A Layered Notch Filter for High-Frequency Dynamic Isolation

J. L. Sackman

J. M. Kelly

Department of Civil Engineering,
University of California, Berkeley,
Berkeley, CA 94720

A. E. Javid

IBM Corporation,
San Jose, CA 95193

An efficient method of isolation from high-frequency vibrations is the use of periodically layered composites acting as a mechanical filter. This device is a periodically layered stack of alternating materials with widely different densities and stiffnesses. The working principle of the device is wave reflection, and the device becomes increasingly effective when there is a large impedance mismatch which leads to rapid attenuation of an input wave for certain frequency ranges. This filter acts only in specific frequency bands. At other frequencies, it will transmit the vibratory energy unmodified, thus acting as a mechanical notch filter. The theoretical development of the mechanical notch filter is based on the theory of waves in periodically layered media. Floquet theory is used to solve the equations for the propagation of plane waves through a laminated system of parallel plates of different materials when the direction of propagation is normal to the plates. Several experiments were conducted to prove the validity of the mechanical notch filter concept. These experiments demonstrated that the theory is correct and that the results have practical application.

1 Introduction

In the design of machinery the control of undesirable vibrations is frequently handled by the use of isolation concepts. In most cases the frequency of these vibrations is low, e.g., less than 100 Hz, and the principles of vibration isolation for this range of frequency are relatively well understood [1]. In some cases, however, there may be a need to protect very sensitive components from higher frequencies and the standard techniques may not be appropriate due to the presence of wave effects. To deal with high-frequency vibration problems it was suggested by Javid [2] that the device known as a mechanical filter [3–5] be used. This device is a periodically layered stack of alternating materials which have wide differences in density and stiffness. A possible application of the device is shown in Fig. 1, where it is used to isolate a sensitive component from the high-frequency vibrations of the base on which the device would normally sit.

The working principle of the device is wave reflection and the device becomes increasingly effective when there is a large impedance mismatch between materials. Large differences in impedance can lead to rapid attenuation of an input wave for certain frequency ranges. This filter acts only in specific frequency bands. At other frequencies it will transmit the vibrations unmodified (as in the case of steady-state vibrations).

The theoretical development of this system is based on the theory of waves in periodically layered media. The equations for the propagation of plane waves through a laminated system of parallel plates of different materials when the direction of propagation is normal to the plates are treated by what is called Floquet theory [4–6]. The problem of a shear wave propagating through an infinite stack of plates each of which is infinite in extent allows a very simple solution which can provide insight into the existence of stopping bands and an understanding of the attenuation of a mechanical filter of this kind. An analysis has been given by Lee and Wang [7] and Lee [8] for cells containing two layers and by Kahrim-Panahi [9] for cells with three layers.

The fact that real filters have plates of finite extent does not appear to affect the response generally, possibly due to the fact that in shear waves the errors introduced by the assumption of infinite plates are due to the presence of shear stresses in the free surfaces at the edge of each plate. If the width-to-thickness ratio of each plate is very large, this error will be small. This supposition appears to be supported by the results of an experimental program. In addition, the theory predicts extremely rapid attenuation of the waves when the frequency of input is in the center of the stopping bands and thus the waves are effectively blocked after only a finite number of layers are traversed. The fact that the real stack is finite, as opposed to infinite as assumed in the theory, seems therefore not to obviate the existence of stopping although it may modify their details.

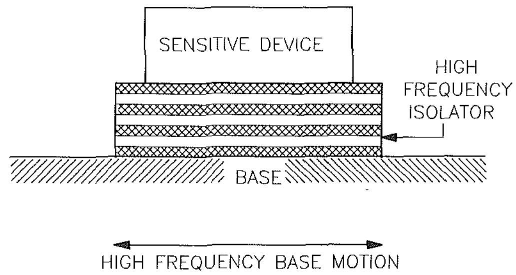

Fig. 1 Typical application of high-frequency isolation device

Contributed by the Pressure Vessels and Piping Division for publication in the JOURNAL OF PRESSURE VESSEL TECHNOLOGY. Manuscript received by the PVP Division, August 24, 1988.

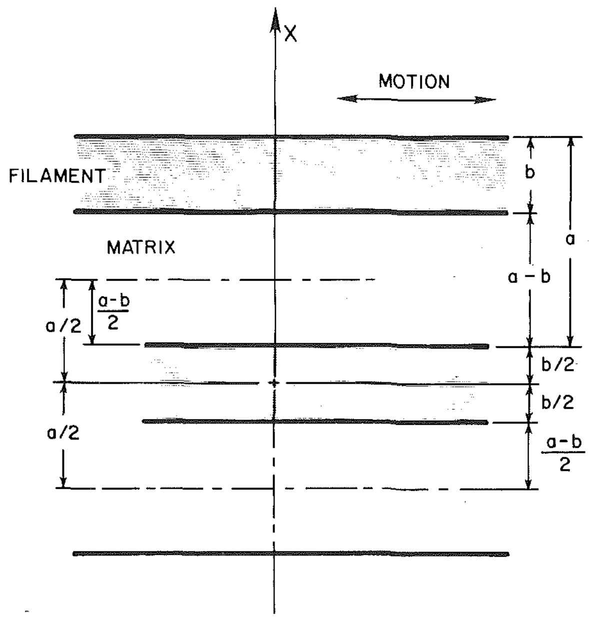

Fig. 2 Infinite, periodically layered composite

2 Theoretical Background

In what follows we use terminology common in the theory of periodically layered composites in which each cell, of thickness a, is made up of two layers of different materials. We refer to that layer which is composed of the denser and stiffer material as the filament, and the other as the matrix. The filament layer is taken to be of thickness b and the matrix of thickness $a-b$. The direction of propagation of plane waves through an infinite stack of these cells is normal to the layering and distance is denoted by x (Fig. 2). The displacement in the parallel direction is denoted by $w(x, t)$ and it is assumed that:

$$w(x,t)=u(x)e^{-i\omega t} \tag{1}$$

where ω is the frequency of steady-state oscillatory input. The densities and shear moduli of the filament and matrix materials will be denoted by ρ_f, ρ_m and G_f, G_m.

