_r < \frac{|A_2|}{2A_1} , \tag{40}$$

which provides the first bound on σ_r for oscillatory unstable modes.

Equation (39) can also be written as

$$\sigma_i\left[2A_1\sigma_r\,|\sigma|^2 - |A_2|\left(|\sigma|^2 - \frac{|A_4|}{|A_2|}\right)\right] = 0 .$$

The consistency of this equation requires that

$$|\sigma|^2 > \frac{|A_4|}{|A_2|} , \tag{41}$$

necessarily. This provides the second bounds on σ_r for oscillatory unstable modes.

Combining the two regions given by (40) and (41), the oscillatory unstable modes lie outside the circle $\sigma_r^2 + \sigma_i^2 > \frac{|A_4|}{|A_2|}$ [given by (41)] and inside the strip $0 < \sigma_r < \frac{|A_2|}{2A_1}$ [given by (40)] shown graphically in figure 2.

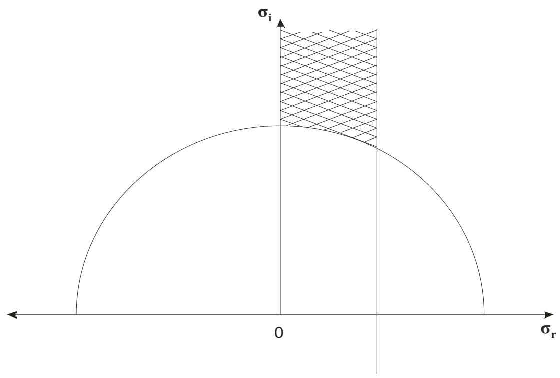


Figure 2.

6. Existence of Variational Principle

Let the Principle of Exchange of Stabilities (PES) be valid at the marginal state so that $\sigma_r = 0 \Rightarrow \sigma_i = 0$. Putting $\sigma = 0$ in equations (20)-(22), we get

$$\left(D^2 - K_1 a^2\right) w = -\left(\theta - N\,\gamma\, Le\right) a^2\, Ra_{Dx}, \tag{42}$$

$$\left(D^2 - a^2\right)\theta = -w \tag{43}$$

and

$$\left(D^2 - a^2\right)\gamma = -w. \tag{44}$$

Elimination of θ and γ from equations (42) to (44) leads to

$$\left[D^4 - \left(1 + K_1\right) a^2 D^2 + K_1 a^4\right] w = \left(1 - N\, Le\right) a^2\, Ra_{Dx} w. \tag{45}$$

On integration of equation (45) after its multiplication by w, we get

$$Ra_{Dx} = \frac{1}{a^2\left(1 - NLe\right)} \frac{I_1}{I_2}, \tag{46}$$

where

$$I_1 = \int \left[\left(D^2 w\right)^2 + (1 + K_1) a^2 \left(Dw\right)^2 + K_1 a^4 \left(w\right)^2\right] dz \tag{47}$$

and

$$I_2 = \int w^2 dz. \tag{48}$$

Let δRa_{Dx} be the change in Ra_{Dx} when w is subjected to a small variation δw which is compatible with the boundary conditions on w. Then, we get from equations (46)-(48),

$$\delta Ra_{Dx} = \frac{1}{a^2\left(1-N\,Le\right)I_2}\left[\delta I_1 - \left(1-N\,Le\right)a^2 Ra_{Dx}\delta I_2\right], \tag{49}$$

where

$$\delta I_1 = 2\int\left\{\left(D^2-a^2\right)\left(D^2-K_1a^2\right)w\right\}\delta w dz \tag{50}$$

and

$$\delta I_2 = 2\int w\delta w dz\,. \tag{51}$$

On simplification, equations (49)-(51) provide

$$\delta Ra_{Dx} = \frac{2}{a^2\left(1-NLe\right)I_2}\left[\int\left\{\left(D^2-a^2\right)\left(D^2-K_1a^2\right)w - \left(1-NLe\right)a^2 Ra_{Dx}w\right\}\delta w dz\right]. \tag{52}$$

It follows from equation (52) that $\delta Ra_{Dx} = 0$ for small arbitrary variations in w if and only if w satisfies equation (45) and the boundary conditions (24). This establishes the existence of variational principle.

