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The Use of Parabolic Arc in Matching Curved Boundary by Point Transformations for Sextic Order Triangular Element

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Abstract

This paper is concerned with curved boundary triangular element having one curved side and two straight sides. The curved element considered here is the 28-node (sextic) triangular element. On using the isoparametric coordinate transformation, the curved triangle in the global (x, y) coordinate system is mapped into a standard triangle: $\{(\xi,\eta)/0 \le \xi,\eta \le 1, \xi+\eta \le 1\}$ in the local coordinate system (ξ,η). Under this transformation curved boundary of this triangular element is implicitly replaced by sextic arc. The equation of this arc involves parameters, which are the coordinates of points on the curved side. This paper deduces relations for choosing the parameters in sextic arc in such a way that each arc is always a parabola which passes through four points of the original curve, thus ensuring a good approximation. The point transformations which are thus determined with the above choice of parameters on the curved boundary and also in turn the other parameters in the interior of curved triangle will serve as a powerful subparametric coordinate transformation for higher order curved triangular elements with one curved side and two straight sides.

Keywords: Finite Element Method, Numerical Integration, Triangular Elements, Point transformations

1. Introduction

The finite element method applied to problems involving a enclosed region R^2, elements with straight sides, usually triangles or quadrilaterals are perfectly satisfactory, if the original domain has a polygonal boundary and suitable basis functions defined on these elements are easy to construct. However, when the problem domain is curved, elements with at least one curved side are desirable. The curved element was introduced into structural analysis by Ergatoudis et al. [10] and reference to it can be found in [2,6,11,12,13]. Mitchell [1] describes three approaches to this problem. One of these involves a transformation of the entire domain onto some standard shape and hence is really a global method as opposed to the standard finite element approach which is local. The other two methods, the isoparametric method and the direct method are local in nature. In the direct method, the basis functions are constructed to match the curved boundaries and integrations are carried out directly in the original plane. This method has been developed with some success by Wachpress [3-5] and Mcleod and Mitchell [14] for triangular elements. The main difficulty with this procedure is that the basis functions in the triangles adjacent to the curved boundary are, in all but a few special cases, no longer polynomials and so the numerical work in these triangles is correspondingly more involved. The major disadvantage of these methods lies in the fact that the basis functions are usually rational functions making the integrations much more difficult. The isoparametric method has advantage of simplicity in defining of transformation and in the fact that the basis functions are polynomials which make the numerical integration easier. In the isoparametric method a triangle with one curved side and two straight sides in global (x, y) space is mapped into a standard triangle i.e. $\{(\xi,\eta)/0 \le \xi,\eta \le 1, \xi+\eta \le 1\}$ in the local parametric space (ξ,η). When the isoparametric coordinates are used to deal with curved boundaries in the finite element method, the original boundary is implicitly replaced by parabolic, cubic arcs. The equations of these arcs involve parameters which are the coordinates of points on the curved side. McLeod and Mitchell [15] determine equations of parabolic and cubic curves using isoparametric coordinate transformations. Further, they also present a simple and systematic procedure to choose the parameters of the cubic curves so that the implicit equations of the curves always represent the parabola passing through four points of the original curves and so is a reasonable approximation to it. The development is put to practical use in the recent works of Rathod and Karim [8-9]. In the recent works of the Rathod et al. [7], they found equations for point transformation of quartic and quintic arcs using isoparametric coordinate transformations and also to choose the parameters in a systematic way so that the implied curves are always a parabola passing through four points (quartic and quintic arcs) of the original curves. It is the purpose of this paper to find equation for point transformations of sextic arc using isoparametric coordinate transformations and also to choose the parameters (coordinates of the points on the curved side) in a systematic way so that the implied curve is always a parabola passing through four points (sextic arc) of the original curve.

2. Point transformations for triangular element with one curved boundary

We consider the triangular elements in which one of the sides is curved and the other two sides are straight as shown in fig.1. The Lagrange interpolants for the field variable u (say) governing the physical problems are:

$$u = \sum_{i=1}^{(n+1)(n+2)/2} N_i^{(n)}(\xi,\eta)u_i^e \,, \quad (n = 2,3,4,5,6) \tag{1}$$

Where $n = 2$ refers to quadratic, $n = 3$ refers to cubic, $n = 4$ refers to quartic, $n = 5$ refers to quintic and $n = 6$ refers to sextic order triangular elements, and $N_i^{(n)}(\xi,\eta)$ refers to the convential triangular element shape functions of order n at the node i. These can be derived easily [7]. Hence the transformation formulae between the physical (Cartesian) and the local (natural) coordinate system are:

$$t = \sum_{i=1}^{(n+1)(n+2)/2} N_i^{(n)}(\xi,\eta)t_i \,, \qquad (t = x, y) \tag{2}$$

Now if we use the standard formulae on dividing a line segment in a given ratio from the plane analytic geometry to the straight sides 3–1 and 3–2, then the Eq. (2) reduces to

$$m^{(n)}\, t(\xi,\eta) = m^{(n)}t_3 + m^{(n)}(t_1 - t_3)\xi + m^{(n)}(t_2 - t_3)\eta + a_{11}^{(n)}(\underline{t})\xi\eta$$

$$+ H(n-3)\sum_{\substack{i+j=n \\ (i \neq j)}} a_{ij}^{(n)}\xi^i\eta^j \,, \quad (1 \le i,\ j \le n-1,\ \ n = 6),\ (t = x, y) \tag{3a}$$

Where, $\underline{t}$ is nodal values of the triangular element and H (n – 3) is the Heaviside step function or unit step function and it has the meaning for the present as

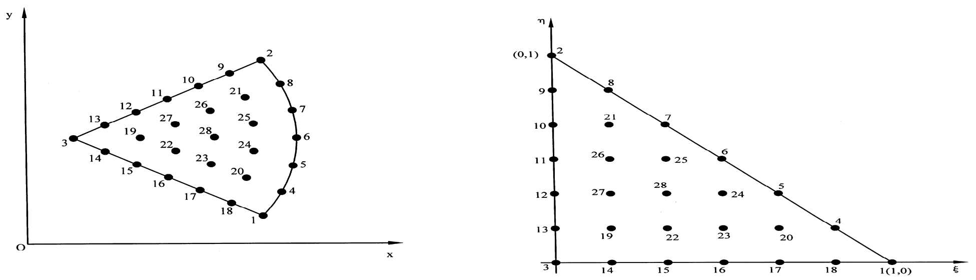
$$H(n-3) = \begin{cases} 0, & n < 3 \ \ ie\, n = 2 \\ 1, & n \ge 3 \ \ ie \ \ n = 3,4,5,6 \end{cases} \tag{3b}$$