The only nonzero strain and stress components are

$$\gamma=\frac{\partial w}{\partial x}, \quad \tau(x,t)=G(x)\gamma(x,t) \tag{2}$$

The equation of motion takes the form

$$\frac{d}{dx}\left[G(x)\frac{du}{dx}\right]+\rho(x)\omega^2 u=0 \tag{3}$$

The functions $G(x)$ and $\rho(x)$ are periodic functions with period a, the cell thickness, and according to Floquet theory the solution takes the form

$$u(x)=v(x)e^{iqx} \tag{4}$$

where $v(x)$ is periodic with period a, and q is a constant wave number. The displacement $w(x,t)$ thus takes the form of a Floquet wave propagating through the stack

$$w(x,t)=v(x)e^{i(qx-\omega t)} \tag{5}$$

The second-order differential equation requires two boundary conditions, which in the Floquet theory are supplied by the quasi-periodic boundary conditions

$$u\left(\frac{a}{2}\right)=u\left(-\frac{a}{2}\right)e^{iqa} \tag{6}$$

$$u'\left(\frac{a}{2}\right)=u'\left(-\frac{a}{2}\right)e^{iqa} \tag{7}$$

It should be noted that these equations arise from the periodicity of $v(x)$ and the requirement of continuity of displacement and shear stress across cell boundaries. The ordinary differential equation (3) and the quasi-periodic boundary conditions constitute a Sturm-Liouville (or eigenvalue) system. Here, the eigenvalue is the wave number q.

Nomenclature

a = cell thickness
b = thickness of filament layer
c = shear wave speed
c_f = shear wave speed in filament material
c_m = shear wave speed in matrix material
f_f = natural frequency of first thickness shear mode of filament layer
f_m = natural frequency of first thickness shear mode of matrix layer
f_m^* = natural frequency of first thickness shear mode of matrix layer of thickness a
f_B = frequency of beginning of first stopping band
f_E = frequency of end of first stopping band
f_M = frequency of middle of first stopping band
i = $\sqrt{-1}$
n = number of cells in filter
p = ratio of acoustic impedances of filament and matrix materials
q = wave number
q_D = imaginary part of wave number q for viscoelastic material
q_I = imaginary part of wave number q for elastic material
q_R = real part of wave number for viscoelastic material
t = time
u = steady-state component of w
u_1, u_2 = fundamental solutions of equation of motion
v = periodic component of u
w = displacement component parallel to layer interfaces
x = coordinate in direction perpendicular to layer interfaces
y = coordinate parallel to layer interfaces and perpendicular to direction of displacement w
z = nondimensional frequency; coordinate in direction of displacement w
G = shear modulus
G_f = shear modulus of filament material
G_m = shear modulus of matrix material
G_I = imaginary part of G
G_R = real part of G
R = reduction factor due to wave trapping
R_D = reduction factor due to damping
R^* = desired degree of reduction
S = ratio of thickness of filament layer to thickness of matrix layer
Z_B = nondimensional frequency of beginning of first stopping band
α = ratio of weight of filament layer to twice the weight of matrix layer
δ = ratio of imaginary part of G to real part of G
γ = shear strain
ω = frequency of vibration
ρ = density
ρ_f = density of filament material
ρ_m = density of matrix material
τ = shear stress

The eigenvalue equation is most conveniently obtained by considering two linearly independent solutions of equation (3), which we can denote by u_1 and u_2, and taking the eigenfunction as $u_1+\alpha u_2$ where α is a constant to be determined from the boundary conditions. Since the functions G and ρ in (3) are constant in each layer in a cell, the basic solutions u_1 and u_2 take particularly simple forms. In the filament, the first solution u_1 is taken to be

$$u_1=\cos\frac{\omega x}{c_f} \quad -\frac{b}{2}\le x\le\frac{b}{2} \tag{8}$$

where $c_f=\sqrt{G_f/\rho_f}$. In the matrix material, $b/2\le x\le a/2$, and u_1 takes the form

$$u_1=A^+\cos\frac{\omega x}{c_m}+B^+\sin\frac{\omega x}{c_m} \tag{9}$$

where A^+, B^+ are determined from continuity of displacement and stress at the interface between filament and matrix at $x=b/2$. These lead to the solution for u_1 for the matrix in the form

$$u_1=\cos\frac{\omega b}{2c_f}\cos\frac{\omega\left(x-\frac{b}{2}\right)}{c_m}-p\sin\frac{\omega_b}{2c_f}\sin\frac{\omega\left(x-\frac{b}{2}\right)}{c_m} \quad \frac{b}{2}\le x\le\frac{a}{2} \tag{10}$$

where

$$p=\frac{G_f}{c_f}\times\frac{c_m}{G_m}=\frac{\rho_f c_f}{\rho_m c_m} \tag{11}$$

The solution for u_1 in the region $-(a/2)\le x\le -(b/2)$ is given by the foregoing using symmetry.

