

The Second Generalized Hamming Weight of Some Evaluation Codes Arising from Complete Bipartite Graphs

M. González Sarabia¹

Unidad Profesional Interdisciplinaria
en Ingeniería y Tecnologías Avanzadas
Instituto Politécnico Nacional
07340 México, D.F., MEXICO
msarabia@itesm.mx

C. Rentería M.²

Escuela Superior de Física y Matemáticas
Instituto Politécnico Nacional
07300 México, D.F., MEXICO
reneri@esfm.ipn.mx

Abstract

In this paper we compute the second generalized Hamming weight of the evaluation codes associated to complete bipartite graphs. The main result depends on the minimum distance and second generalized Hamming weight of the generalized Reed-Solomon codes.

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1 Introduction

In [20] V. K. Wei introduced a generalization of the minimum distance of a binary code as a consequence of his studies of the wire-tap channel of type II. The r th generalized Hamming weight of a linear code C is the size of the

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²Partially supported by COFFA-IPN and SNI-SEP, México.

smallest support of an r - dimensional subcode of C . Moreover, for an $[n, k]$ linear code C we define the support of C by

$$\text{supp}(C) := \{j : \text{there is an element } (x_1, \dots, x_n) \in C \text{ with } x_j \neq 0\}$$

In fact, if $1 \leq r \leq k$, the r th generalized Hamming weight of C is defined by

$$d_r(C) := \min \{|\text{supp}(D)| : D \text{ is a linear subcode of } C \text{ and } \dim(D) = r\}$$

Obviously, $d_1(C)$ is the minimum distance of the code C . In this paper we will work with the case $r = 2$ and we will compute this value in the case of some evaluation codes.

2 Evaluation Codes

Let K be a finite field with q elements, let \mathbb{P}_K^l be the l - projective space over K and $X = \{P_1, \dots, P_s\}$ be a subset of \mathbb{P}_K^l . We always use the standard representation for the points in \mathbb{P}_K^l , i.e., $P = (0, 0, \dots, 0, 1, a_i, \dots, a_l)$. Let \mathcal{L} be a finite dimensional K - linear space of functions which are defined on the set X and take values on K . Thus the evaluation map

$$\begin{aligned} ev : \mathcal{L} &\rightarrow K^s, \\ ev(f) &= (f(P_1), \dots, f(P_s)) \end{aligned}$$

defines a K -linear code: $C_X = ev(\mathcal{L})$.

Let $S = K[X_0, \dots, X_l] = \bigoplus_{d \geq 0} S_d$ be the polynomial ring over the finite field K with the natural graduation. If $\mathcal{L} = S_d$ is the d -graded homogeneous component of the polynomial ring S , the corresponding linear code $C_X(d) := ev(S_d)$ will be called the evaluation linear code over the set X , which is isomorphic to $S_d/I_X(d)$, where $I_X = \bigoplus_{d \geq 0} I_X(d)$ is the graded vanishing ideal of X . The dimension of these codes is given by the Hilbert function of S/I_X , i.e., $\dim_K C_X(d) = H_X(d)$.

This kind of codes has been studied in many particular cases (cf. [1], [2], [4], [5], [6], [7], [8], [9], [10], [14], [15], [16],[17], [18]) and their main parameters have been computed. Especific examples have been given with the help of Macaulay 2 (cf. [11]). In fact, a generating matrix of these codes can be obtained by finding a Gröbner basis for the ideal I_X , and then the cosets module $I_X(d)$ of monomials of degree d not belonging to the leading terms ideal $LT(I_X)$ of I_X , forms a K -basis for $S_d/I_X(d)$. If $B \subseteq S_d$ is this set of monomials then $(ev(h))_{h \in B}$ is a generating matrix for $C_X(d)$.

3 Toric Varieties

In this paper we will work with the case where X is a toric variety. Our definition of a toric variety agrees with the given by R.H. Villarreal in [19].

Let $A = (a_{ij})$ be a fixed $m \times (n+1)$ matrix with non negative integer entries a_{ij} and with non-zero columns. Let $K[X_0, \dots, X_n]$ and $K[t_1, \dots, t_m]$ the two polynomial rings over K , and φ the graded homomorphism of K -algebras

$$\varphi : K[X_0, \dots, X_n] \rightarrow K[t_1, \dots, t_m]$$

induced by

$$\varphi(X_i) = t_1^{a_{1i}} \dots t_m^{a_{mi}}$$

The kernel of φ , denoted by I_A , is called the toric ideal associated with the matrix A .

Remark 3.1 *When the field K is algebraically closed, we can use Macaulay 2 (cf. [11]) to compute the toric ideal I_A (cf. [3]).*

The toric variety determined by the matrix A is the subset of the projective space \mathbb{P}_K^n given by

$$X = \{(t_1^{a_{11}} \dots t_m^{a_{m1}}, \dots, t_1^{a_{1(n+1)}} \dots t_m^{a_{m(n+1)}}) \in \mathbb{P}_K^n \mid t_1, \dots, t_m \in K\}$$

Of course, we take the values of $t_1, \dots, t_m \in K$ so that they define a point in \mathbb{P}_K^n .