$m^{(6)} = 10,$ for sextic curved triangular element (3c)

and in Appendix-A the coefficients are listed

$$\left(a_{11}^{(6)}(\underline{t})\right),\ \left(a_{21}^{(6)}(\underline{t}), a_{12}^{(6)}(\underline{t})\right),\ \left(a_{31}^{(6)}(\underline{t}), a_{22}^{(6)}(\underline{t}), a_{13}^{(6)}(\underline{t})\right),\ \left(a_{41}^{(6)}(\underline{t}), a_{32}^{(6)}(\underline{t}), a_{23}^{(6)}(\underline{t}),\ a_{14}^{(6)}(\underline{t})\right)$$

$$\left(a_{51}^{(6)}(\underline{t}), a_{42}^{(6)}(\underline{t}), a_{33}^{(6)}(\underline{t}),\ a_{24}^{(6)}(\underline{t}), a_{15}^{(6)}(\underline{t})\right) \tag{3d}$$



a)Unmapped sextic triangle b)mapped sextic triangle

Fig.1: Mapping of a 28-node sextic curve triangle into right isosceles triangle

3. Triangle with one parabolic boundary

The point transformation Eqs. (3a-d) will reduce to two parametric equations of the degree 6 in local variate $\xi \; or \; \eta$ along the curved boundary for which $\xi + \eta = 1$. We would now like to approximate the curved boundary of the triangle by a parabolic arc i.e. by two parametric equations for x and y by a quadratic polynomial in $\xi \; or \; \eta$. This is possible only if we neglect the higher order terms in Eq. (3a) i.e. the terms $\sum_{\substack{i+j=n \\ (i \neq j)}} a_{ij}^{(n)} \xi^i \eta^j$. Hence we may assume without loss of generality that the point transformation over the curved triangle is given by

$$t(\xi,\eta) = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + A_{11}^{(n)}\xi\eta, \quad t = x, y \qquad (n = 6) \tag{4}$$

Where $A_{11}^{(n)} = \dfrac{a_{11}^{(n)}}{m^{(n)}}$ and $m^{(n)}$ are integral constant which are defined in Eq. (3c).

4. Explicit form of point transformations and Jacobians

We note that Eq. (4) reduces to a pair of parametric equations for x and y along the curved boundary and they are quadratic polynomials, either in ξ or η. Let us assume that the given curved boundary can be approximated by a general conic [9], that is, the equation

$$f(x,y) = p_{00} + p_{10}x + p_{01}y + p_{20}x^2 + p_{11}xy + p_{02}y^2 = 0 \tag{5}$$

We have also from Eq. (4) the parametric equation along the curved boundary is of the form (say):

$$x(\xi,1-\xi) = r_0(\underline{t}) + r_1(\underline{t})\xi + r_2(\underline{t})\xi^2, \; y(\xi,1-\xi) = s_0(\underline{t}) + s_1(\underline{t})\xi + s_2(\underline{t})\xi^2 \tag{6a}$$

If we substitute from Eq. (6a) into Eq. (5), then on the curved boundary f has the form:

$$f(\xi,1-\xi) = f_0 + f_1\xi + f_2\xi^2 + f_3\xi^3 + f_4\xi^4 = 0 \tag{6b}$$

Clearly, Eq. (6b) is a polynomial in ξ, of degree four, since it has to pass through the end points of the curved boundary, $\xi = 0,1$ are definitely two of its roots. The other two roots in 0<ξ<1, determine two intermediate points on the curved boundary. Thus, we can only determine the curved boundary by a parabolic arc which passes through two intermediate points in 0<ξ<1 and two end points at ξ = 0 and ξ = 1. If we have more than two intermediate points on the parabolic arc of this curved boundary, then they will be all expressible in terms of the two intermediate points which only lie on the original curved boundary. We shall now determine the relations among the nodal points along the curved boundary, if the curved triangle has more than four nodes along the curved boundary.

Lemma: Let the point transformation for the curved triangle with one parabolic curved boundary side and two straight sides be expressible as:

$$t(\xi,\eta) = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + A_{11}^{(n)}(\underline{t})\xi\eta, \tag{7}$$

then it can be shown that: Sextic case $(n = 6)$

$$A_{11}^{(6)} = \frac{a_{11}^{(6)}}{10} = \begin{cases} (3.6)\{(t_4 + t_8) - (t_1 + t_2)\} \\ OR \\ (2.25)\{(t_5 + t_7) - (t_1 + t_2)\} \end{cases} \tag{8}$$

and $(t_4 + t_8) = \frac{1}{8}[5(t_5 + t_7) + 3(t_1 + t_2)],\ (t_5 + t_7) = \frac{1}{5}[8(t_4 + t_8) - 3(t_1 + t_2)]$

$$t_6 = \frac{1}{10}(9(t_4 + t_8) - 4(t_1 + t_2)) \tag{9}$$

Proof: The proof follows from the foregoing analysis of point transformations to match the parabolic arc discussed in section 3 of the paper and alternatively it also follows from the global to local transformation of coordinates and geometric considerations.

5. Analysis of point transformations

The triangle is spanned by a total of 28 nodes and has 7 nodes respectively along the curved side. The global coordinates (x, y) and the local coordinates (ξ, η) under the subparametric coordinate transformation which map this curved triangle fig.1(a) into isosceles right triangles are as shown in fig.1(b) and they are related by equations 3(a)-3(d) as derived in the previous section. The parametric equations of the curved side in fig.1(a) can be obtained by substituting $\eta = 1 - \xi$ in Eqs. 3(a) – 3(d). This leads to equations of the form:

$$m^{(n)}\, t(\xi, 1-\xi) = \alpha_0^{(n)}(\underline{t}) + \alpha_1^{(n)}(\underline{t})\xi + \alpha_2^{(n)}(\underline{t})\xi^2 + \ldots\ldots\ldots\ldots\ldots + \alpha_k^{(n)}(\underline{t})\xi^n, \tag{10}$$

$(\alpha_k^{(n)}(\underline{t}),\ k = 0, 1, 2, \ldots\ldots\ldots n)$ can be obtained from $a_{ij}^{(n)}(\underline{t})$ values as listed in Appendix-A. In sextic case, the curved side of the triangle is spanned by the coordinates $(t_i, i = 1,4,5,6,7,8,2)$. The point transformation for this case can be obtained from Eq. (3a). Hence, on the curved side, we obtain the following equation on substituting $\eta = 1 - \xi$ in Eq. (3a):

$$10t(\xi, 1-\xi) = \alpha_0^{(6)}(\underline{t}) + \alpha_1^{(6)}(\underline{t})\xi + \alpha_2^{(6)}(\underline{t})\xi^2 + \alpha_3^{(6)}(\underline{t})\xi^3 + \alpha_4^{(6)}(\underline{t})\xi^4 + \alpha_5^{(6)}(\underline{t})\xi^5 + \alpha_6^{(6)}(\underline{t})\xi^6 \tag{11}$$

Now, the choice for the location of points $(t_i, i = 4,5,6,7,8)$ to make the above sextic curve to reduce to a unique parabola can be achieved by setting:

$$\alpha_3^{(6)}(\underline{t}) = 0,\ \alpha_4^{(6)}(\underline{t}) = 0, \alpha_5^{(6)}(\underline{t}) = 0, \alpha_6^{(6)}(\underline{t}) = 0 \tag{12}$$