7. Results and Discussion

a. Stationary Convection

For stationary convection at marginal state $\left(\sigma_r = 0, \sigma_i = 0\right)$, the corresponding characteristic value of the Darcy Rayleigh number, Ra_{Dx} is given by

$$Ra_{Dx} = Rs_{Dx} + \frac{\left(m^2\pi^2 + a^2\right)\left(m^2\pi^2 + K_1a^2\right)}{a^2}, \tag{53}$$

Equation (53) constitutes the marginal stability curve. As m increases, Ra_{Dx} will increase rapidly and since we are interested in the most dangerous mode. Therefore, we are confined to the lowest order mode producing the minimum of Ra_{Dx} for a given wave number. Therefore we set m = 1 so that minimization of Ra_{Dx} with respect to wave number a yields the critical Darcy Rayleigh number for stationary convection as

$$Ra_{Dx,\min}^{stat.} = Rs_{Dx} + \pi^2\left(1 + K_1^{1/2}\right)^2 \tag{54}$$

and the corresponding critical wave number is given by

$$a = \pi\,K_1^{-1/4}.$$

We observe that the stationary mode is depending on anisotropy parameter and the concentration parameter and is independent of the viscoelastic parameter and the specific heat ratio. Therefore, in the absence of salt $\left(Rs_{Dx}=0\right)$ for isotropic porous material it is identical with Yoon et al. (2004) and in agreement with Malashetty and Swamy (2007) for anisotropic material with isotropic mono diffusion thermal convection. The critical wave number is also independent of viscoelastic parameter and depends on anisotropic parameter only.

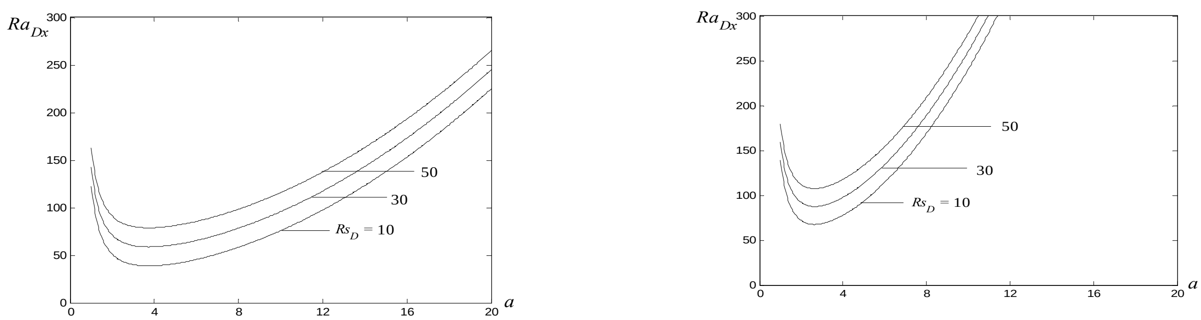
Equation (53) can also be written as for lowest mode m = 1,

$$Ra_T = \frac{Rs_{Dx}}{K_1} + \frac{\left(\pi^2+a^2\right)\left(\dfrac{\pi^2}{K_1}+a^2\right)}{a^2},$$