The second solution is obtained by taking

$$u_2=\sin\frac{\omega x}{c_f} \quad -\frac{b}{2}\le x\le\frac{b}{2} \tag{12}$$

in the filament material and using the same technique to extend it to the matrix material in the regions $b/2\le x\le a/2$, $-a/2\le x\le -b/2$. When we take both solutions in the form

$$u=u_1+\alpha u_2 \tag{13}$$

and use the first quasi-periodic boundary condition, we obtain α and the solution for u in the form

$$u=\cos\frac{\omega x}{c_f}+\frac{i\left[\cos\frac{\omega b}{2c_f}\cos\frac{\omega(a-b)}{2c_m}-p\sin\frac{\omega b}{2c_f}\sin\frac{\omega(a-b)}{2c_m}\right]}{\left[\sin\frac{\omega b}{2c_f}\cos\frac{\omega(a-b)}{2c_m}+p\cos\frac{\omega b}{2c_f}\sin\frac{\omega(a-b)}{2c_m}\right]}\times\tan\frac{qa}{2}\sin\frac{\omega x}{c_f} \tag{14}$$

for the filament region $-b/2\le z\le b/2$ with a corresponding expression in the matrix region. When these two solutions are substituted into the second quasi-periodic boundary condition, we obtain the final eigenvalue equation relating the eigenvalue q to the input frequency ω

$$\cos qa=\cos\frac{\omega(a-b)}{c_m}\cos\frac{\omega b}{c_f}-\frac{1}{2}\left(p+\frac{1}{p}\right)\sin\frac{\omega(a-b)}{c_m}\sin\frac{\omega b}{c_f} \tag{15}$$

It is the basic equation which reveals the existence of a wave-blocking filter. In order that the composite system transmit waves of a specific frequency ω, the eigenvalue q must be real. If for some value of ω the right-hand side of equation (15) is of magnitude greater than 1, then q will be complex and the imaginary part will give a solution in the form of a decaying exponential. Under these circumstances, a wave will penetrate in effect only a finite difference into the stack. It is clear from equation (15) that if p is large enough, the value of the right-hand side can exceed 1 for a wide range of ω. The design of the filter follows from this equation and a detailed analysis of equation (15) is given in the next section.

3 Application to Filter Design

In the design of practical filters, it is essential that there be a large impedence mismatch between the two materials and this means that the parameter p in equation (15) will be very large. We will also be considering very thin layers of either material. It follows that the quantity

$$\frac{\omega b}{c_f}=\pi\left[\frac{\frac{\omega}{2\pi}}{\frac{c_f}{2b}}\right] \tag{16}$$

is the ratio of two frequencies: the first, $\omega/2\pi$, is the input frequency, and $c_f/2b$ is the natural frequency of the first thickness shear mode of the filament layer. This ratio will be very small in most realistic designs, and we thus will approximate $\sin\omega b/c_f$ by $\omega b/c_f$ and $\cos\omega b/c_f$ by 1. If $1/p$ with respect to p is neglected, then equation (15) becomes

$$\cos qa=\cos\frac{\omega(a-b)}{c_m}-\frac{1}{2}p\frac{\omega b}{c_f}\sin\frac{\omega(a-b)}{c_m} \tag{17}$$

It is convenient to rewrite this equation in the form

$$\cos qa=\cos z-\alpha z\sin z \tag{18}$$

where

$$\alpha=\frac{1}{2}\frac{\rho_f b}{\rho_m(a-b)}=\frac{1}{2}\frac{\rho_f}{\rho_m}S \tag{19}$$

$$S=\frac{b}{(a-b)} \tag{20}$$

$$z=\frac{\omega(a-b)}{c_m}=\pi\left(\frac{\frac{\omega}{2\pi}}{\frac{c_m}{2(a-b)}}\right) \tag{21}$$

In the foregoing, S is a thickness ratio (viz., the ratio of the thickness of the filament layer to that of the matrix layer), and z is essentially the ratio of the input frequency $\omega/2\pi$ to that of the natural frequency of the first thickness shear mode of the compliant layer, $c_m/2(a-b)$. For the regime of practical interest, equation (18) is the key relationship and the basis for the design of the band-stopping mechanical filters.

The design concepts involved are quite straightforward. Given that one wishes to block frequencies in a certain band, the material properties and the thicknesses of the layers making up each cell must be chosen so that the right-hand side of equation (18) has magnitude greater than 1 in the frequency band of interest. Thus, the edges of the stop bands will be given by the condition that $\cos qa=-1$ or $+1$. We see from the right-hand side of equation (18) that for small values of z (corresponding to small values of ω), $\cos qa$ will be positive and slightly less than 1. As ω (and therefore z) increase, $\cos qa$ diminishes, becoming negative, reaching the value -1, which

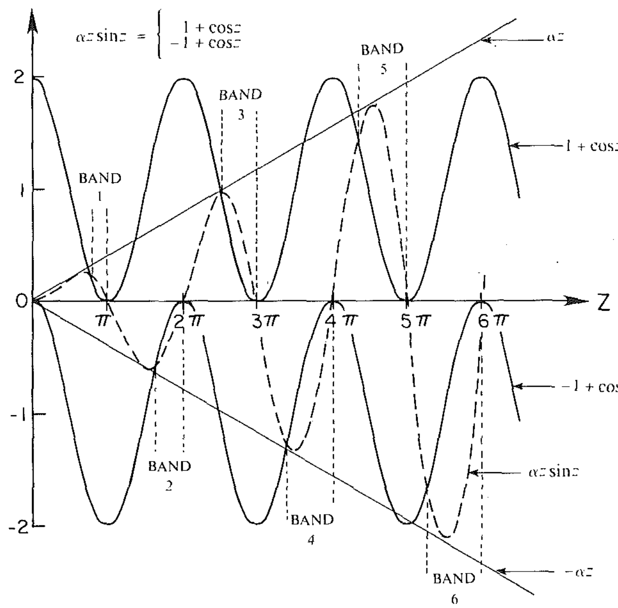

Fig. 3 Determination of stopping bands

is the start of the first stop band, and then decreasing from -1 to a minimum and increasing back to -1 when $z=\pi$, which is the end of the first stop band. As ω (and z) increase, cos qa will become positive, then attain the value $+1$, which is the start of the second stop band, then increase from $+1$ to a maximum, and then decrease back to $+1$ at $z=2\pi$, which is the end of the second stop band. In this way, an infinite sequence of stop bands is generated, each terminating at an integral multiple of π.

The start of each stop band must be determined numerically from equation (18). In practice, we would be interested in only the lowest few bands, and also designing the system in such a way that the start of the first stop band is as low as possible. This will produce the widest possible first stop band.

