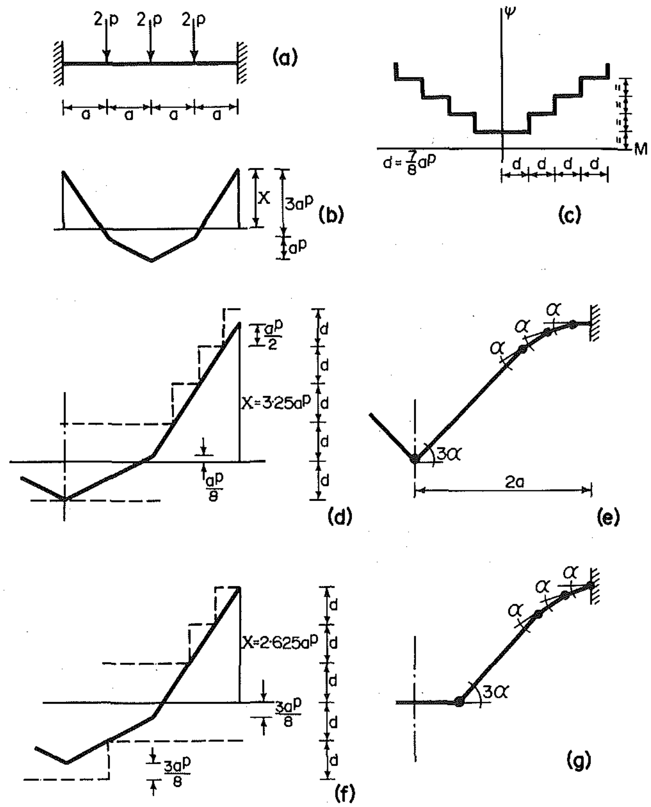

Fig. 2

tistically admissible stress field $M(X)$ can only be optimal if at moment values M_i ($i = 1, 2, \ldots, n$) the rotations given by equations (1)–(3) result in a kinematically admissible displacement field (necessary condition).

Example. For a clamped beam with three point loads, Fig. 2*(a),* the moment diagram and specific cost function are given in Figs. 2*(b,c).* Note that cost discontinuities occur at multiples of $7aP/8$. The redundant moment value χ that minimizes the cost is to be determined.

Since the slope of the moment diagram in the outer half of the beam is three times that in the inner half, equation (1) will give three times greater rotations in the inner half, the stress in the cost function being equal.

The foregoing theorem admits an infinite number of solutions corresponding to $2.625\, aP \leq \chi \leq 3.25\, aP$. The moment diagrams for the limiting cases are shown in Figs. 2*(d,f)* and the corresponding displacement fields in Figs. 2*(e,g).* It can be checked easily that solutions outside the foregoing range are both kinematically inadmissible and nonoptimal and that within that range the cost value is constant.

Remark. The proposed theorem is not related directly to Foulkes' theorem [2] and its extensions [6] because the latter preassign a specified strength distribution to *given* subsets of the structure. In the problem considered, the cost function is discontinuous but the subsets of the structure over which various regimes of the cost function apply are not preassigned. Hence the proposed extension of the Prager-Shield theory gives a more economical design than Foulkes' method.

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Nonlinear Axisymmetric Free Vibration in Simply Supported Cylindrical Shells[1]

J. H. Ginsberg[2]

Introduction

The problem of nonlinear resonant and free vibration of cylinders has received considerable attention in the published literature; a partial bibliography may be found in [1].[3] Previous investigations such as [2] and [3] have considered the case of axisymmetric motion; however they suffer from a dual weakness. First, being based upon Donnell's shell theory their validity is limited to very long axial wavelengths [4]. Equally important, the solutions obtained in these studies were not uniform asymptotic representations of the response, e.g., the boundary conditions were not fully satisfied and some significant terms were omitted from the expressions for the displacement [5].

This analysis of undamped axisymmetric free vibration follows the method of [5] in presenting a uniform asymptotic solution, for which the only assumption necessary is that transverse shear and rotatory inertia are negligible.