Now, Eq. (12) can be explicitly written as:

$$-a_{21}^{(6)}(\underline{t}) + a_{12}^{(6)}(\underline{t}) + a_{31}^{(6)}(\underline{t}) - 2a_{22}^{(6)}(\underline{t}) + 3a_{13}^{(6)}(\underline{t}) + a_{32}^{(6)}(\underline{t}) - 3a_{23}^{(6)}(\underline{t}) + 6a_{14}^{(6)}(\underline{t}) + a_{33}^{(6)}(\underline{t})(\underline{t}) - 4a_{24}^{(6)} + 10a_{15}^{(6)}(\underline{t}) = 0$$

$$-a_{31}^{(6)}(\underline{t}) + a_{22}^{(6)}(\underline{t}) - a_{13}^{(6)}(\underline{t}) + a_{41}^{(6)}(\underline{t}) - 2a_{32}^{(6)}(\underline{t}) + 3a_{23}^{(6)}(\underline{t}) - 4a_{14}^{(6)}(\underline{t}) + a_{42}^{(6)}(\underline{t}) - 3a_{33}^{(6)}(\underline{t})(\underline{t}) + 6a_{24}^{(6)} - 10a_{15}^{(6)}(\underline{t}) = 0$$

$$-a_{41}^{(6)}(\underline{t})+a_{32}^{(6)}(\underline{t})-a_{23}^{(6)}(\underline{t})+a_{14}^{(6)}(\underline{t})+a_{51}^{(6)}(\underline{t})-2a_{42}^{(6)}(\underline{t})+3a_{33}^{(6)}(\underline{t})-4a_{24}^{(6)}(\underline{t})+5a_{15}^{(6)}(\underline{t})=0$$
$$-a_{51}^{(6)}(\underline{t})+a_{42}^{(6)}(\underline{t})-a_{33}^{(6)}(\underline{t})+a_{24}^{(6)}(\underline{t})-a_{15}^{(6)}(\underline{t})=0 \quad (13)$$

We can equivalently express the Eq. (13) as:

$$a_{33}^{(6)}(\underline{t})=-(a_{15}^{(6)}(\underline{t})+a_{51}^{(6)}(\underline{t}))+(a_{24}^{(6)}(\underline{t})+a_{42}^{(6)}(\underline{t}))$$
$$a_{22}^{(6)}(\underline{t})=-\frac{1}{2}(a_{24}^{(6)}(\underline{t})+a_{42}^{(6)}(\underline{t}))-\frac{1}{2}(a_{32}^{(6)}(\underline{t})+a_{23}^{(6)}(\underline{t}))+\frac{3}{2}(a_{41}^{(6)}(\underline{t})+a_{14}^{(6)}(\underline{t}))$$
$$(a_{15}^{(6)}(\underline{t})-a_{51}^{(6)}(\underline{t}))+(a_{14}^{(6)}(\underline{t})-a_{41}^{(6)}(\underline{t}))+(a_{12}^{(6)}(\underline{t})-a_{21}^{(6)}(\underline{t}))+(a_{13}^{(6)}(\underline{t})-a_{31}^{(6)}(\underline{t}))=0$$
$$(a_{24}^{(6)}(\underline{t})-a_{42}^{(6)}(\underline{t}))+(a_{23}^{(6)}(\underline{t})-a_{32}^{(6)}(\underline{t}))+(a_{14}^{(6)}(\underline{t})-a_{41}^{(6)}(\underline{t}))+2(a_{12}^{(6)}(\underline{t})-a_{21}^{(6)}(\underline{t}))$$
$$+2(a_{13}^{(6)}(\underline{t})-a_{31}^{(6)}(\underline{t}))=0 \quad (14)$$

From the Eq. (3d) and Eq. (14), it can be shown that:

$$10t(\xi,\eta)=10t_3+10(t_1-t_3)\xi+10(t_2-t_3)\eta+a_{11}^{(6)}(\underline{t})\xi\eta$$
$$+\frac{1}{2}(\xi^2\eta+\xi\eta^2)(a_{21}^{(6)}(\underline{t})+a_{12}^{(6)}(\underline{t}))+\frac{1}{2}(\xi^3\eta+\xi\eta^3)(a_{31}^{(6)}(\underline{t})+a_{13}^{(6)}(\underline{t}))$$
$$+\frac{1}{2}(\xi^4\eta+\xi\eta^4+3\xi^2\eta^2)(a_{41}^{(6)}(\underline{t})+a_{14}^{(6)}(\underline{t}))$$
$$+\frac{1}{2}(\xi^2\eta^3+\xi^3\eta^2-\xi^2\eta^2)(a_{32}^{(6)}(\underline{t})+a_{23}^{(6)}(\underline{t}))$$
$$+\frac{1}{2}(\xi^4\eta^2+\xi^2\eta^4-\xi^2\eta^2+2\xi^3\eta^3)(a_{42}^{(6)}(\underline{t})+a_{24}^{(6)}(\underline{t}))$$
$$+\frac{1}{2}(\xi^5\eta+\xi\eta^5-2\xi^3\eta^3)(a_{51}^{(6)}(\underline{t})+a_{15}^{(6)}(\underline{t}))$$
$$+\frac{1}{2}(-\xi\eta^2+\xi^2\eta+2\xi^2\eta^4-2\xi^4\eta^2-\xi^5\eta+\xi\eta^5)(a_{21}^{(6)}(\underline{t})-a_{12}^{(6)}(\underline{t}))$$
$$+\frac{1}{2}(-\xi\eta^3+\xi^3\eta+2\xi^2\eta^4-2\xi^4\eta^2-\xi^5\eta+\xi\eta^5)(a_{31}^{(6)}(\underline{t})-a_{13}^{(6)}(\underline{t}))$$
$$+\frac{1}{2}(\xi^3\eta^2-\xi^2\eta^3-\xi^4\eta^2+\xi^2\eta^4)(a_{32}^{(6)}(\underline{t})-a_{23}^{(6)}(\underline{t})) \quad (15)$$

Now, we have to determine the coordinate points t_4,t_5,t_6,t_7 and t_8 along the curved boundary and also the points in the interior of the triangle viz: $t_{19},t_{20},t_{21},t_{22},t_{23},t_{24},t_{25},t_{26},t_{27}$ and t_{28}. We first note here that the equations $a_{21}^{(6)}(\underline{t})-a_{12}^{(6)}(\underline{t})=0$, $a_{31}^{(6)}(\underline{t})-a_{13}^{(6)}(\underline{t})=0$, $a_{41}^{(6)}(\underline{t})-a_{14}^{(6)}(\underline{t})=0$ and $a_{32}^{(6)}(\underline{t})-a_{23}^{(6)}(\underline{t})=0$, when used in Eq. (14), implies that, $a_{42}^{(6)}(\underline{t})-a_{24}^{(6)}(\underline{t})=0$ and $a_{51}^{(6)}(\underline{t})-a_{15}^{(6)}(\underline{t})=0$.

We then note here that the equations $a_{21}^{(6)}(\underline{t})+a_{12}^{(6)}(\underline{t})=0$, $a_{31}^{(6)}(\underline{t})+a_{13}^{(6)}(\underline{t})=0$, $a_{41}^{(6)}(\underline{t})+a_{14}^{(6)}(\underline{t})=0$, $a_{32}^{(6)}(\underline{t})+a_{23}^{(6)}(\underline{t})=0$, $a_{42}^{(6)}(\underline{t})+a_{24}^{(6)}(\underline{t})=0$ and $a_{51}^{(6)}(\underline{t})+a_{15}^{(6)}(\underline{t})=0$, when used in Eq. (14) also implies that $a_{22}^{(6)}(\underline{t})=0$, $a_{33}^{(6)}(\underline{t})=0$. Thus, to determine the 15-unknowns $t_4,t_5,t_6,t_7,t_8,t_{19},t_{20},t_{21},$ $t_{22},t_{23},t_{24},t_{25},t_{26},t_{27}$ and t_{28}, we have to use the above 14-linear equation:

$$a_{ij}^{(6)}(\underline{t})\mp a_{ij}^{(6)}(\underline{t})=0,\ ij\in\{21,31,41,32,42,51\} \text{ and } a_{ij}^{(6)}(\underline{t})=0,\ ij\in\{22,23\} \quad (16)$$