Considering the stop bands produced by cos $qa=-1$, the corresponding values of z are, from equation (18), given by roots of

$$\alpha z \sin z = 1 + \cos z \tag{22}$$

shown graphically in the sketch, Fig. 3. For the stop bands produced by cos $qa=+1$, we get the equation

$$\alpha z \sin z = -1 + \cos z \tag{23}$$

whose roots are also shown in Fig. 3.

The specific roots giving the beginning of each stop band may be determined numerically by a relatively simple procedure. However, for certain conditions it is possible to obtain closed-form approximations for the first root. For example, if α is small, the first band is narrow and the first root for z is close to π and a simple approximation is

$$z=\pi(1-2\alpha) \tag{24}$$

More pertinent to filter design are large values of α; here, the band is wide and the first root is approximately given by

$$z=\sqrt{2/\alpha} \tag{25}$$

Most real filter designs will involve intermediate values of α, necessitating a numerical analysis of equation (18). Examples will be given in a subsequent section.

In any real filter we will have a finite (and probably a relatively small) number of cells. It is important to know how much decay will take place as the wave propagates through a filter. For a filter of infinite extent this is given by q_1, the imaginary part of cos qa when this has a magnitude greater than 1. For a filter of finite extent, wave reflections will occur at the free surfaces at the top and bottom of the filter so that the aforementioned q_1 will no longer be the exact measure of the decay. Nevertheless, we can use this as an indicator of how effectively waves are blocked by the finite element filter when they are in a stopping band. For a fixed-frequency ω and cos $qa<-1$, we obtain

$$q=\frac{\pi}{a}+iq_I \tag{26}$$

where

$$q_I a=\cosh^{-1}[-\cos z+\alpha z \sin z] \tag{27}$$

Since $w(x,t)=v(x)e^{i(qx-\omega t)}$ and $v(x)$ is periodic with period a

$$\frac{|w(na,t)|_{\max}}{|w(0,t)|_{\max}}=e^{-nq_I a} \tag{28}$$

where n is the number of cells in the filter. We interpret the left-hand side of equation (28) as measuring the ratio of the maximum of the response at the top of the filter to the maximum of the input at the bottom of the filter. We denote this ratio as $1/R$, where R is the reduction factor associated with the filter. The number of cells required to obtain a reduction of R is thus given by

$$n=\frac{\ln R}{q_I a} \tag{29}$$

which is an important relationship for filter design. For a given choice of cell materials and geometry, the number of layers can be selected to provide the necessary degree of attenuation, and therefore reduction, at the prescribed frequency. This frequency must lie within the stop band, since when $|\cos qa|\le 1$, there is no attenuation, and hence no reduction.

The degree of attenuation for fixed n will vary with frequency within the stop band. By determining how the reduction factor R diminishes as ω approaches either edge of the stop band ($R=1$ at any edge of the stop band), we can define the *effective* range of the stop band, i.e., that range of the frequency ω within the stop band for which $R\ge R^*$, where R^* would be the desired degree of reduction for the filter. Once the filter has been designed, it is relatively easy to determine its effective range by a straightforward numerical evaluation of equation (18), also using equation (28). An illustration of this will be given later.

4 Example

To verify physically the fundamental phenomenon of interest, namely the existence of stop bands in a periodically layered medium, some experiments were performed on a block of such material. The block, a square 20.32 cm on edge in planform, was 13 cells deep. Each cell was 1.27 cm thick and consisted of a 0.953-cm-thick layer of rubber bonded to a steel layer of 0.318 cm thickness. The properties of these materials were

$$G_m=1.38\text{ MPa},\quad \rho_m=1.16\text{ gm/cm}^3$$

$$G_f=82.7\text{ GPa},\quad \rho_f=7.7\text{ gm/cm}^3$$

for the rubber and steel materials, respectively.

Before experiments were commenced, computations were performed to determine the first (lowest frequency) stop band. This required the use of the following quantities:

$$c_m = \sqrt{G_m/\rho_m}=3.45\times10^3\text{ cm per s}$$

$$c_f = \sqrt{G_f/\rho_f}=328\times10^3\text{ cm per s}$$

$$p = \frac{G_f c_m}{G_m c_f} = 631 \gg \frac{1}{p}$$

$$f_f = \frac{c_f}{2b} = 516.5 \times 10^3 \text{ Hz}$$

We expected that the lowest stop band of this system would have a frequency f of order 10^3 Hz. Therefore $f/f_f \ll 1$, so that we are justified in using the simplified equation (17), or equivalently equation (18), rather than the more complex equation (15). Thus, we have to find the lowest root z of the equation

$$\cos qa = \cos z - \alpha z \sin z = -1 \tag{30}$$

where α is given by equation (19) and in this case has the value 1.106.

A simple numerical solution of equation (30) by the elementary scheme of the method of chords gives for the lowest root, z, a value of 1.2515, which represents the start of the first stop band. As mentioned previously, the end of the first stop band is given by $z=\pi$. From equation (21), we recall that

$$f = \frac{\omega}{2\pi} = \frac{z}{\pi} f_m \tag{31}$$

where f is the frequency of vibration in Hz, and

$$f_m = \frac{c_m}{2(a-b)} \tag{32}$$

is the natural frequency of the lowest thickness shear mode of the rubber layer. This yields for the beginning of the first stop band the frequency $f_B = 722$ Hz, and for the end of the first stop band the frequency $f_E = 1808$ Hz. The frequency f_M at the middle of the first stop band is 1265 Hz. Thus, our preliminary estimate that the frequency of the first stop band would be of order 10^3 is correct.

To see how large the attenutation is when the frequency of the vibration f is within the stop band, we examine the situation at the middle of the stop band where $f = f_M = 1265$ Hz. Using (31), we compute for the right-hand side of (18) that cos $qa = -2.55$, whence, from (27) $q_I a = 1.59$.