Formulation

The length of the shell is L and the radius is R; let $\xi = x/L$ and θ be nondimensional cylindrical coordinates. The axial and radial components of displacement of a point on the middle surface are u and w, respectively, and there is no circumferential displacement for axisymmetric motions. The expressions for the strain energy V and kinetic energy T are detailed in [5] and are not repeated here, except to note that

$$V = V_{\text{membrane}} + V_{\text{bending}} \tag{1}$$

and that only V_{membrane} contains terms of higher order than quadratic.

The boundary conditions for the simply supported shell are

$$w = w,_{\xi\xi} = N_\xi = 0 \text{ at } \xi = \pm 1/2, \quad u = 0 \text{ at } \xi = 0 \tag{2}$$

where N_ξ is the axial stress resultant referred to the deformed configuration. This requirement on N_ξ leads to the following nonlinear boundary condition for the displacements [5]:

$$\frac{1}{R}\left[\frac{R}{L}u,_\xi + \nu w\right] + \frac{R^2}{2L^2}\left[(3u,_\xi{}^2 + w,_\xi{}^2) + \nu w^2 + \nu\frac{R}{L}u,_\xi w\right] = 0 \text{ at } \xi = \pm\frac{1}{2} \tag{3}$$

If the displacement components are expressed as infinite sums of the axisymmetric modes of linear nontorsional free vibration, only the linear portion of (3) will be satisfied. To satisfy the non-

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[3] Numbers in brackets designate References at end of Note.

Manuscript received by ASME Applied Mechanics Division, August, 1973; final revision, October, 1973.

linear portion a term $\hat{u}$ is added to the expression for u. Thus, in the case of displacements which are symmetric about $\xi = 0$,

$$u = \hat{u} + \sum_{j\,\text{odd}}^{\infty} \sum_{k=1}^{2} \alpha_{0,j,k} c_{0,j,k}(\tau) \sin j\pi\xi$$
$$w = \sum_{j\,\text{odd}}^{\infty} \sum_{k=1}^{2} \gamma_{0,j,k} c_{0,j,k}(\tau) \cos j\pi\xi \quad (4)$$

where τ is the nondimensional time Ωt and a dot will indicate a derivative with respect to τ. Equation (4) follows the notation of [5]. Axisymmetric motion is indicated by the first subscript being zero and the value k denotes the particular mode of nontorsional motion corresponding to the axial wave number j. For the case of antisymmetric motion about $\xi = 0$ the sum over j should extend over the even values $j \geq 2$. The values $\alpha_{0,j,k}$ and $\gamma_{0,j,k}$ are the modal amplitudes for axisymmetric free vibration, normalized so that

$$\alpha_{0,j,k}^{2} + \gamma_{0,j,k}^{2} = 1 \quad (5)$$

The linear analog of the nonlinear problem considered herein is harmonic free vibration at the natural frequency $\Omega_{0,m,p}$ (m and p are fixed values). To establish this correspondence the perturbation parameter ϵ is introduced by the initial condition

$$c_{0,m,p}(\tau = 0) = \epsilon, \quad \dot{c}_{0,m,p}(\tau = 0) = 0 \quad (6)$$

It is then required that

$$c_{0,j,k} \leq 0(\epsilon^2) \text{ unless } j = m \text{ and } k = p \quad (7)$$

and

$$c_{0,j,k}(\tau + 2\pi) = c_{0,j,k}(\tau) \quad (8)$$

Condition (7) allows for the determination of a function $\hat{u}$ which, in conjunction with (4), satisfies the boundary conditions to $0(\epsilon^2)$, specifically

$$\hat{u} = -\frac{m\pi R}{4L} \alpha_{0,m,p}{}^{2} c_{0,m,p} \sin 2m\pi\xi + 0(\epsilon^3) \quad (9)$$