4 Complete Bipartite Graphs

Let $K_{m,n}$ be a complete bipartite graph (cf. [12]). The incidence matrix associated to $K_{m,n}$ is the $(m+n) \times (mn)$ matrix $B = (b_{ij})$ with $b_{ij} = 1$ if the vertex v_i and the edge a_j are incident and $b_{ij} = 0$ otherwise.

In the general case, the toric variety X associated to the incidence matrix of the complete bipartite graph $K_{m,n}$ is given by

$$X = \{(t_1 t_{m+1}, t_1 t_{m+2}, \dots, t_1 t_{m+n}, t_2 t_{m+1}, t_2 t_{m+2}, \dots, t_2 t_{m+n}, \dots, t_m t_{m+1}, t_m t_{m+2}, \dots, t_m t_{m+n} : t_i \in K^* \text{ for all } i = 1, \dots, m+n\}$$

And in fact, it can be written as

$$X = \{(1, \alpha_1, \alpha_2, \dots, \alpha_{n-1}, \beta_1, \alpha_1 \beta_1, \alpha_2 \beta_1, \dots, \alpha_{n-1} \beta_1, \beta_2, \alpha_1 \beta_2, \alpha_2 \beta_2, \dots, \alpha_{n-1} \beta_2, \dots, \beta_{m-1}, \alpha_1 \beta_{m-1}, \alpha_2 \beta_{m-1}, \dots, \alpha_{n-1} \beta_{m-1}) : \alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{m-1} \in K^*\}$$

Let $s = \#(X)$ and consider the following evaluation map

$$\begin{aligned}\theta : K[Z_{00}, \dots, Z_{(m-1)(n-1)}]_d &\rightarrow K^s \\ \theta(f) &= (f(P_1, \dots, f(P_s)))\end{aligned}$$

where $X = \{P_1, \dots, P_s\}$.

In this case, the evaluation code of order d , $C_X(d)$, associated to the incidence matrix of the complete bipartite graph $K_{m,n}$ is the image of the last evaluation map.

From now on we will use the following notation:

$$X_1 = \{(1, \alpha_1, \dots, \alpha_{n-1}) : \alpha_1, \dots, \alpha_{n-1} \in K^*\}$$

and

$$X_2 = \{(1, \beta_1, \dots, \beta_{m-1}) : \beta_1, \dots, \beta_{m-1} \in K^*\}$$

Obviously, $\#(X_1) = (q-1)^{n-1}$ and $\#(X_2) = (q-1)^{m-1}$.

Let

$$\begin{aligned}\theta_1 : K[X_0, \dots, X_{n-1}]_d &\rightarrow K^{s_1} \\ \theta_1(g) &= (g(Q_1), \dots, g(Q_{s_1}))\end{aligned}$$

where $s_1 = (q-1)^{n-1}$ and $X_1 = \{Q_1, \dots, Q_{s_1}\}$. Then $C_{X_1}(d)$ (the generalized Reed-Solomon code of order d associated to X_1) is the image of the last map and

$$C_{X_1}(d) \simeq K[X_0, \dots, X_{n-1}]_d / I_{X_1}(d)$$

In the same way, we define

$$\begin{aligned}\theta_2 : K[Y_0, \dots, Y_{m-1}]_d &\rightarrow K^{s_2} \\ \theta_2(h) &= (h(R_1), \dots, h(R_{s_2}))\end{aligned}$$

where $s_2 = (q-1)^{m-1}$ and $X_2 = \{R_1, \dots, R_{s_2}\}$. Then $C_{X_2}(d)$ (the generalized Reed-Solomon code of order d associated to X_2) is the image of the last map and

$$C_{X_2}(d) \simeq K[Y_0, \dots, Y_{m-1}]_d / I_{X_2}(d)$$

In the next section we will compute the second generalized Hamming weight in the case of the evaluation codes associated to complete bipartite graphs and it will depend on the results about generalized Reed-Solomon codes.

5 Main Result

Let ψ the Segre map (cf. [13]) restricted to the set $X_1 \times X_2 \subset \mathbb{P}_K^{n-1} \times \mathbb{P}_K^{m-1}$, and with values in \mathbb{P}_K^{mn-1} . Therefore the image of this map is X . This means that for each $P_{ij} \in X$ we can find two points $Q_i \in X_1$ and $R_j \in X_2$ such that $P_{ij} = \psi(Q_i, R_j)$ for all $i = 1, \dots, s_1$ and $j = 1, \dots, s_2$. Moreover if $\Lambda = (f(P_{11}), \dots, f(P_{s_1 s_2})) \in C_X(d)$ then

$$f(P_{ij}) = f(X_0 Y_0, \dots, X_{n-1} Y_{m-1})(Q_i, R_j)$$

for all $i = 1, \dots, s_1$ and $j = 1, \dots, s_2$. In fact let $f(X_0 Y_0, \dots, X_{n-1} Y_{m-1}) = \sum_{I,J} a_{I,J} X^I Y^J$.