It is now clear that the use of these 14-linear equations reduces the Eq. (15) to the form:

$$10t(\xi,\eta)=10t_3+10(t_1-t_3)\xi+10(t_2-t_3)\eta+a_{11}^{(6)}(\underline{t})\xi\eta \tag{17}$$

Using the six equations: $a_{ij}^{(6)}(\underline{t})-a_{ji}^{(6)}(\underline{t})=0$ from the above set of Eq. (16), we obtain:

$$-20\alpha_{4,8}^{t}(\underline{t})-5\alpha_{5,7}^{t}(\underline{t})+132\alpha_{20,21}^{t}(\underline{t})+480\alpha_{22,27}^{t}(\underline{t})-360\alpha_{23,26}^{t}(\underline{t})+24\alpha_{24,25}^{t}(\underline{t})=15\alpha_{1,2}^{t}(\underline{t}),$$
$$42\alpha_{4,8}^{t}(\underline{t})+18\alpha_{5,7}^{t}(\underline{t})-282\alpha_{20,21}^{t}(\underline{t})-888\alpha_{22,27}^{t}(\underline{t})+756\alpha_{23,26}^{t}(\underline{t})-84\alpha_{24,25}^{t}(\underline{t})=-17\alpha_{1,2}^{t}(\underline{t}),$$
$$-4\alpha_{4,8}^{t}(\underline{t})-\alpha_{5,7}^{t}(\underline{t})+24\alpha_{20,21}^{t}(\underline{t})+60\alpha_{22,27}^{t}(\underline{t})-56\alpha_{23,26}^{t}(\underline{t})+4\alpha_{24,25}^{t}(\underline{t})=\alpha_{1,2}^{t}(\underline{t}),$$
$$0\alpha_{4,8}^{t}(\underline{t})-3\alpha_{5,7}^{t}(\underline{t})+6\alpha_{20,21}^{t}(\underline{t})+30\alpha_{22,27}^{t}(\underline{t})-28\alpha_{23,26}^{t}(\underline{t})+14\alpha_{24,25}^{t}(\underline{t})=0\alpha_{1,2}^{t}(\underline{t}),$$
$$0\alpha_{4,8}^{t}(\underline{t})+\alpha_{5,7}^{t}(\underline{t})-2\alpha_{20,21}^{t}(\underline{t})-8\alpha_{22,27}^{t}(\underline{t})+8\alpha_{23,26}^{t}(\underline{t})-4\alpha_{24,25}^{t}(\underline{t})=0\alpha_{1,2}^{t}(\underline{t}),$$
$$6\alpha_{4,8}^{t}(\underline{t})+0\alpha_{5,7}^{t}(\underline{t})-30\alpha_{20,21}^{t}(\underline{t})-60\alpha_{22,27}^{t}(\underline{t})+60\alpha_{23,26}^{t}(\underline{t})+0\alpha_{24,25}^{t}(\underline{t})=-\alpha_{1,2}^{t}(\underline{t}), \tag{18}$$

Where $\alpha_{i,j}^{t}(\underline{t})=(t_i-t_j)$ and the solution to the above set of Eq. (18) is:

$$a_{4,8}^{t}(\underline{t})=\frac{4}{6}a_{1,2}^{t}(\underline{t}),\ a_{5,7}^{t}(\underline{t})=\frac{2}{6}a_{1,2}^{t}(\underline{t}),\ a_{20,21}^{t}(\underline{t})=\frac{3}{6}a_{1,2}^{t}(\underline{t}),\ a_{22,27}^{t}(\underline{t})=\frac{1}{6}a_{1,2}^{t}(\underline{t}),$$
$$a_{23,26}^{t}(\underline{t})=\frac{2}{6}a_{1,2}^{t}(\underline{t}),\ a_{24,25}^{t}(\underline{t})=\frac{1}{6}a_{1,2}^{t}(\underline{t}) \tag{19}$$

The remaining eight linear equations are:

$$264t_3+15\beta_{1,2}^{t}-20\beta_{4,8}^{t}-17\beta_{5,7}^{t}-16t_6-1368t_{19}+156\beta_{20,21}^{t}+1072\beta_{22,27}^{t}-536\beta_{23,26}^{t}+136\beta_{24,25}^{t}-492t_{28}=0 \tag{R1}$$
$$-246t_3-17\beta_{1,2}^{t}+42\beta_{4,8}^{t}+18\beta_{5,7}^{t}+16t_6+1428t_{19}-282\beta_{20,21}^{t}-1248\beta_{22,27}^{t}+804\beta_{23,26}^{t}-132\beta_{24,25}^{t}+432t_{28}=0 \tag{R2}$$
$$12t_3+\beta_{1,2}^{t}-4\beta_{4,8}^{t}-\beta_{5,7}^{t}+0t_6-72t_{19}+24\beta_{20,21}^{t}+68\beta_{22,27}^{t}-56\beta_{23,26}^{t}+4\beta_{24,25}^{t}-12t_{28}=0 \tag{R3}$$
$$14t_3+0\beta_{1,2}^{t}+0\beta_{4,8}^{t}-3\beta_{5,7}^{t}-4t_6-108t_{19}+6\beta_{20,21}^{t}+102\beta_{22,27}^{t}-40\beta_{23,26}^{t}+26\beta_{24,25}^{t}-84t_{28}=0 \tag{R4}$$
$$-2t_3+0\beta_{1,2}^{t}+0\beta_{4,8}^{t}+\beta_{5,7}^{t}+0t_6+16t_{19}-2\beta_{20,21}^{t}-16\beta_{22,27}^{t}+8\beta_{23,26}^{t}-4\beta_{24,25}^{t}+12t_{28}=0 \tag{R5}$$
$$-10t_3-\beta_{1,2}^{t}+6\beta_{4,8}^{t}+0\beta_{5,7}^{t}+0t_6+60t_{19}-30\beta_{20,21}^{t}-60\beta_{22,27}^{t}+60\beta_{23,26}^{t}+0\beta_{24,25}^{t}+0t_{28}=0 \tag{R6}$$
$$-70t_3+0\beta_{1,2}^{t}+0\beta_{4,8}^{t}+11\beta_{5,7}^{t}+12t_6+476t_{19}-22\beta_{20,21}^{t}-416\beta_{22,27}^{t}+148\beta_{23,26}^{t}-92\beta_{24,25}^{t}+324t_{28}=0 \tag{R7}$$
$$-t_3+0\beta_{1,2}^{t}+0\beta_{4,8}^{t}+0\beta_{5,7}^{t}+t_6+9t_{19}+0\beta_{20,21}^{t}-9\beta_{22,27}^{t}+3\beta_{23,26}^{t}-3\beta_{24,25}^{t}+9t_{28}=0 \qquad \text{(R8)} \tag{20}$$

Where $\beta_{i,j}^{t}=(t_i+t_j)$. Now it can be shown that either from Eqs. (R5), (R6) and (R8) on the relation of Eq. (14): $a_{33}^{(6)}(\underline{t})=-(a_{15}^{(6)}(\underline{t})+a_{51}^{(6)}(\underline{t}))+(a_{24}^{(6)}(\underline{t})+a_{42}^{(6)}(\underline{t}))$ that the choice for t_6 is given by:

$$t_6=\frac{1}{20}(\beta_{1,2}^{t}-6\beta_{4,8}^{t}+15\beta_{5,7}^{t}) \tag{21}$$