For a filter n cells deep, this gives a reduction factor R of exp $[n\ q_I a]$ (see equation (28)). Thus, we have

n	13	7	3
R	9.48×10^8	68,200	118

Even for a filter of only 3 cells, we get a considerable reduction (over 100).

Experiments were performed for $n = 13$, then $n = 7$, and finally for $n = 3$. The beginning and end of the first stop band occurred at frequencies very close to the predicted ones given in the foregoing.

It is important to see how the quantity $q_I a$, which controls the reduction factor R, varies as we approach the edges of the stop band. As discussed previously, this will define the *effective* width of the stop band. We have determined $q_I a$ and the corresponding values of R for 3 layers (i.e., $n = 3$) at the 1/8th point of the stop band (where $f = 858$ Hz) and at the 7/8th point of the stop band (where $f = 1672$ Hz). We obtain

f(Hz)	722	858	1265	1672	1808
$q_I a$	0	1.012	1.59	1.141	0
R	1	20.8	118	30.6	1

Thus, we see that even when we are close to the edges of the stop band, a three-cell filter gives a considerable degree of reduction.

A certain amount of reduction, unaccounted for in the present theory, will also occur due to the viscoelastic nature of the rubber. This reduction due to material damping can be estimated and compared to that caused by wave trapping produced by the layering. A shear wave propagating through the rubber alone would have a displacement field given by

$$w(x,t) = e^{i(qx-\omega t)} \tag{33}$$

with

$$q = q_R + iq_D \tag{34}$$

where q_R is the real part of the wave number and q_D represents the attenuation factor due to material damping. This factor can be written in terms of the real (G_R) and imaginary (G_I) parts of the shear modulus of the rubber as

$$q_D = \frac{\omega}{\sqrt{G_R/\rho}} \sin\left(\frac{\theta}{2}\right)\left[1 + \left(\frac{G_I}{G_R}\right)^2\right]^{-1/4} \tag{35}$$

with

$$\theta = \arctan \delta \tag{36}$$

$$\delta = \frac{G_I}{G_R} \tag{37}$$

and ρ is the density of the rubber. Since $G_R \gg G_I$ for the rubber used in the bearing, we get

$$q_D = \frac{\omega\delta}{2c} \tag{38}$$

where $c = \sqrt{G_R/\rho}$ is the speed of shear waves in the rubber. For this rubber, $\delta = 0.15$, and if we consider a filter with three cells, the reduction factor, R_D, due to damping is at the midpoint frequency of the stop band (1265 Hz)

$$R_D = e^{3q_D(a-b)} = e^{(3)(.439)(.375)} = 1.64 \tag{39}$$

The reduction factor R due to wave trapping was, for 3 cells, at this frequency, 118. Clearly, material damping plays a negligible role in reducing the magnitude of the output; the major reduction is caused by wave trapping.

The block was also cut so that its dimensions in the y direction was halved, and vibration experiments were performed on this reduced block. This reduction in the planform of the block should have no effect on its performance as a filter since all planes where y is equal to a constant are traction free. Experiments appear to bear out this conclusion.

The block was then cut along the plane z equal to a constant (see Fig. 2) so that its dimension in the z direction was halved. This reduction could have an effect on its performance, since the planes for which z is a constant are not traction free. As discussed in the Introduction to this paper, edge effects will thus be present. It was the purpose of these experiments to get a preliminary estimate of the importance of edge effects.

The experimental results will be discussed in Section 7.

5 Filter Design: Trial and Error

As an example, consider first the design of a filter with a relatively low-frequency stop band. It is desired to design a filter that will block a frequency of 400 Hz. The dimensions of the filter are to be 3.81 cm × 10.16 cm in planform and 2.54 cm deep. To get an effective filter, it should be designed so that 400 Hz will fall near the middle of the first stop band.

Let use begin by assuming that a sufficiently high reduction factor will be obtained by using 3 cells. Each cell will be composed of a layer of rubber bonded to a layer of steel. Since the frequencies to be blocked are relatively low, it is desirable to make the frequency at which the first stop band begins as low as possible. Thus, the most compliant rubber available must be used. An oil-extended rubber has been produced with the following properties [10]:

$$G_m = 0.260 \text{ MPa}; \quad \rho_m = 0.93 \text{ gm/cm}^3$$

For steel, we have the following properties:

$$G_f = 82.7 \text{ GPa}; \quad \rho_m = 7.7 \text{ gm/cm}^3$$

This yields

$$c_f = 328 \times 10^3 \text{ cm per s}$$

$$c_m = 1.672 \times 10^3 \text{ cm per s}$$

$$p = 1623 \gg \frac{1}{p}$$

Note that $\omega b/c_f \approx 0.02 \ll 1$, so that use of the approximate relation (18) is justified.

For our first design attempt we chose $S = b/a - b = 1/2$. Thus, since $a = 0.847$ cm, $b = 0.282$ cm, and $a - b = 0.564$ cm. Then, $\alpha = 2.07$ and solving equation (18) with this value of α, we obtain for the lowest root z_B, $z_B = 0.945$. Then, $f_B = 0.945/\pi f_m$, $f_m = c_m/2(a-b) = 1482$ Hz, which yields the value $f_B = 446$ Hz. The 400-Hz frequency is not within this stop band, and thus the design is not appropriate.

For the second attempt, try $S = 1/3$. Going through the same foregoing procedure, $z_B = 1.136$ is obtained, which yields $f_B = 476$ Hz, a value even larger than the first attempt. Thus, the movement is in the wrong direction, and a value of S larger than 1/2 must be chosen.

In the third attempt, use $S = 1$. This leads to $z_B = 0.6815$ and $f_B = 429$ Hz, which is an improvement over the two previous designs, but is still not suitable. Therefore, again a larger value of S is required.

For the fourth attempt use $S = 3$, which leads to $z_B = 0.3986$ and $f_B = 501$ Hz, which makes this design worse than any of the previous three. However, these four designs indicated that there must be some particular value of S which, for a given cell size a, minimizes f_B. We now wish to determine that value of S and the corresponding minimum value of f_B. For our purposes, this would be the optimal filter.