Of course in this notation we are using that if $I = (i_0, \dots, i_{n-1}) \in \mathbb{N}^n$, X^I will denote the monomial $X_0^{i_0} \cdots X_{n-1}^{i_{n-1}}$ and if $J = (j_0, \dots, j_{m-1}) \in \mathbb{N}^m$, Y^J will denote the monomial $Y_0^{j_0} \cdots Y_{m-1}^{j_{m-1}}$.

For each point $Q \in X_1 (R \in X_2)$ let

$$f_Q(Y) = \sum_{I,J} a_{I,J} Q^I Y^J \in K[Y_0, \dots, Y_{m-1}]_d$$

(respectively $f_R(X) = \sum_{I,J} a_{I,J} X^I R^J \in K[X_0, \dots, X_{n-1}]_d$). The codeword $\Lambda \in C_X(d)$ can be seen as a matrix

$$\begin{pmatrix} f_{Q_1}(R_1) & \cdots & f_{Q_1}(R_{s_2}) \\ f_{Q_2}(R_1) & \cdots & f_{Q_2}(R_{s_2}) \\ \cdots & \cdots & \cdots \\ f_{Q_{s_1}}(R_1) & \cdots & f_{Q_{s_1}}(R_{s_2}) \end{pmatrix} \tag{1}$$

where the rows are elements of $C_{X_2}(d)$ and the columns are elements of $C_{X_1}(d)$.

The following theorem is the main result of this work.

Theorem 5.1 *The second generalized Hamming weight of the evaluation codes of order d associated to the incidence matrix of a complete bipartite graph is given by*

$$\delta_2(C_X(d)) = \min \{ \delta(C_{X_1}(d)) \cdot \delta_2(C_{X_2}(d)), \delta_2(C_{X_1}(d)) \cdot \delta(C_{X_2}(d)) \}$$

where $\delta(C_{X_i}(d))$ means the minimum distance of the code $C_{X_i}(d)$. Of course $\delta_2(C_{X_i}(d))$ means the second generalized Hamming weight of the code $C_{X_i}(d)$ for each $i = 1, 2$.

Proof. (A) Let C_1 be a 1–dimensional subcode of $C_{X_1}(d)$ and C_2 be a 2–dimensional subcode of $C_{X_2}(d)$. Then the tensor product of linear spaces $C_1 \otimes_K C_2$ is a 2–dimensional subcode of $C_X(d)$. Thus

$$\delta_2(C_X(d)) \leq |\text{supp}(C_1 \otimes_K C_2)| = |\text{supp}(C_1)| \cdot |\text{supp}(C_2)|$$

and therefore

$$\delta_2(C_X(d)) \leq \delta(C_{X_1}(d)) \cdot \delta_2(C_{X_2}(d))$$

In exactly the same way we can prove that

$$\delta_2(C_X(d)) \leq \delta_2(C_{X_1}(d)) \cdot \delta(C_{X_2}(d))$$

and then

$$\delta_2(C_X(d)) \leq \min \{ \delta(C_{X_1}(d)) \cdot \delta_2(C_{X_2}(d)), \delta_2(C_{X_1}(d)) \cdot \delta(C_{X_2}(d)) \} \quad (2)$$

(B) Let D be a 2–dimensional subcode of $C_X(d)$. Any codeword can be seen as a matrix of the form (1). Let D_R be the subcode of $C_{X_2}(d) \subseteq K^{s_2}$ spanned by the rows of this matrix when we consider the complete matrix corresponding to any element of D . In a similar way, let D_C be the subcode of $C_{X_1}(d) \subseteq K^{s_1}$ spanned by the columns of the same matrix.

We observe that if $\dim_K D_R = \dim_K D_C = 1$ then $\dim D \neq 2$. Therefore $\dim_K D_R \geq 2$ or $\dim_K D_C \geq 2$. If $\dim_K D_R \geq 2$ then $|\text{supp}(D_R)| \geq \delta_2(C_{X_2}(d))$. If some element of the matrix is nonzero then there are, at least, $\delta(C_{X_1}(d))$ nonzero components in the corresponding column. Thus

$$|\text{supp}(D)| \geq \delta_2(C_{X_2}(d)) \cdot \delta(C_{X_1}(d))$$

and therefore

$$\delta_2(C_X(d)) \geq \delta_2(C_{X_2}(d)) \cdot \delta(C_{X_1}(d)) \quad (3)$$

In the case that $\dim_K D_C \geq 2$ we obtain

$$\delta_2(C_X(d)) \geq \delta(C_{X_2}(d)) \cdot \delta_2(C_{X_1}(d)) \quad (4)$$

and the result follows from inequalities (2), (3) and (4). ■

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