where

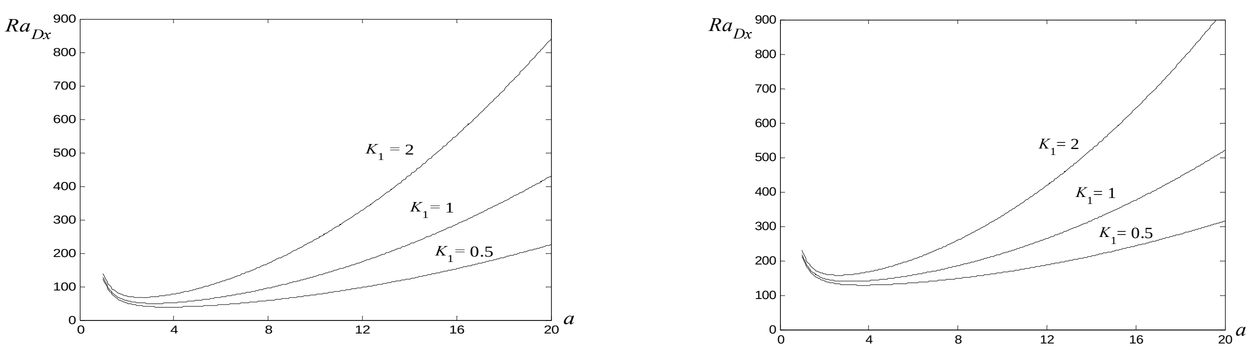
$$Ra_T = \frac{Ra_{Dx}}{K_1} = \frac{k_z g\alpha\beta d^4}{\kappa\nu}.$$

For a single component system $\left(Rs_{Dx}=0\right)$, our equation coincides with Bhadauria (2012) for the case of thermal isotropy.



(a) (b)

Figure 3. Variations of critical Darcy Rayleigh number with wave number for different solutal Darcy Rayleigh number for stationary convection when (a) K_1 = 0.5, (b) 2



(a) (b)

Figure 4. Variations of critical Darcy Rayleigh number with wave number for different anisotropy parameter (K_1) for stationary convection when (a) Rs_{Dx} = 10, (b) Rs_{Dx} = 100

In the present case, it is clear from (54) that the solutal Darcy Rayleigh number Rs_{Dx} postpones instability when the fluid layer is soluted from below. However, if the fluid layer is soluted from above $(\beta' < 0)$, the solutal Darcy Rayleigh number promotes instability. For anisotropic porous medium, when the horizontal permeability is more than the vertical permeability, i.e., $K_1 > 1$, the instability is postponed, whereas it is promoted for $K_1 < 1$.

Furthermore, isotropic porous material, i.e., $K_1 = 1$ yields the critical Darcy Rayleigh number

$$Ra_{D'\min}^{stat.} = Rs_D + 4\pi^2 \tag{55}$$

and the corresponding critical wave number

$$a_c = \pi \ .$$

In the absence of solute $(Rs_{Dx} = 0)$, classical results, $Ra_D = 4\pi^2$ and $a_c = \pi$ [Horton and Rogers (1945), Lapwood (1948), Combarnous and Bories (1975), Cheng (1978) and Yoon et al. (2004)] are recovered.

It is important to note that the same result was found by Wang and Tan (2008) while discussing double diffusion in Darcy Maxwell fluid in isotropic porous medium.

Figures 3 (a) and (b) respectively provide critical Darcy Rayleigh number when K_1=0.5 and 2. A comparison of these two graphs shows a stabilizing character of anisotropy parameter when it exceeds 1. Both the graphs, however, indicate stabilizing effect of solutal Darcy Rayleigh number, consistence with equation (55). Figures 4 (a) and (b) also provide the information about K_1 and Rs_D.

b. General Discussion

The important observations regarding different parameters are

Figures 5 (a) and (b) demonstrate the effect of relaxation time λ for K_1= 0.5 and 2 respectively, for fixed ε (=0.25), Le (=0.9), Ra_{Dx} (=50) and Rs_{Dx} (=10). As λ increases the range of unstable wave numbers increases, indicating that λ acts as a catalyst of instability. The characterization of modes remains essentially the same for K_1= 0.5 but for K_1 = 2, it is observed that initially the modes grow through oscillations and after λ=0.75, the periodically growing modes are also introduced.

For fixed λ (=1), Le (=0.25), Ra_{Dx} (=50) and Rs_{Dx} (=10), the unstable region between critical wave numbers a_{c_1} and a_{c_2} is shown in figures 6 (a) and (b). These demonstrate the stabilizing character of retardation time ε for K_1 = 0.5 and 2 respectively.