6 Optimal Design

There are several ways to investigate the question of what is the optimal value of S so as to minimize f_B. We begin with the simplest approach. It is clear from equation (18) and Fig. 2, that one way to obtain a small value of f_B is to choose material properties and a value of S which make α as large as possible. This can be accomplished for values of S of order unity by choosing materials such that $\rho_f/\rho_m \gg 1$.

Assume that $\rho_f/\rho_m \to \infty$ which implies that $\alpha \to \infty$. As $\alpha \to \infty$, $z_B \to 0$. Thus,

$$f(z) = \cos z - \alpha z \sin z \to \left(1 - \frac{1}{2}z^2\right) - \alpha z^2$$

That is,

$$f(x) \to 1 - \alpha z^2$$

since $\alpha \gg 1$. The beginning of the first stop band, z_B, occurs when $f(z) = -1$. Thus

$$-1 = 1 - \alpha z^2$$

whence, $z_B = \sqrt{2/\alpha}$. Now,

$$f_B = \frac{z_B}{\pi} \frac{c_m}{2(a-b)} = \frac{z_B}{\pi} f_m^* \frac{a}{a-b}$$

where $f_m^* = c_m/2a$ is the natural frequency of vibration of the first thickness shear mode of a layer of the rubber used in the filter with a depth equal to a, the cell depth. The factor f_B may be rewritten as

$$f_B = \frac{f_m^*}{\pi}[z_B(1+S)]$$

But $z_B = \sqrt{2/\alpha}$ with $\alpha = 1/2\ \rho_f/\rho_m \times S$. Putting this into the expression for f_B

$$f_B = \left[\frac{2}{\pi}\sqrt{\rho_m/\rho_f} f_m^*\right]\left[\sqrt{S} + \frac{1}{\sqrt{S}}\right]$$

Once the materials and cell thickness have been chosen, the first bracket is a constant. The dependence of f_B on the thickness ratio S is expressed solely in the second bracket. Therefore, in order to minimize f_B, that value of $S > 0$ which minimizes the second bracket must be selected. The second bracket is a well-known function whose minimum can be found by elementary means, and it occurs when $S = 1$. This is an interesting result, and certainly not an obvious one. The optimal filter occurs when the layers of rubber and steel are of equal thickness.

However, it must be kept in mind that the preceding analysis is correct only when $\alpha \gg 1$. For our choice of materials, if we set $S = 1$, we get $\alpha = 4.14$, which is *not* $\gg 1$. Thus, we can use $S = 1$ only as a rough guide to optimal design when α is not $\gg 1$. Our four previous design attempts bear out the validity of $S = 1$ leading to a near optimal design. Of all our designs, the one which gave the lowest value for f_B was the one where $S = 1$.

The optimal design problem can be dealt with in a more general manner. Let us attempt to construct an algorithm that will lead to an optimal design for any value of $\alpha > 0$. We have

$$f_B(S) = \frac{f_m^*}{\pi} z_B(S)[1+S] \tag{39}$$

where $z_B(S)$ is the lowest root of equation (18) with $\cos qa = -1$:

$$\cos z - \frac{1}{2}\left(\frac{\rho_f}{\rho_m}\right) Sz \sin z + 1 = 0 \tag{40}$$

In the sequel, as a matter of convenience, the subscript B on z_B will be dropped. For a minimum value of f_B, we must have

$$\frac{df_B}{dS} = 0 \tag{41a}$$

which implies that

$$(1+S)\frac{dz(S)}{dS} + z(S) = 0 \tag{41b}$$

From equation (40), we get

$$\frac{dz}{dS} = \frac{-z\sin z}{S\sin z + Sz\cos z + 2\frac{\rho_m}{\rho_f}\sin z} \tag{42}$$

Substituting (42) into (41), and simplifying, the optimality condition is obtained as

$$S = [1 - 2(\rho_m/\rho_f)]\left[\frac{\tan z}{z}\right] \tag{43}$$

It is important to recall the meaning of z: it is the lowest root of equation (40).

Based on the optimality condition, equation (43), a simple iterative design procedure can now be constructed which will lead to the optimal value for S. Say that we have arrived at the ith design attempt. It is associated with the value S_i. We obtain the root z_i which is the lowest root of equation (40) with $S = S_i$. How do we choose S_{i+1} to give us a better design? We use the optimality condition, equation (43). In the right-hand side of equation (43) we substitute for z the value z_i. This then produces the value S_{i+1}. That is,

$$S_{i+1}=[1-2(\rho_m/\rho_f)]\left[\frac{\tan z_i}{z_i}\right] \qquad (44)$$

This process is continued until convergence to the optimal value of S occurs. As a partial check on the validity of this formula, consider what happens as $\rho_f/\rho_m \rightarrow \infty$, which implies that $\alpha \rightarrow \infty$. Then, equation (43) yields $S=1$, which was the result previously obtained by a direct solution.

As an illustration of the application of this more refined optimality criterion, consider the example discussed in Section 5. Let us start with case 3 (where $S_1=1$) because our crude criterion states that $S=1$ is the optimal condition. We can use this as input to the more refined criterion, equation (44). It was seen that $S_1=1$ gives $z_1=0.6815$ and $f_B=429$ Hz, which leads to $S_2=0.9027$. With this value of S_2 we obtain for z_2 the value 0.7157 and for f_B, 429 Hz. This then leads to $S_3=0.9214$, which then gives $z_3=0.7075$ and $f_B=429$ Hz. At this stage it is seen that the iterative optimization scheme has converged. Note that f_B is not very sensitive to small changes in S when S is near the optimal point. For $S=1$ or 0.9027 or 0.9214, we get $f_B=429$ Hz. There is no point in trying to seek greater accurracy. (The true properties of the rubber are not known all that accurately.) From a practical point of view, the easiest filter to manufacture would be for $S=1$, i.e., equal thicknesses of steel and rubber. (If there are slight errors in the manufacturing process, so that S is not exactly equal to 1, but close to it, then our previous results tell us that it is not critical, since when S is near 1, f_B is insensitive to small changes in S.) The lowest value of f_B attainable for the materials utilized and a cell thickness of 0.847 cm is 429 Hz. Thus, such a filter cannot be used to stop transmission of a disturbance with a frequency of 400 Hz. Therefore, we must try a larger cell size, which implies, for the fixed-depth dimension of 2.54 cm, a smaller number of cells.