Lets us further choose $\beta_{20,21}^{t}$ from Eq. (R6) of Eq. (20) as:

$$90\beta_{20,21}^{t}=-30t_3-3\beta_{1,2}^{t}+18\beta_{4,8}^{t}+180t_{19}-180\beta_{22,27}^{t}+180\beta_{23,26}^{t} \tag{22}$$

If we now use Eqs. (21) - (22) in (R8) of Eq. (20) then the three Eqs. (R5), (R6) and (R8) of Eq. (20) are identical. Hence, we select only one among the Eqs. (R5), (R6) and (R8). Thus, we retain Eq. (R6) and (R8). Thus, from the set of Eq. (20), we retain the five Eqs. (R1),(R2),(R3),(R4) and (R7) on using Eqs. (21)-(22) to determine the five unknowns $t_{19}, \beta^t_{22,27}, \beta^t_{23,26}, \beta^t_{24,25}$ and t_{28}. We note that $\beta^t_{20,21}$ can be then obtained from Eq. (R6), i.e. Eq. (22). These five equations, thus obtained are:

$$-132\, t_{19} + 95\,\beta^t_{22,27} - 28\,\beta^t_{23,26} + 7\,\beta^t_{24,25} - 61.5\,t_{28} = w_1,\; 72\,t_{19} - 57\,\beta^t_{22,27} + 20\,\beta^t_{23,26}$$

$$-11\beta^t_{24,25} + 36t_{28} = w_2,\; -6\,t_{19} + 5\,\beta^t_{22,27} - 2\,\beta^t_{23,26} + \beta^t_{24,25} - 3t_{28} = w_3,$$

$$-48\,t_{19} + 45\,\beta^t_{22,27} - 14\,\beta^t_{23,26} + 13\,\beta^t_{24,25} - 42\,t_{28} = w_4,$$

$$108\,t_{19} - 93\,\beta^t_{22,27} + 26\,\beta^t_{23,26} - 23\,\beta^t_{24,25} + 81\,t_{28} = w_5 \qquad (23)$$

Where

$$w_1 = \frac{1}{8}(-212\,t_3 - 9\,\beta^t_{1,2} - 16\,\beta^t_{4,8} + 29\,\beta^t_{5,7}),\; w_2 = \frac{1}{30}(380\,t_3 + 17\,\beta^t_{1,2} + 48\,\beta^t_{4,8} - 75\,\beta^t_{5,7}),$$

$$w_3 = \frac{1}{20}(-20\,t_3 - \beta^t_{1,2} - 4\,\beta^t_{4,8} + 5\,\beta^t_{5,7}),\; w_4 = \frac{1}{5}(-30\,t_3 + 15\,\beta^t_{1,2} - 6\,\beta^t_{4,8} + \beta^t_{5,7}),$$

$$w_5 = \frac{1}{3}(47\,t_3 - 15\,\beta^t_{1,2} - \beta^t_{4,8} + 6\,\beta^t_{5,7}) \qquad (24)$$

The solution to the above set of Eqs. (23)-(24) is:

$$t_{19} = \frac{1}{540}(360\,t_3 - 55\,\beta^t_{5,7} + 142\,\beta^t_{4,8} + 3\,\beta^t_{1,2}),$$

$$\beta^t_{20,21} = \frac{1}{540}(180\,t_3 - 200\,\beta^t_{5,7} + 752\,\beta^t_{4,8} - 102\,\beta^t_{1,2}),$$

$$\beta^t_{22,27} = \frac{1}{360}(360\,t_3 - 135\,\beta^t_{5,7} + 360\,\beta^t_{4,8} - 45\,\beta^t_{1,2}),$$

$$\beta^t_{23,26} = \frac{1}{360}(240\,t_3 - 165\,\beta^t_{5,7} + 480\,\beta^t_{4,8} - 75\,\beta^t_{1,2}),$$

$$\beta^t_{24,25} = \frac{1}{360}(120\,t_3 - 75\,\beta^t_{5,7} + 552\,\beta^t_{4,8} - 177\,\beta^t_{1,2}),$$

$$t_{28} = \frac{1}{540}(180\,t_3 - 145\,\beta^t_{5,7} + 448\,\beta^t_{4,8} - 123\,\beta^t_{1,2}) \qquad (25)$$

The complete solution for the sextic curved triangle can be now obtained from Eqs. (19), (21) and (25) and this is summarized as in the following:

$$t_4 - t_8 = \frac{4}{6}(t_1 - t_2),\; t_5 - t_7 = \frac{2}{6}(t_1 - t_2),\; t_6 = \frac{1}{20}(\beta^t_{1,2} - 6\,\beta^t_{4,8} + 15\,\beta^t_{5,7}),$$

$$t_{20} = \frac{1}{540}(90\,t_3 - 51\,\beta^t_{1,2} + 376\,\beta^t_{4,8} - 100\,\beta^t_{5,7}) + \frac{1}{4}(t_1 - t_2),$$

$$t_{21} = \frac{1}{540}(90\,t_3 - 51\,\beta^t_{1,2} + 376\,\beta^t_{4,8} - 100\,\beta^t_{5,7}) + \frac{1}{4}(t_1 - t_2),$$

$$t_{22} = \frac{1}{16}(8t_3 - \beta^t_{1,2} + 8\,\beta^t_{4,8} - 3\,\beta^t_{5,7}) + \frac{1}{12}(t_1 - t_2),$$

$$t_{27} = \frac{1}{16}(8t_3 - \beta^t_{1,2} + 8\,\beta^t_{4,8} - 3\,\beta^t_{5,7}) - \frac{1}{12}(t_1 - t_2),$$

$$t_{23} = \frac{1}{48}(16\,t_3 - 5\,\beta^t_{1,2} + 32\,\beta^t_{4,8} - 11\,\beta^t_{5,7}) + \frac{1}{6}(t_1 - t_2),$$

$$t_{26} = \frac{1}{48}(16\,t_3 - 5\,\beta^t_{1,2} + 32\,\beta^t_{4,8} - 11\,\beta^t_{5,7}) - \frac{1}{6}(t_1 - t_2),$$

$$t_{24} = \frac{1}{240}(40\,t_3 - 59\,\beta^t_{1,2} + 184\,\beta^t_{4,8} - 25\,\beta^t_{5,7}) + \frac{1}{12}(t_1 - t_2),$$
$$t_{25} = \frac{1}{240}(40\,t_3 - 59\,\beta^t_{1,2} + 184\,\beta^t_{4,8} - 25\,\beta^t_{5,7}) - \frac{1}{12}(t_1 - t_2),$$
$$t_{19} = \frac{1}{1080}(720\,t_3 + 6\beta^t_{1,2} + 248\,\beta^t_{4,8} - 110\,\beta^t_{5,7}),$$
$$t_{28} = \frac{1}{1080}(360\,t_3 - 246\,\beta^t_{1,2} + 896\,\beta^t_{4,8} - 290\,\beta^t_{5,7}) \quad (26)$$

We have observed in Eqs. (19)and(21) that the location of points $(t_i, i = 4,5,6,7,8)$ can be obtained from the relations:

$$t_4 - t_8 = \frac{4}{6}(t_1 - t_2),\ t_5 - t_7 = \frac{2}{6}(t_1 - t_2),\ t_6 = \frac{1}{20}[(t_1 + t_2) - 6(t_4 + t_8) + 15(t_5 + t_7)] \quad (27)$$

These relations lead to the following two solutions, which refer to symmetric location of boundary coordinates.