Two following important observations are made:

(i) There is sharp decline in the upper curve as ε increases from 0.01 to 0.05 reducing sharply the range of unstable wave numbers.

(ii) The same characterization of modes prevails when the horizontal permeability is less than the vertical permeability (K_1 <1). However, when K_1 =2, the aperiodically growing modes ceases to exist beyond ε = 0.1 and the modes grow only through oscillations, if we take into account the information from the figure 6 (b).

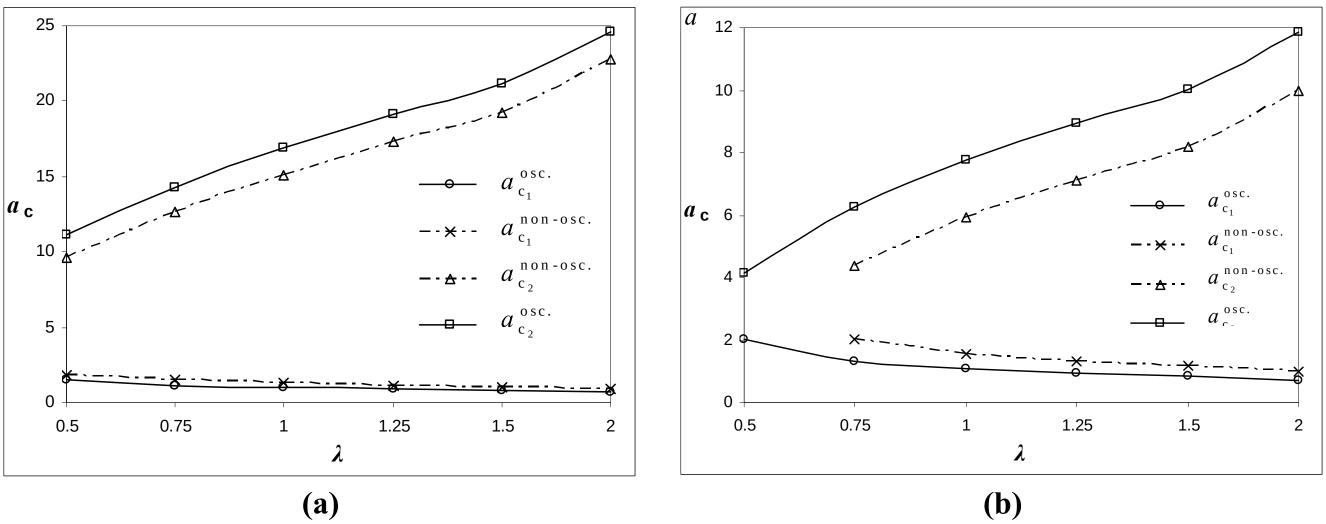


Figure 5. Variations of critical wave number a_c with relaxation time λ for (a) K_1 = 0.5 and (b) K_1 = 2.

