ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.9(2010) No.4,pp.444-447

Differential Transform Method for Quadratic Riccati Differential Equation

J. Biazar[1] *, M. Eslami[2]
[1,2]Department of Mathematics, Faculty of Sciences, University of Guilan
P.O. Box 413351914 P.C. 4193833697, Rasht, Iran
(Received 4 January 2010, accepted 9 March 2010)

Abstract: In this article differential transform method (DTM) is considered to solve quadratic Riccati differential equation. The results derived by differential transform method will be compared with the results of homotopy analysis method and Adomian decomposition method. It would be shown that this method used for quadratic Riccati differential equation is more effective and promising than homotopy analysis method and Adomain decomposition method. An efficient recurrence relation for solving these equations will be obtained.

Keywords: differential transform method; Riccati differential equation

1 Introduction

The concept of the differential transform was first proposed by Zhou [1], and its main applications therein are to solve both linear and nonlinear initial value problems in electric circuit analysis. This method constructs a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for high order equation. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. It can be said that Differential transform method is a universal one, and is able to solve various kinds of functional equations. For example, it was applied to two point boundary value problems [10], to differential-algebraic equations [11], to the KdV and mKdV equations [12], to the Schrdinger equations [13] and to fractional differential equations [14]. In this paper, an analytical solution for the Riccati differential equation [2] will be discussed using differential transform method

$$\frac{dy}{dt} = Q(t)y + R(t)y^2 + P(t), \quad y(0) = G(t), \tag{1}$$

where $Q(t), R(t)\,, P(t), G(t)$ are known scalar functions.

The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676-1754). The book of Reid [3] contains the fundamental theories of Riccati equation, with applications to random processes, optimal control, and diffusion problems. Beside important engineering and science applications that today are known as the classical proved, such as stochastic realization theory, optimal control, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics [4, 5]. The solution of this equation can be reached using classical numerical methods such as the forward Euler method and Runge-Kutta method. An unconditionally stable scheme was presented by Dubois and Saidi [6]. El-Tawil et al. [7] presented the usage of Adomian decomposition method (ADM) to solve the nonlinear Riccati in an analytic form. Very recently, Tan and Abbasbandy [8] employed the analytic technique called Homotopy Analysis Method (HAM) to solve a quadratic Riccati equation. Abbasbandy [9] solved one example of the quadratic Riccati differential equation (with constant coefficient) by He's variational iteration method by using Adomian's polynomials. We think that, this paper can be used to convey to students the idea that the DTM is a powerful tool for approximately solving linear and nonlinear problems.

* **Corresponding author**. *E-mail address*: jafar.biazar@gmail.com(J. Biazar), eslami_mostafa@yahoo.com(M. Eslami)

IJNS.2010.06.30/371

2 Basic idea of differential transform method

The basic definitions and fundamental operations of differential transform are given in [1, 10-14]. For convenience of the reader, we will present a review of the differential transform method. The differential transform of the derivative of a function is defined as follows

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0}, \tag{2}$$

where $u(x)$ is the original function and $U(k)$ is the transformed function. The inverse differential transform of$U(k)$ is difined as

$$u(x) = \sum_{k=0}^{\infty} U(k)(x-x_0)^k, \tag{3}$$

in a real application, and when x_0 is taken 0, then the function $u(x)$ is expressed by a finite series and Eq. (3) can be written as

$$u(x) = \sum_{k=0}^{\infty} U(k)x^k. \tag{4}$$

The fundamental mathematical operations performed by one-dimensional differential transform method can be readily obtained and are listed in Table 1.

Table 1: The fundamental mathematical operations

Original function	Transformed function
$u(x) = f(x) \pm g(x)$	$U(k) = G(k) \pm H(k)$
$u(x) = \lambda g(x)$	$U(k) = \lambda G(k)$
$u(x) = \partial g(x)/\partial x$	$U(k) = (k+1)G(k+1)$
$u(x) = \partial^m g(x)/\partial x^m$	$U(k) = (k+1)\ldots(k+m)G(k+m)$
$u(x) = x^m$	$U(k) = \delta(k-m) = \begin{cases} 1 & k=m \\ o & otherwise \end{cases}$
$u(x) = f(x)g(x)$	$U(k) = \sum_{r=0}^{k} F(r)G(k-r)$
$u(x) = f_1(x)f_2(x)\ldots f_m(x)$	$U(k) = \sum_{k_{m-1}=0}^{k} \ldots \sum_{k_1=0}^{k_2} F_1(k_1)F_2(k_2-k_1)\ldots F_m(k-k_{m-1})$

3 Numerical examples

To illustrate the ability and reliability of the method for the quadratic Riccati differential equation some examples are provided. The results reveal that the method is very effective and simple.