Next, consider a filter with 2 cells, so that $a=1.27$ cm. Based on our previous work, we also choose $S=1$. This leads to $z_B=0.6815$, as for our previous case 3. But now this leads to $f_B=286$ Hz. It is known that z_E, the root associated with the end of the first stop band, is $z_E=\pi$, which leads to $f_E=1317$ Hz. Thus, the frequency of interest, 400 Hz, lies within the stop band, and this design is potentially acceptable.

Now, we must check to see if the reduction factor for this design is sufficiently high for practical purposes. For this design, we obtain $q_I a=0.627$. Thus, for a filter with 2 cells, we have

$$R=e^{nq_I a}=26$$

If this is not a sufficiently high reduction factor, then the filter cannot be designed successfully with this choice of materials. One would have to find more suitable materials, e.g., the stiff material should have a larger density and a larger shear wave speed while the compliant material should have a lower density and a slower shear wave speed.

It is interesting to note that if we could make the filter 1 cell thicker (i.e., 3.81 cm instead of 2.54 cm), the reduction factor R would increase to the value 132. This value would appear to be sufficient high to meet most practical requirements.

7 Experimental Results

To verify certain aspects of the foregoing theoretical results, experiments were performed on several specimens of periodically layered composites. Due to time and cost constraints, only a limited program could be pursued. The purpose of these experiments was to ascertain whether the wave-stopping phenomenon predicted for the infinite composite medium would be observed in a block, or mechanical filter, of finite dimensions. If wave blocking were observed in the finite filter, we then wanted to determine whether the frequencies of the start and end of the observed stopping band were adequately predicted by the theory based on the infinite medium.

Two composites were used in the experiments. The first was the composite described in Section 4 and the second was made of alternating steel and rubber layers, each with a thickness of 0.635 cm. The rubber employed was an oil-extended one as described in Section 5. The experiments were performed by attaching the bottom of a specimen (or filter) to the rigid base of a vibration table which was then excited, first by a sine sweep input and next by a random input. A transducer attached to the rigid base of the vibration table measured the input at the bottom of the specimen, and one attached to the top of the specimen measured the output there. These measurements were fed into a Fourier analyzer which then displayed and plotted the transmissibility of the specimen filter. In this way we could directly observe whether any wave stopping occurred.

The tests performed on the first composite were with a specimen having 13 cells and a planform of 20.32 cm × 20.32 cm. Wave stopping was observed in a frequency band close to that of the lowest predicted from the theory for the infinite medium. To assess whether decreasing the planform of the filter would obviate the wave-stopping action, the specimen was cut in half to produce a filter of 10.16 cm × 20.32 cm. The behavior of this filter was essentially the same as that of the larger one. Next, this filter was cut so as to remove 6 cells, leaving a filter of 7 cells. Tests on that specimen yielded a stopping band whose starting and ending frequencies differed little from those obtained in the previous test. This 7-cell filter was then cut to reduce its planform to a square 10.16 cm on edge. Again, tests demonstrated wave blocking with essentially the same frequency interval for the stopping band. For the final series of tests on this composite, the 7 cells of the 10.16-cm specimen were shaved to 3. Tests on this filter yielded results similar to those described in the foregoing.

Typical experimental data are shown in Figs. 4 and 5 for the 3-cell filter. The input and output oscilloscope traces under sinusoidal excitation at a frequency below that of the first stopping band are shown in Fig. 4. The output is not reduced. The input and output traces when the excitation frequency is within the stopping band of the filter are shown in Fig. 5 and the reduction in the output is marked. The transmissibility (output/input) versus the frequency of this filter are shown in Fig. 6 in which the stopping band is obvious. Within the stopping band, this filter can produce a drop in the transmissibility of as much as 40 dB.

Tests on the second composite were performed on a specimen with 3 cells and a planform of 7.62 cm by 12.7 cm. The results were similar to those for the first specimen except, of course, that different stopping bands and attenuation values were obtained. The transmissibility versus frequency for this filter is plotted in Fig. 7 where a reduction of as much as 60 dB is observed.

These results demonstrate that the fundamental wave-stopping phenomenon predicted by the theory for a mechanical filter of infinite extent composed of a periodically layered composite is not obviated when the filter is of finite extent. By reducing the filter to finite extent, the details of its behavior are modified, but its basic action is not.

8 Concluding Remarks

The study summarized in this paper indicates that it may be feasible to use periodically layered composites as mechanical filters for a variety of vibration isolation purposes. In comparison to conventional vibration isolation methods, the use of such mechanical filters would be primarily in the higher ranges of frequency.