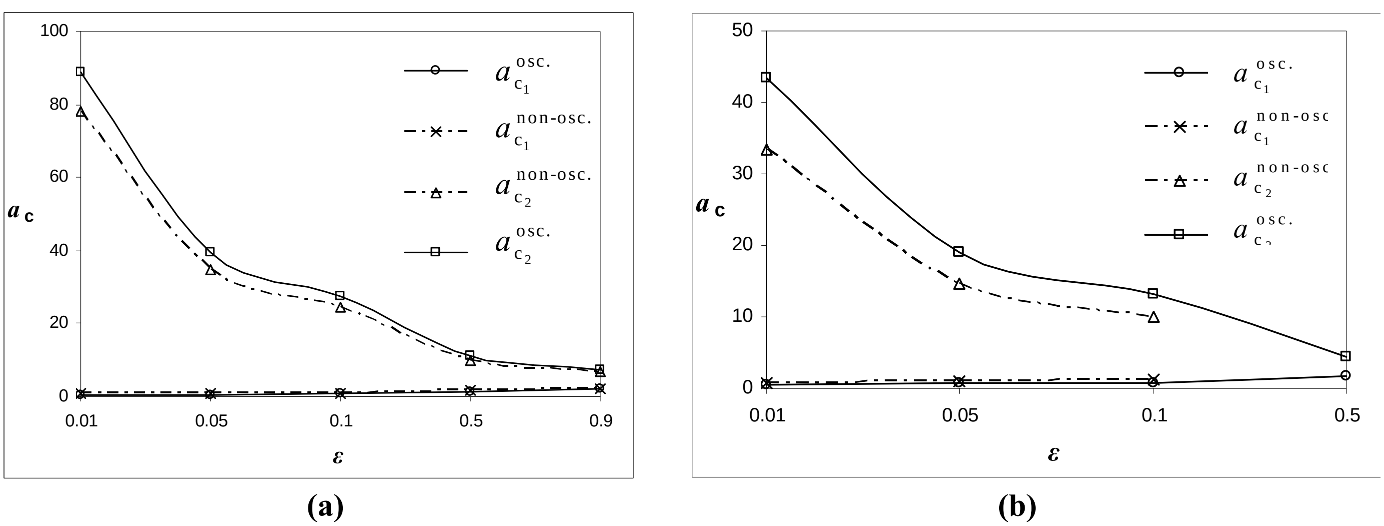


Figure 6. Variations of critical wave number a_c with retardation time ε for (a) K_1 = 0.5 and (b) K_1 = 2.

Figure 7 shows the unstable region between critical wave numbers a_{c_1} and a_{c_2} for fixed ε (=0.25), Le (=0.9), λ (=0.5), Ra_{Dx} (= 50) and Rs_{Dx} (= 10). It shows that as the anisotropy parameter $K_1(k_x/k_z)$ increases beyond 1 ($K_1 >$ 1), the range of unstable wave numbers decreases, showing, thereby, that if the permeability in horizontal direction is more than the permeability in vertical direction, the system becomes more stable. In the initial small range of K_1 (for $K_1 < 1$), there is sharp decline in the values of a_{c_2} whereas the increase in the values of a_{c_1} is slow, the range of unstable wave numbers therefore reduces sharply in this range of K_1. Moreover, for large values of K_1, a_{c_1} and a_{c_2} coincide, no mode grows through oscillations or periodically and the system becomes completely stable. Instability therefore, predicted on behalf of isotropic porous medium may not exist if the medium is anisotropic and the horizontal permeability is more than the vertical permeability ($K_1 >$ 1). Moreover, if the horizontal permeability is less than the vertical permeability ($K_1 <$ 1), wave numbers predicted to be stable on the assumption of isotropic porous medium may infect be unstable.

Further investigations of unstable range of wave numbers ($a_{c_1} < a < a_{c_2}$) leads to its characterization of mode into oscillatory and non-oscillatory. For a given anisotropy parameter, the middle range of unstable modes grows periodically whereas the wave numbers outside on both side of this middle range grow through oscillations.