First solution: Let us ensure that the points t_5 and t_7 to lie on the original curved boundary, then the remaining points may lie off the original curved boundary. This is done by using equations $t_5 - t_7 = \frac{1}{3}(t_1 - t_2)$ in conjunction with the equation of the original curved boundary. Then the remaining points are determined by equations:

$$t_4 - t_8 = \frac{4}{6}(t_1 - t_2),\ t_4 + t_8 = \frac{1}{8}[5(t_5 + t_7) + 3(t_1 + t_2)],$$
$$t_6 = \frac{1}{20}[(t_1 + t_2) - 6(t_4 + t_8) + 15(t_5 + t_7)] \quad (28)$$

Second solution: Let us ensure that the points t_4 and t_8 to lie on the original curved boundary, then the remaining points may lie off the original curved boundary. This is done by using equation $t_4 - t_8 = \frac{1}{3}(t_1 - t_2)$ in conjunction with the equation of the original curved boundary. Then the remaining points are determined by equations:

$$t_5 - t_7 = \frac{2}{7}(t_1 - t_2),\ t_5 + t_7 = \frac{1}{5}[8(t_4 + t_7) - 3(t_1 + t_2)],$$
$$t_6 = \frac{1}{20}[(t_1 + t_2) - 6(t_4 + t_8) + 15(t_5 + t_7)] \quad (29)$$

Explicit form of the point transformations

Theorem: The point transformation for the curved triangular element with one curved side and two straight sides can be expressed in terms of the four points $(t_i, i = 1,2,3,4)$, $(t = x, y)$ as:

$$t(\xi,\eta) = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + \frac{n}{(n-1)}[nt_4 - ((n-1)t_2 + t_1)]\xi\eta \quad (30)$$

Where $n = 6$ for the sextic curved triangular element.

Proof: This follows from Lemma and the linear relation between the nodal coordinates along the curved boundary derived in the previous sections.

Explicit form of the Jacobians

By using the transformation Eq. (30), the Jacobian $J(\xi,\eta)$ can be expressed as:

$$J(\xi,\eta)=\frac{\partial(x,y)}{\partial(\xi,\eta)}=\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\eta}-\frac{\partial x}{\partial\eta}\frac{\partial y}{\partial\xi}, \quad J(\xi,\eta)=\alpha_0+\alpha_1\xi+\alpha_2\eta \tag{31}$$

Where $\alpha_0=(x_1-x_2)(y_2-y_3)-(x_2-x_3)(y_1-y_3)$,

$\alpha_1=(x_1-x_3)A_{11}^{(n)}(\underline{y})-(y_1-y_3)A_{11}^{(n)}(\underline{x})$, $\alpha_2=(y_2-y_3)A_{11}^{(n)}(\underline{x})-(x_2-x_3)A_{11}^{(n)}(\underline{y})$,

$$A_{11}^{(n)}(\underline{t})=\frac{n}{(n-1)}[nt_4-((n-1)t_2+t_1)], \ t=x,y \qquad (n=6) \tag{32}$$

6. Application Example

6.1 Determination of points over the curved Triangle

To determine the application of derived solutions of curved boundary triangular elements, we consider a domain consisting of the quarter ellipse defined by: $\frac{x^2}{36}+\frac{y^2}{4}=1$.The location of points along the curved boundary, which reduce the isoparametric transformation to parametric equations of the form: $t=\alpha_0^{(n)}(\underline{t})+\alpha_1^{(n)}(\underline{t})\xi+\alpha_2^{(n)}(\underline{t})\xi^2$ is discussed in the previous sections of this paper in full detail. Further, the location of the points in the interior of the curved triangle which reduce the isoparametric transformations from sextic order to the quadratic transformation is: $t(\xi,\eta)=t_3+(t_1-t_3)\xi+(t_2-t_3)\eta+A_{11}^{(n)}(\underline{t})\xi\eta, \ (n=6)$ under the subparametric concept, is also fully described in the previous sections. The determination of the points along the curved boundary of the triangle and the points located in the interior of the curved triangle is of utmost importance for us to proceed with the application of higher order curved triangular elements under the subparametric transformation. Hence, we have tabulated these points for the sextic order curved triangular element in the Table I a,b,c,d.

6.2 Determination of Arc Length for the Curved Triangle

Calculating the length of a given curve between two end points is useful in many applications. We propose to determine the arc length of the quarter ellipse (as triangular element). We have shown that the parametric equations along the curved boundary are:

$$x(\xi)=\alpha_0^{(n)}(x)+\alpha_1^{(n)}(x)\xi+\alpha_2^{(n)}(x)\xi^2, \ y(\xi)=\alpha_0^{(n)}(y)+\alpha_1^{(n)}(y)\xi+\alpha_2^{(n)}(y)\xi^2, \tag{33}$$

We can find the arc lengths from the above as:

$$s=\text{arc length}=\int_0^1\sqrt{\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dy}{d\xi}\right)^2}\,d\xi \tag{34}$$

We have described the parametric equations along the curved boundary of the ellipse (under subparametric point transformation) and the computed values of the arc length in Table II for the sextic ordered curved triangular element. We note that the theoretical (exact) value of the arc length s [7] of a quarter ellipse:

$$\frac{x^2}{36}+\frac{y^2}{4}=1 \text{ is } s=6.688222104 \quad (35)$$

We have then compared the theoretical value of s in Eq. (35) with finite element approximation of s (expressed as an integral Eq. (34)) by straight forward application of numerical integration and subparametric mapping proposed in the present paper. These findings are given in Table II. We find very good agreement in both of these values.

6.3 Determination of Center of Gravity (Centroid) of Curved Triangular Element

Mass property calculations are one of the earliest engineering applications implemented into CAD/CAM systems. One of these properties is the centroid of an area bounded by a curve. Let us consider the area A of one quadrant of the ellipse: $\frac{x^2}{6^2}+\frac{y^2}{2^2}=1$ then the centroid $(\bar{x},\bar{y})$ of the area A is given by

$$\bar{x}=\iint_A x\,dxdy \Big/ \iint_A dxdy\,,\ \bar{y}=\iint_A y\,dxdy \Big/ \iint_A dxdy \quad (36)$$

We note that the theoretical (exact) values[7]for the ellipse considered here are $\bar{x}=2.546479089$, $\bar{y}=0.848826363$, $I_x=\iint_A x\,dxdy=24$, $I_y=\iint_A y\,dxdy=8$,

$$\bar{A}=\iint_A dxdy=9.424777961 \quad (37)$$

We shall now use the subparametric point transformations and explicit form of Jacobian derived in Eqs. (36)-(37) to obtain the above physical quantities:

$$\iint_A dxdy=\frac{\alpha_0}{2}+\frac{(\alpha_1+\alpha_2)}{6} \quad (38a)$$

$$\iint_A t\,dxdy=\alpha_0[\frac{t_3}{2}+\frac{(t_1-t_3)}{6}+\frac{(t_2-t_3)}{6}+\frac{A_{11}^{(n)}(\underline{t})}{24}]$$
$$+\alpha_1[\frac{t_3}{6}+\frac{2(t_1-t_3)}{24}+\frac{(t_2-t_3)}{24}+\frac{2A_{11}^{(n)}(\underline{t})}{120}]$$
$$+\alpha_2[\frac{t_3}{6}+\frac{(t_1-t_3)}{24}+\frac{2(t_2-t_3)}{24}+\frac{2A_{11}^{(n)}(\underline{t})}{120}] \quad (38b)$$