Example 1. Consider the following equation [7]

$$\frac{dy}{dt} = -y^2(t) + 1, \tag{5}$$

subject to the initial condition

$$y(0) = 0. \tag{6}$$

With the exact solution

$$y(t) = \frac{e^{2t}-1}{e^{2t}+1}.$$

The Taylor expansion of $y(t)$ about $t = 0$ gives

$$y(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 - \cdots.$$

Taking the differential transform of (5), leads to

$$(k+1)Y(k+1) = -\sum_{r=0}^{k} Y(r)Y(k-r) + \delta(k). \tag{7}$$

From the initial condition given by Eq. (6) we have

$$Y(0) = 0. \tag{8}$$

Substituting Eq. (7) into Eq. (8) and by recursive method, the results are listed as follows

$$Y(1) = 1,$$
$$Y(2) = 0,$$
$$Y(3) = -\frac{1}{3},$$
$$Y(4) = 0,$$
$$Y(5) = \frac{2}{15},$$
$$\vdots$$

Therefore, the closed form of the solution can be easily written as

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 - \ldots = \frac{e^{2t}-1}{e^{2t}+1}.$$

Which is the exact solution.

Example 2. Consider the following quadratic Riccati differential equation

$$\frac{dy}{dt} = 2y(t) - y^2(t) + 1, \tag{9}$$

subject to the initial condition

$$y(0) = 0. \tag{10}$$

The exact solution was found to be [7]

$$y(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2t} + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right).$$

The Taylor expansion of $y(t)$ about $t = 0$ gives $y(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \cdots$. Taking the differential transform of (9), we have

$$(k+1)Y(k+1) = 2Y(k) - \sum_{r=0}^{k} Y(r)Y(k-r) + \delta(k). \tag{11}$$

From initial condition given by Eq. (10)

$$Y(0) = 0. \tag{12}$$

Substituting Eq. (12) into Eq. (11), and by recursive method, the results are listed as follows

$$Y(1) = 1,$$
$$Y(2) = 1,$$
$$Y(3) = \frac{1}{3},$$

$$Y(4) = \frac{-1}{3},$$
$$Y(5) = \frac{-7}{15},$$
$$\vdots$$

Therefore, the form of the solution can be easily written as

$$y(t) = \sum_{k=0}^{\infty} Y(k)t^k = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 - \frac{7}{45}t^6 + \frac{53}{315}t^7 + \ldots = 1 + \sqrt{2}\tanh\left(\sqrt{2t} + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})\right).$$

Which is the exact solution.

4 Conclusion

In this paper, differential transform method is proposed for solving quadratic Riccati differential equation. The small size of computation in comparison with the computational size required in numerical methods [7-9], and the rapid convergence show that the method is reliable and introduces a significant improvement in solving differential equations over existing method. In comparison with A.D.M the main advantage of the DTM is that this method provides the solution of the problem without the need to calculate Adomian's polynomials [7]. This method is apt to be utilized as an alternative approach to current techniques being employed to a wide variety of physical problems. Computations in this paper are performed using Maple 12.

References

[1] J. K. Zhou. Differential transformation and its application for electrical circuits. *Huarjung University Press*, Wuuhahn, China,(in Chinese). 1986.

[2] A. D. Polyanin and V. F. Zaitsev. Handbook of exact solutions for ordinary differential equations. Florida: CRC. 2003.

[3] W. T. Reid. Riccati differential equations (Mathematics in science and engineering, New York: Academic Press. 86:1972.

[4] B. D. Anderson and J. B. Moore. Optimal control-linear quadratic methods. Prentice-Hall, New Jersey. 1999.

[5] I. Lasiecka and R.Triggiani. Differential and algebraic Riccati equations with application to boundary/point control problems: continuous theory and approximation theory. (Lecture notes in control and information sciences, Berlin: Springer. 164:1991.

[6] F. Dubois and A. Saidi. Unconditionally stable scheme for Riccati equation. *ESAIM Proceedings*, 8(2000):39-52.

[7] A. A. Bahnasawi, M. A. El-Tawil and A. Abdel-Naby. Solving Riccati differential equation using Adomian's decomposition method. *Appl. Math. Comput.,* 157(2004):503-514.

[8] Y. Tan and S. Abbasbandy. Homotopy analysis method for quadratic Riccati differential equation. *Commun. Nonlin. Sci. Numer. Simul.,* 13(2008)(3):539-546.

[9] S. Abbasbandy. A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials. *J. Comput. Appl. Math.,*207(2007)(1):59-63.

[10] C. L. Chen and Y. C. Liu. Solution of two point boundary value problems using the differential transformation method. *J Opt Theory Appl.*, 99(1998):23-35.

[11] Fatma Ayaz. Applications of differential transform method to to differential-algebraic equations. *Applied Mathematics and Computation*, 152(2004):649-657.

[12] Figen Kangalgil and Fatma Ayaz. Solitary wave solutions for the KdV and mKdV equations by differential transform method. *Chaos, Solitons and Fractals*, 41(2009)(1):464-472.

[13] S. V. Ravi Kanth and K. Aruna. Two-dimensional differential transform method for solving linear and non-linear Schroinger equations. *Chaos, Solitons and Fractals*, 41(2009)(5):2277-2281.

[14] A. Arikoglu and I. Ozkol. Solution of fractional differential equations by using differential transform method. *Chaos, Solitons and Fractals*, 34(2007):1473-1481.