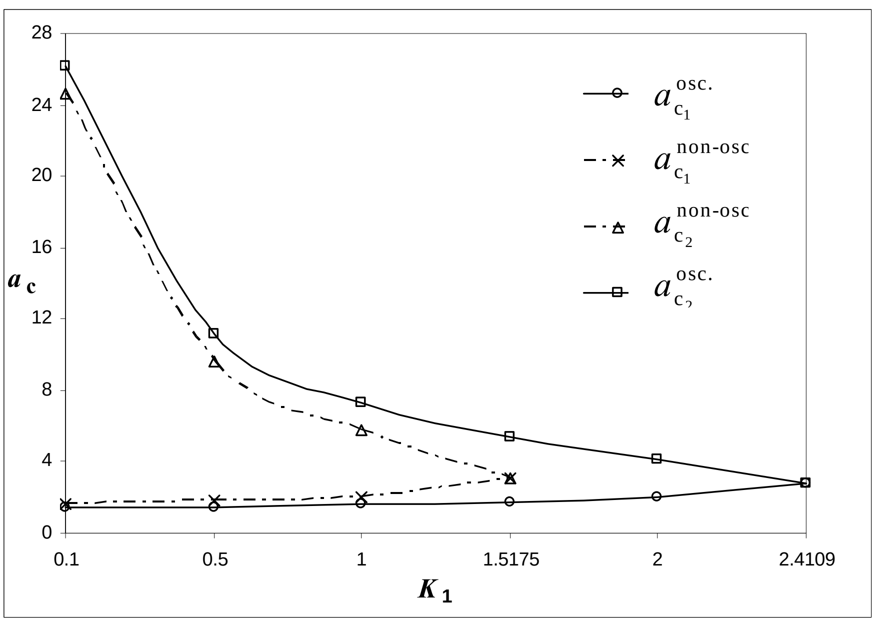


Figure 7. Variations of critical wave number a_c with anisotropy parameter K_1

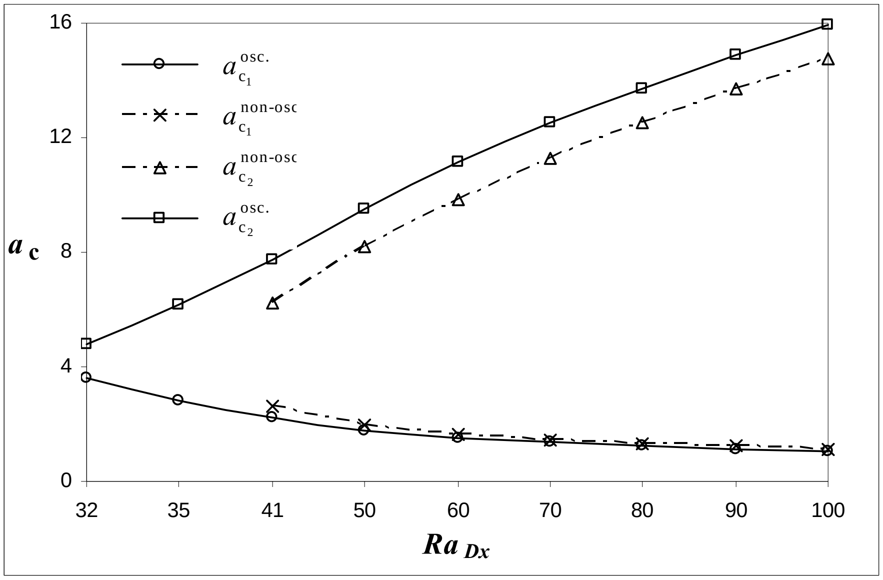


Figure 8. Variations of critical wave number a_c with critical Darcy Rayleigh number Ra_{Dx} for anisotropy parameter K_1 = 0.5.

Figure 8 provides the effect of Darcy Rayleigh number Ra_{Dx} on the stability or instability of the system for fixed ε (= 0.25), Le (=0.25), λ (=0.4), Rs_{Dx} (= 10) and K_1 (=0.5). It is concluded that the system is completely stable for $Ra_{Dx} < 32$, oscillatory unstable modes are introduced in the middle range of wave number $a_{c_1}^{osc.} < a_c < a_{c_2}^{osc.}$ for 32 < Ra_{Dx} < 41 and as Ra_{Dx} further increases, the unstable range of wave numbers is divided into oscillatory unstable and non-oscillatory unstable modes. One periodically growing mode exists in the middle range $a_{c_1}^{non\text{-}osc.} < a_c < a_{c_2}^{non\text{-}osc.}$ which increases with Ra_{Dx} and two oscillatory unstable modes are squeezed in two narrow layers on both sides outside this middle range given by $a_{c_1}^{osc.} < a_c < a_{c_1}^{non\text{-}osc.}$ and $a_{c_2}^{non\text{-}osc.} < a_c < a_{c_2}^{osc.}$. The middle range $a_{c_1}^{non\text{-}osc.} < a_c < a_{c_2}^{non\text{-}osc.}$ which contains the periodically growing modes is further subdivided as shown in Figure 8, the middle layer has one growing and two decaying modes whereas the narrow layers on both sides of this middle layer has two growing and one decaying modes. The unstable region between critical wave numbers a_{c_1} and a_{c_2} is shown.