The displacements (4) and (9) are substituted into the equations for the strain and kinetic energies, and requirement (7) is utilized to retain only terms in the energy expressions which are $0(\epsilon^4)$ or larger. The result of this substitution is

$$V = \frac{\pi ERhL}{2(1-\nu^2)} \Big[\sum_{j\,\text{odd}}^{\infty} \sum_{k=1}^{2} \left(\omega_{0,j,k}{}^{2} c_{0,j,k}{}^{2} + K_{0,j,k} C_{0,m,p}{}^{2} C_{0,j,k} \right) + K_1 c_{0,m,p} \Big]$$

$$T = \frac{\pi ERhL}{2(1-\nu^2)} \omega^2 \Big\{ \sum_{j\,\text{odd}}^{\infty} \sum_{k=1}^{2} \big[(\dot{c}_{0,j,k})^2 + M_{0,j,k} c_{0,m,p} \dot{c}_{0,m,p} \dot{c}_{0,j,k} \big] + M_1 c_{0,m,p}{}^{2} (\dot{c}_{0,m,p})^2 \Big\} \quad (10)$$

where

$$\omega = \left[\frac{(1-\nu^2)\rho R^2}{E} \right]^{1/2} \Omega$$

$$\omega_{0,j,k} = \left[\frac{(1-\nu^2)\rho R^2}{E} \right]^{1/2} \Omega_{0,j,k} \quad (11)$$

The terms in (10) associated with $K_{0,m,p}$ and $M_{0,m,p}$ are $0(\epsilon^3)$, while all other terms are either $0(\epsilon^2)$ or $0(\epsilon^4)$. These terms result in a quadratic nonlinearity in the equations for $c_{0,m,p}$ as well as the usual cubic nonlinearity.

The equations of motion are obtained from Lagrange's equations. To solve the equations of motion, the perturbation technique of Lindstedt is used. Let

$$c_{0,m,p} = \epsilon c_{0,m,p}{}^{(1)} + \epsilon^2 c_{0,m,p}{}^{(2)} + \epsilon^3 c_{0,m,p}{}^{(3)} + \ldots$$
$$c_{0,j,k} = \epsilon^2 c_{0,j,k}{}^{(2)} + \ldots \; ; \; k \neq p \text{ if } j = m$$
$$\omega = \omega_{0,m,p}(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots) \quad (12)$$

The foregoing are then substituted into the equations of motion and each perturbation step is solved such that the initial condition (6) and periodicity condition (8) are satisfied. The solutions for the first two steps are

$$c_{0,m,p}{}^{(1)} = \cos\tau$$
$$\omega_1 = 0$$
$$c_{0,m,p}{}^{(2)} = \frac{1}{4\omega_{0,m,p}{}^{2}} \big[(\omega_{0,m,p}{}^{2} M_{0,m,p} - 3K_{0,m,p}) + 2K_{0,m,p} \cos\tau - (\omega_{0,m,p}{}^{2} M_{0,m,p} - K_{0,m,p}) \cos 2\tau \big]$$
$$c_{0,j,k}{}^{(2)} = -\frac{K_{0,j,k}}{4\omega_{0,j,k}{}^{2}} + \frac{(2\omega_{0,m,p}{}^{2} M_{0,j,k} - K_{0,j,k})}{4(\omega_{0,j,k}{}^{2} - 4\omega_{0,m,p}{}^{2})} \cos 2\tau \quad (13)$$

A secular term is found to occur in the equation for $c_{0,m,p}$ at the third perturbation step and the value ω_2 which eliminates this term is determined. Hence the detuning factor is

$$\frac{\omega}{\omega_{0,m,p}} - 1 = \omega_2 \epsilon^2 \quad (14)$$

This result is analogous to that found for free vibration of a simple one-degree-of-freedom oscillator whose spring has linear, quadratic, and cubic characteristics. The coefficients appearing in the foregoing equation depend upon the values of L/R, H/R, ν, m, and p. Typical values of $\omega_{0,m,p}$ and ω_2 are given in Table 1 for the common case $m = 1$.