Where, $(t=x,y)$ and $(n=6)$ for sextic order curved triangle. We can then obtain the required integrals, viz, $\iint_A x\,dxdy, \iint_A y\,dxdy$ from Eq. (38b).We have then compared the theoretical values of $\bar{x},\bar{y}$ and that of the centroid as found in Eq. (37) with finite element approximation of $\bar{x},\bar{y}$ and that of the centroid (as expressed in Eq. (36) and Eqs. (38a, b)) by explicit subparametric transformation and Jacobian proposed in the present paper. These results are tabulated in Table III, Which compare very well with the analytical results in Eq. (37).

7. Conclusions

This paper concerns the use of isoparametric coordinate transformation to deal with the curved boundaries in the finite element method. This involves the transformation of each triangle in global/physical coordinate system (x, y) with one curved side and two straight sides into a standard triangle: $\{(\xi,\eta)/0 \le \xi,\eta \le 1, \xi+\eta \le 1\}$ in the local or natural coordinate system (ξ,η). Isoparametric coordinate transformation for each curved triangle is obtained through point transformation of global (x, y) coordinates and so the original curves are implicitly replaced by sextic curve depending on the degree of parametric coordinates. It is shown in this paper to find equation of sextic curve in terms of isoparametric coordinate transformations and to choose the coordinate points on the curved sides in a systematic way so that the implied curve is always a parabola passing through four points of the original curved boundary and so is a reasonable approximation to it. We have also shown that the point transformations are expressible as

$$t(\xi,\eta) = t_3 + (t_1 - t_3)\xi + (t_2 - t_3)\eta + \frac{n}{(n-1)}[nt_4 - \{(n-1)t_2 + t_1\}]\xi\eta\,,\ (t = x, y)$$

$(n = 6)$ and the Jacobian required in the evaluation of integrals is also easily expressed as $J = \alpha_0 + \alpha_1\xi + \alpha_2\eta$. Finally we have considered an application example, which consists of the quarter ellipse: $\left\{(x, y)/x = 0, y = 0, \frac{x^2}{36} + \frac{y^2}{4} = 1\right\}$

We take this as a curved triangle in the physical coordinate system (x, y). We have demonstrated the use of point transformations to determine the points along the curved boundary of the triangle and also the points in the interior of the curved triangle. These findings are tabulated in Table I. We have next demonstrated the use of point transformation to determine the arc length of the curved boundary and this is summarized in Table-II. An additional demonstration which uses the point transformation and the Jacobian is considered. We have thus evaluated certain integrals, for example, $\iint_A t^\alpha dxdy,\ (t = x, y, \alpha = 0,1)$ and found the physical quantities like area and centroid of the curved triangular elements. These findings are tabulated in table III. We hope that this study gives us the required impetus in the use of higher order curved triangular elements under the subparametric coordinate transformation.

Table I a (First solution)

Nodal points-.	x - coordinate	y - coordinate	$\frac{x_i^2}{36}+\frac{y_i^2}{4}$
4	5.701941016	0.567313672	0.98357596
5	5.123105625	1.041035208	0.99999999
6	4.263493828	1.421164609	1.009854423
7	3.12305625	1.707701875	0.999999999
8	1.701941016	1.421164609	0.98357596
19	1.140388203	0.38012940	0.07224918
20	4.561552813	0.520517604	0.64572809
21	1.561552813	1.520517604	0.64572809
22	2.280776407	0.426925468	0.1900647
23	3.42116461	0.473721536	0.381224337
24	3.842329219	0.947443072	0.634509144
25	2.842329219	1.280776406	0.634509144
26	1.42116461	1.140388203	0.381224337
27	1.280776407	0.760258802	0.1900647
28	2.561552813	0.853850937	0.364530711

Table I b (Second solution)

Nodal points-.	x - coordinate	y - coordinate	$\frac{x_i^2}{36}+\frac{y_i^2}{4}$
4	5.741657387	0.580552462	1
5	5.186651819	1.062217273	1.029336303
6	4.334983296	1.444994432	1.044004454
7	3.186651819	1.72888394	1.029336303
8	1.741657387	1.913885796	1
19	1.148331477	0.382777159	0.073259176
20	4.59332591	0.531108636	0.65659251
21	1.59332591	1.531108637	0.65659251
22	2.296662955	0.432220985	0.193222098
23	3.444994423	0.48166481	0.387666541
24	3.889988864	0.963329621	0.652334694
25	2.889988864	1.296662955	0.652334694
26	1.444994423	1.148331478	0.387666542
27	1.296662955	0.765554318	0.193222098
28	2.5933258198	0.86444197	0.373629958

Note: We wish to note here that the third and fourth solutions are found impossible to obtain for the application example considered here, hence in the Table I c and I d we tabulate the fifth and sixth solutions as described in (26).

Table I c (Fifth solution)

Nodal points-.	x - coordinate	y - coordinate	$\frac{x_i^2}{36}+\frac{y_i^2}{4}$
4	5.785013542	0.530578125	1.000000000
5	5.256021667	0.9822583331	1.008590185
6	4.413024375	1.355040625	0.999999999
7	3.256021667	1.648925	0.974229444
8	1.785013542	1.863911458	0.957049073
19	1.157002708	0.372782291	0.071926527
20	4.6280101833	0.491129166	0.65525986
21	1.628010833	1.491129166	0.629489305
22	2.314005417	0.41223125	0.191223125
23	3.471008125	0.451680208	0.385667569
24	3.94201625	0.903360416	0.635667569
25	2.94201625	1.23669375	0.622782291
26	1.471008125	1.118346875	0.372782291
27	1.314005417	0.745564583	0.186928032
28	2.628010834	0.8244625	0.361780185

Table I d (Sixth solution)

Nodal points-.	x - coordinate	y - coordinate	$\frac{x_i^2}{36}+\frac{y_i^2}{4}$
4	5.591734375	0.595004513	0.957049073
5	4.946775	1.085340555	0.974229444
6	4.065121875	1.471008125	0.999999999
7	2.946775	1.752007221	1.008590184
8	1.591734375	1.928337847	0.999999999
19	1.118346875	0.385667569	0.071926527
20	4.4733875	0.542670277	0.6294890305
21	1.4733875	1.542670277	0.65525986
22	2.23669375	0.438001805	0.186928032
23	3.3550540625	0.490336041	0.372782291
24	3.71008125	0.980672082	0.62278229
25	2.71008125	1.314005416	0.635667568
26	1.355040625	1.157002708	0.385667569
27	1.23669375	0.771335138	0.191223124
28	2.4733875	0.87600361	0.361780184

Table II (Arc Length values)

Parametric $x(\xi), y(\xi)$ of the curved boundary and arc length for the triangle.