Figure 9 plots Lewis number Le against the wave number a_c for fixed ε (= 0.25), λ (=0.4), Ra_{Dx} (=50), Rs_{Dx} (=10) and K_1 (=0.5). The effect of Lewis number Le is found to be destabilizing. The behavior of modes is essentially the same as in Figure 8. The unstable region between critical wave numbers a_{c_1} and a_{c_2} slightly increases as Lewis number Le increases as shown in table 1 and figure 9.

Table 1. Variations of critical wave number a_c with Lewis number Le

Le	$a_{c_1}^{osc.}$	$a_{c_1}^{non\text{-}osc.}$	$a_{c_2}^{non\text{-}osc.}$	$a_{c_2}^{osc.}$
0.001	1.794	2.0053	8.1491	9.5169
0.1	1.7886	2.0018	8.1668	9.5195
0.5	1.765	1.9862	8.2379	9.5333
0.9	1.7381	1.9693	8.3078	9.5537

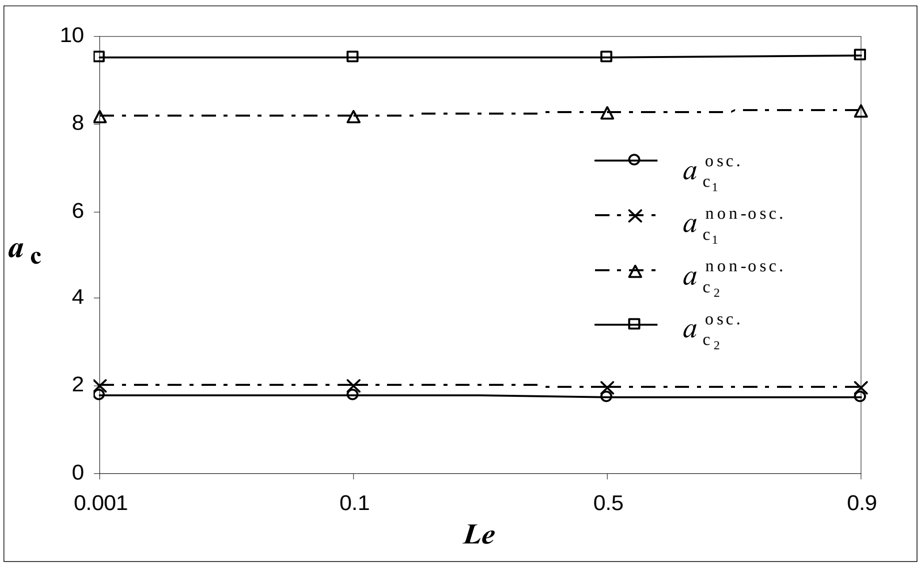


Figure 9. Variations of critical wave number a_c with Lewis number Le for anisotropy parameter $K_1 = 0.5$.