Table 1 Coefficients when $h/R = 0.01$, $\nu = 0.3$, and $m = 1$

	Fundamental Frequency ($p = 1$)				
L/R	1	2	4	8	16
$\omega_{0,1,1}$	0.949	0.928	0.711	0.372	0.187
ω_2	-11.295	0.129	-0.209	-0.130	-0.031
ω_2($c_{0,m,p}$ only)	10.994	0.044	-0.203	-0.099	-0.026
	Higher Frequency ($p = 2$)				
L/R	1	2	4	8	16
$\omega_{0,1,2}$	3.157	1.614	1.055	1.008	1.002
ω_2	-13.000	-3.057	-1.292	-1.162	-1.115
ω_2($c_{0,m,p}$ only)	-6.494	-1.446	-0.506	-0.542	-0.537

Discussion of Results

The first point of consideration is whether the Galerkin procedure of [2, 3] can yield accurate results for this problem. These two studies in effect retained only $c_{0,m,p}$ in the determination of the axisymmetric response. The present analysis was used to check this assumption by letting $c_{0,m,p}$ be the only nonzero generalized coordinate. As can be seen from Table 1, this procedure can be expected to yield reasonable results only in the case of vibration of long shells at their fundamental frequency.

It is useful to make a comparison with the results for nonaxisymmetric motion [5]. The first noteworthy feature is that the magnitude of ω_2 is significantly lower for axisymmetric motion. Also, it was found here that satisfying the nonlinear portion of the boundary condition (6) is less important than it is for nonaxisymmetric motion. Finally, nonaxisymmetric motions did not show any quadratic nonlinearity. This effect arises because in the nonlinear theory the increase in the shell diameter at any instant is $0(\epsilon)$ in the case of axisymmetric motion, whereas it is $0(\epsilon_2)$ for non-axisymmetric motion. Similarly, quadratic nonlinearities were found to occur in the case of prestressed shells [6], where the diametral change is also of first-order significance.

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The Effect of Pulse Duration on the Transient Response of Cylindrical Shells Subjected to Axial Impact

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Introduction

Wave propagation in isotropic cylindrical shells has been the subject of numerous papers. When studied together, these papers describe the response of cylindrical shells having a range of thickness-to-radius ratios, subjected to pulses of various shapes (e.g., sine squared, triangular, rectangular, step) and a spectrum of pulse durations. In addition the analyses described in these papers involved the solution of the governing equations of different theories (e.g., uniaxial, membrane, or bending) by various mathematical techniques (transforms, finite difference, method of characteristics, modal, etc.). No major conclusions regarding the effects of pulse shape and duration on the choice of shell theory needed for a transient analysis can be gleaned from these papers. In addition, there is a need to understand the impact response of a shell, as a function of the thickness to equivalent pulse wavelength ratio.

This Note presents a portion of a study designed to respond to some of the previously mentioned needs. The effect of pulse duration on cylindrical shell response to impact loading, together with the adequacy of the uniaxial, membrane, and bending theories to handle pulses of different durations will be presented. In this study, the cylindrical shell governing equations of motion for each of the three theories are solved, in conjunction with axial impact boundary conditions, by the method of characteristics. All conclusions and discussions are based on this type of analysis.

Equations of Motion

In this study, three systems of equations of motion for a cylindrical shell were used. The equations, as presented herein, have been nondimensionalized with respect to h such that

$$\bar{u} = \frac{u}{h}, \bar{\psi} = \psi, \bar{w} = \frac{w}{h}, \bar{x} = \frac{x}{h}; \bar{\tau} = \frac{tc_p}{h}$$

where u, ψ, and w are the axial, rotary, and radial displacements, respectively; h, x, and t are the shell thickness, axial coordinate, and time, respectively, and c_p is the plate velocity, $[E/\rho\ (1 - \nu^2)]^{1/2}$; ν is Poisson's ratio and ρ is the mass density.