Triangle order/ Discretisation on Type	Location of Points (x_4, y_4) on boundary curve	Parametric of the curved boundary		Arc Length s
SexticTriangle		$x(\xi) =$	$y(\xi) =$	
First solution	(5.701941016, 0.567313672)	11.053975315ξ -5.053975315ξ^2	2-0.315341563ξ -1.684658437ξ^2	6.656077036
Second solution	(5.741657387, 0.580552462)	11.339933186ξ -5.339933 186ξ^2	2-0.22002227ξ -1.779977729ξ^2	6.700775893
Fifth solution	(5.7850 13542,	11.65097502ξ -5.65097502ξ^2	2-0.579837501ξ -1.420162499ξ^2	6.675275273
Sixth solution	(5.591734375, 0.395004513)	10.2604875ξ -4.2604875ξ^2	2-0.115967504ξ -1.884032496ξ^2	6.61679063
			Exact arc length	$s = 6.688222104$

Table III

Triangle order	Location of Points (x_4, y_4) on boundary curve	Explicit form of Parametric eqns. $x = a_{10}\xi + a_{11}\xi\eta$ $y = b_{10}\eta + b_{11}\xi\eta$	Explicit form of Jacobian $J = \alpha_0 + \alpha_1\xi + \alpha_2\eta$	$\bar{x}$ (centroid) $\bar{y}$ (centroid)	
Sextic Curved Triangle First solution	$x_4 = 5.70194101$ $y_4 = 0.56731367$	$a_{10} = 6$ $a_{11} = 5.053975313$ $b_{01} = 2$ $b_{11} = 1.684658437$	$\alpha_0 = 12$ $\alpha_1 = 10.10795062$ $\alpha_2 = 10.10795062$	$\bar{A}$	9.369316875
				I_x	23.81079506
				I_y	7.936931685
				$\bar{x}$	2.541398711
				$\bar{y}$	0.84711957
Second solution	$x_4 = 5.74165738$ $y_4 = 0.58055246$	$a_{10} = 6$ $a_{11} = 5.339933186$ $b_{01} = 2$ $b_{11} = 1.779977729$	$\alpha_0 = 12$ $\alpha_1 = 10.67986637$ $\alpha_2 = 10.67986637$	$\bar{A}$	9.559955457
				I_x	24.5808585
				I_y	8.193619595
				$\bar{x}$	2.57123152
				$\bar{y}$	0.857077172
Fifth solution	$x_4 = 5.78501354$ $y_4 = 0.53057812$	$a_{10} = 6$ $a_{11} = 5.652097502$ $b_{01} = 2$ $b_{11} = 1.4201625$	$\alpha_0 = 12$ $\alpha_1 = 8.520975$ $\alpha_2 = 11.304195$	$\bar{A}$	9.304195
				I_x	23.78014823
				I_y	7.773444383
				$\bar{x}$	2.555852304
				$\bar{y}$	0.835477371
Sixth solution	$x_4 = 5.59173437$ $y_4 = 0.59500451$	$a_{10} = 6$ $a_{11} = 4.2604875$ $b_{01} = 2$ $b_{11} = 1.88432496$	$\alpha_0 = 12$ $\alpha_1 = 11.304195$ $\alpha_2 = 8.520975$	$\bar{A}$	9.304194996
				I_x	23.32033314
				I_y	7.926716071
				$\bar{x}$	2.506432115
				$\bar{y}$	0.851950767
Exact values: $\bar{A} = 9.424777961$, $I_x = 24$, $I_y = 8$, $\bar{x} = 2.546479089$, $\bar{y} = 0.848826363$					

APPENDIX-A

$a_{11}^{(6)} =$ [-137 t_1-137t_2-1350t_3 +72 t_4 +45t_5 +40 t_{6+}45t_7 +72 t_8 +5400t_{19}-540t_{20}-540t_{21}-3600t_{22} +1800t_{23}-360t_{24}-360t_{25}+1800t_{26}-3600t_{27}+1350t_{28}],

$a_{21}^{(6)} =$45[0 t_1+15t_2 +132t_3-20t_4 -11t_5 -8 t_6 -6 t_7 -684t_{19} +144t_{20} +12t_{21}+776t_{22}-48t_{23}+80t_{24}+56t_{25} -88t_{26} +296t_{27} -246t_{28}],

$a_{12}^{(6)} =$45[15 t_1 +0t_2 +132t_3 -6t_5 -8 t_6 – 1 1 t_7 -20t_8 -684t_{19} +12t_{20} +144t_{21}+296t_{22}-8t_{23}+56t_{24}+80t_{25} -448t_{26}+776t_{27} -246t_{28}],

$a_{31}^{(6)} =$180[0 t_1- 9t_2 -61t_3 +21 t_4 +9t_5 +4t_6 +357t_{19} -141t_{20} - 534t_{22} +390t_{23} -54t_{24}-12t_{25}+12t_{26} - 90t_{27}+108t_{28}],

$a_{22}^{(6)} =$270[0 t_1+0t_2 -70t_3+11t_5 +12t_6 +11 t_7 +476t_{19}-22t_{20} -22t_{21}-416t_{22}+148t_{23}- 92t_{24} -92t_{25}+148t_{26}- 416t_{27}+324t_{28}],

$a_{13}^{(6)} =$ 180[-9 t_1+0t_2-61t_3+4t_6 +9t_7 +21t_8 +357t_{19} -141t_{21} - 90t_{22} +12t_{23} -12t_{24}-54t_{25}+390t_{26} - 534t_{27}+108t_{28}],

$a_{41}^{(6)} =$540[0 t_1 +3t_2 +18t_3 -12 t_4 -3t_5 -108t_{19} +72t_{20} +192t_{22}-168t_{23}+12t_{24}+12t_{27} -18t_{28}],

$a_{32}^{(6)} =$ 1080[0 t_1+0t_2 +21t_3 -9t_5 -6t_6 -162t_{19} +18t_{20}+198t_{22} -102t_{23}+60t_{24} +18t_{25} -18t_{26}+108t_{27}- 126t_{28}],

$a_{23}^{(6)} =$1080[0 t_1+0t_2 +21t_3 -6 t_6 -9 t_7 -162t_{19} +18t_{21}+ 108t_{22} -18t_{23}+18t_{24}+60t_{25}-102t_{26}+198t_{27}- 126t_{28}],

$a_{14}^{(6)} =$540[3t_1+0t_2+18t_3-3t_7-12t_8 -108t_{19} +72t_{21} +12t_{22}+12t_{25}-168t_{26}+192t_{27} -18t_{28}],

$a_{51}^{(6)} =$3888[0t_1+0t_2-t_3+t_4+5t_{19} -5t_{20}-10t_{22}+10t_{23}], $a_{42}^{(6)} =$9720[0 t_1+0t_2-t_3+t_5+8t_{19}-2t_{20}-12t_{22}+8t_{23}- 4t_{24}-4t_{27}+6t_{28}],

$a_{33}^{(6)} =$12960[0 t_1+0t_2-t_3+ t_6 +9t_{19}-9t_{22}+3t_{23}-3t_{24}-3t_{25}+3t_{26}-9t_{27}+9t_{28}], $a_{24}^{(6)} =$9720[0t_1+0t_2-t_3+t_7 +8t_{19}-2t_{21}-4t_{22}-4t_{25}+8t_{26}-12t_{27}+6t_{28}],

$a_{15}^{(6)} =$3888[0 t_1+ 0t_2-t_3+t_8+5t_{19}-5t_{21}+10t_{26}-10t_{27}]

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