The theory employed in this study is based on wave propagation in a periodically layered composite of infinite extent and composed of two materials. Given such a medium, the

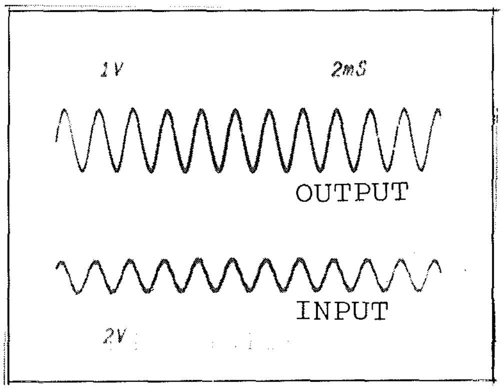

Fig. 4 Input and output for filter when frequency of vibration is below first stopping band

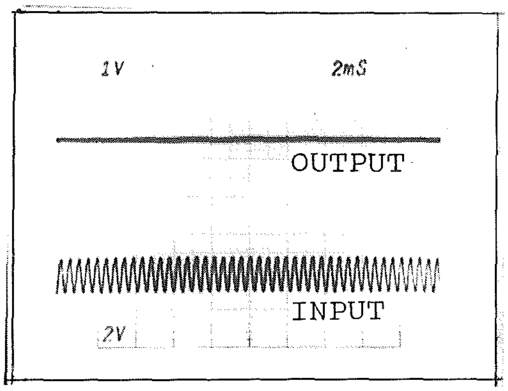

Fig. 5 Input and output for filter when frequency of vibration is within first stopping band

prediction of frequencies of the stop bands of any particular composite is quite straightforward. These stop bands define the mechanical filtering action of the medium. Theoretical predictions of the beginning and ending frequencies of stop bands appear to be corroborated by the experiments.

However, in practical applications, mechanical filters are of finite extent. This then introduces edge effects and reflections from the top and bottom face of the filter the consequences of which have not been explored analytically. Such studies appear to be nontrivial. The limited results from the small experimental program seem to indicate that these effects do not obviate the basic physical phenomenon in the layered composite. The filtering effects (i.e., the stop bands) still appear to exist as the size of the mechanical filter is reduced to reasonable dimensions. This is an area of study which needs further investigation, both analytically and experimentally.

The experiments established that the lowest stopping band of the finite filter was predicted quite well by the theory for the infinite model. Whether that would be true for the higher frequency stopping bands is an open question. Because of the limitations of the experimental equipment, higher frequency ranges could not be explored. But in practical application, the filter would be designed so that its first stopping band blocked the frequency range of interest. The designer would not normally depend on higher frequency stopping bands to filter out a frequency range which needed to be blocked.

The inverse problem, that of designing a mechanical filter to produce a desired stop band, does not appear to be difficult if it is based on a model of a periodically layered medium of infinite extent (i.e., assuming that edge effects are negligible and that reflections from the top and bottom faces of the filter are negligible). The equation upon which mechanical filter design

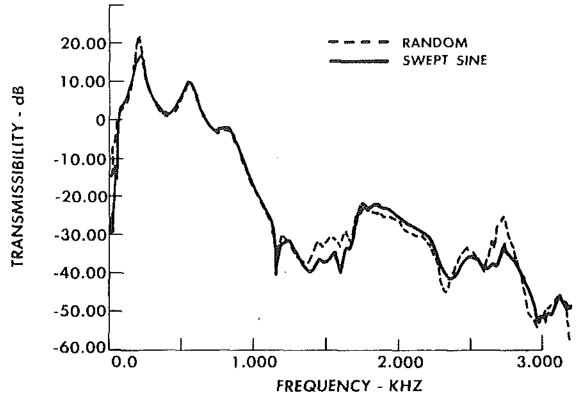

Fig. 6 Transmissibility versus frequency for first composite ($a = 1/2$ in., $b = 1/8$ in.)

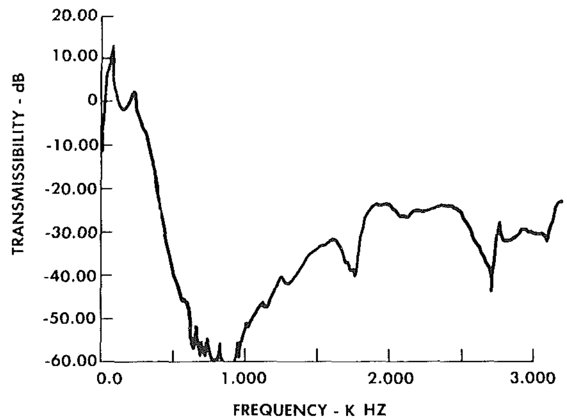

Fig. 7 Transmissibility versus frequency for second composite ($a = 1/2$ in., $b = 1/4$ in.)

is based can, for a large class of problems of interest, be simplified so that it is easy to use in practice.

Acknowledgments

The authors gratefully acknowledge the support of the IBM Corporation which enabled this research. The experiments reported herein were conducted at the IBM facility in San Jose, California.

References

1 Snowdon, J. C., *Vibration Isolation: Use and Characterization*, NBS Handbook 128, National Bureau of Standards, Washington, D.C., 1979.

2 Javid, A. E., personal communication, Apr. 1985.

3 Vincent, *Phil. Magazine*, Vol. 46, 537, 1898.

4 Brillouin, L., *Wave Propagation in Periodic Structures*, McGraw-Hill Book Company, Inc., New York, 1946.

5 Delph, T. J., Herrmann, G., and Kaul, R. K., "Harmonic Wave Propagation in a Periodically Layered, Infinite Elastic Body: Antiplane Strain," ASME *Journal of Applied Mechanics*, Vol. 45, 1978, pp. 343–349.

6 Floquet, G., *Annals of Ec. Norm.* (2), Vol. 13, 1883, p. 47.

7 Lee, E. H., and Wang, W. H.,"On Waves in Composite Materials with Periodic Structures," *SIAM Journal of Applied Mathematics*, Vol. 25, No. 3, 1973, pp. 492–499.

8 Lee, E. H., "Wave Propagation in Composites with Periodic Structures," *Proceedings of the Fifth Canadian Congress of Applied Mechanics*, G49–G59, 1975.

9 Kahrim-Panahi, K., "Antiplane Strain Harmonic Waves in Infinite, Elastic, Periodically Triple-Layered Media," *Journal of the Acoustical Society of America*, Vol. 74, No. 1, 1983, pp. 314–319.

10 Derham, C. J., et al., *New Rubber Technology*, Vol. 8, Part 3, 1977, p. 63.