The first system of equations, hereafter referred to as the bending theory, includes bending, transverse shear, and rotary inertia effects and is given by [1][3]

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - \frac{\partial^2 \bar{u}}{\partial \bar{\tau}^2} = -\nu \frac{h}{R}\frac{\partial \bar{w}}{\partial \bar{x}}$$

$$\frac{\partial^2 \bar{\psi}}{\partial \bar{x}^2} - \frac{\partial^2 \bar{\psi}}{\partial \bar{\tau}^2} = \left(\frac{h}{R}\right)^2 \frac{g}{\eta(1-\eta)}\bar{\psi} + \left(\frac{h}{R}\right)^2 \frac{(g+\eta\nu)}{\eta(1-\eta)}\frac{\partial \bar{w}}{\partial \bar{x}}$$

$$\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \left(\frac{c_p}{c_s}\right)^2 \frac{\partial^2 \bar{w}}{\partial \bar{\tau}^2} = \left(\frac{h}{R}\right)\frac{\nu}{g}\frac{\partial \bar{u}}{\partial \bar{x}} - \left(1 + \frac{\eta\nu}{g}\right)\frac{\partial \bar{\psi}}{\partial \bar{x}} + \left(\frac{h}{R}\right)^2 \frac{(1+\eta)}{g}\bar{w} \quad (1)$$

The second system of equations results from a modified membrane theory (transverse shear effect included) and is given by [2]

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - \frac{\partial^2 \bar{u}}{\partial \bar{\tau}^2} = -\nu \frac{h}{R}\frac{\partial \bar{w}}{\partial \bar{x}}$$
$$\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \left(\frac{c_p}{c_s}\right)^2 \frac{\partial^2 \bar{w}}{\partial \bar{\tau}^2} = \left(\frac{h}{R}\right)\frac{\nu}{g}\frac{\partial \bar{u}}{\partial \bar{x}} + \left(\frac{h}{R}\right)^2 \frac{(1+\eta)}{g}\bar{w} \quad (2)$$

The final system of equations is the simple uniaxial theory and is given by

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} - \frac{\partial^2 \bar{u}}{\partial \bar{\tau}^2} = 0 \quad (3)$$

where R is the radius of the midsurface of the shell, c_s is the shear velocity, $k\ (G/\rho)^{1/2}$, and k^2 is the shear correction factor; $\eta = h^2/12R^2$, $g = k^2(1 - \nu)/2$.

Analysis

Each of the three systems of equations, equations (1)–(3) are completely hyperbolic and are amenable to solution by the method of characteristics. A computer code, MCDIT 21 [3], based on the method of characteristics, was used to obtain the transient solution of each of these systems. Axial velocity impact boundary conditions were utilized at the impacted end of the semi-infinite cylindrical shell and are given by

Bending Theory:

$$\frac{\partial \bar{u}}{\partial \bar{\tau}}(0,\bar{\tau}) = \sin \pi\left(\frac{\bar{\tau}}{\bar{\tau}_0}\right); \quad 0 \leq \bar{\tau} \leq \bar{\tau}_0$$
$$\bar{N}_x(0,\bar{\tau}) = 0; \quad \bar{\tau}_0 < \bar{\tau} \quad (4)$$
$$\bar{Q}_x(0,\bar{\tau}) = \bar{M}_x(0,\bar{\tau}) = 0; \quad 0 \leq \bar{\tau}_0$$

Modified Membrane Theory:

$$\frac{\partial \bar{u}}{\partial \bar{\tau}}(0,\bar{\tau}) = \sin \pi\left(\frac{\bar{\tau}}{\bar{\tau}_0}\right); \quad 0 \leq \bar{\tau} \leq \bar{\tau}_0$$
$$\bar{N}_x(0,\bar{\tau}) = 0; \quad \bar{\tau}_0 < \bar{\tau} \quad (5)$$
$$\bar{Q}_x(0,\bar{\tau}) = 0; \quad 0 \leq \bar{\tau}_0$$

Uniaxial Theory:

$$\frac{\partial \bar{u}}{\partial \bar{\tau}}(0,\bar{\tau}) = \sin \pi\left(\frac{\bar{\tau}}{\bar{\tau}_0}\right); \quad 0 \leq \bar{\tau} \leq \bar{\tau}_0$$
$$\bar{N}_x(0,\bar{\tau}) = 0; \quad \bar{\tau}_0 < \bar{\tau} \quad (6)$$

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Manuscript received by ASME Applied Mechanics Division, May, 1973; final revision, October, 1973.

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