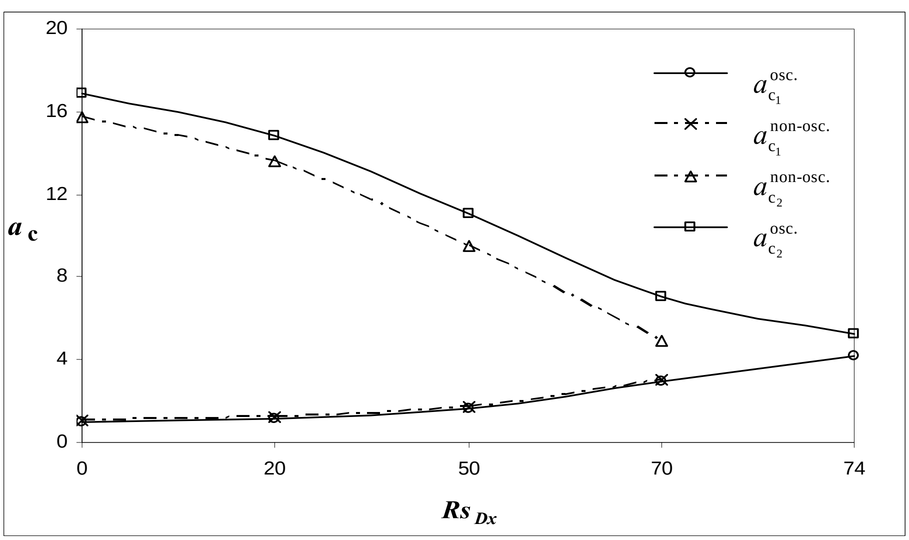


Figure 10. Variations of critical wave number a_c with Solutal Darcy Rayleigh number Rs_{Dx} for anisotropy parameter $K_1 = 0.5$.

Figure 10 plots the solutal Darcy Rayleigh number Rs_{Dx} against wave number a_c for fixed ε (=0.25), Le (=0.25), λ (=0.4), Ra_{Dx} (=100) and K_1 (=0.5). This shows the stabilizing character of the solutal Darcy Rayleigh number. The characterization of modes into stable or unstable, growing periodically or through oscillations and one periodically growing or two periodically growing modes obey the same pattern as in figure 8. The unstable region between critical wave numbers a_{c_1} and a_{c_2} is shown.

8. Conclusion

The paper has critically examined the effect of anisotropy on the onset of stationary and oscillatory convection in a horizontal porous layer saturated with viscoelastic liquid heated and soluted from below. If the horizontal permeability is less than the vertical permeability i.e. the case when $K_1 < 1$, the critical Darcy Rayleigh number is reduced implying, thereby, a destabilizing effect of anisotropy whereas the critical Darcy Rayleigh number is increased for the case when $K_1 > 1$. The characterization of unstable modes into oscillatory and non-oscillatory is also explained numerically through graphs.

NOMENCLATURE

a wave number
C solute concentration
d height of the fluid layer
$\boldsymbol{K}$ permeability tensor of the porous medium, $\left[k_x\left(\hat{i}\hat{i}+\hat{j}\hat{j}\right)+k_z\left(\hat{k}\hat{k}\right)\right]$
K_1 anisotropy parameter $\left(=\dfrac{k_x}{k_z}\right)$
$\boldsymbol{k_1}$ (0, 0, 1)
Le Lewis number $\left(=\dfrac{\kappa}{\kappa'}\right)$
p pressure
$\boldsymbol{q}$ velocity vector (u, v, w)
Ra_{Dx} Darcy Rayleigh number $\left(=\dfrac{k_x g\alpha\beta d^4}{\kappa\nu}\right)$
Rs_{Dx} solutal Darcy Rayleigh number $\left(=\dfrac{k_x g\alpha'\beta' d^4}{\kappa'\nu}\right)$
T temperature
t time
x, y, z space coordinates

Greek Symbols

α thermal expansion coefficient

α' solute expansion coefficient
β $\left(\frac{\Delta T}{d}\right)$
β' $\left(\frac{\Delta C}{d}\right)$
κ thermal diffusivity
κ' solutal diffusivity
$\bar{\varepsilon}$ retardation time
ε dimensionless retardation time
$\bar{\lambda}$ relaxation time
λ dimensionless relaxation time
μ viscosity
ν kinematic viscosity
ρ density

Subscripts

c critical
0 reference value
min minimum

Superscripts
* dimensionless quantity
' perturbed quantity
stat. stationary
osc. oscillatory
non-osc. non-oscillatory

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