A survey of model reduction by balanced truncation and some new results*

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Abstract

Balanced truncation is one of the most common model reduction schemes. In this note, we present a survey of balancing related model reduction methods and their corresponding error norms, and also introduce some new results. Five balancing methods are studied: **(1)** Lyapunov balancing, **(2)** Stochastic balancing **(3)** Bounded real balancing, **(4)** Positive real balancing and **(5)** Frequency weighted balancing. For positive real balancing, we introduce a multiplicative-type error bound. Moreover, for a certain subclass of positive real systems, a modi£ed positive-real balancing scheme with an absolute error bound is proposed. We also develop a new frequency-weighted balanced reduction method with a simple bound on the error system based on the frequency domain representations of the system gramians. Two numerical examples are illustrated to verify the ef£ciency of the proposed methods.

1 Introduction

Direct numerical simulation of dynamical systems has been a successful means for studying complex physical phenomena. In this paper, we will examine linear time invariant dynamical systems in state space form:

$$G(s): \left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t)+Du(t) \end{array} \right. \quad \Leftrightarrow \quad G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \quad \Leftrightarrow \quad G(s) :\overset{s}{=} C(sI-A)^{-1}B+D \qquad (1.1)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. We note that by abuse of notation, both the underlying dynamical system and its transfer function are denoted by $G(s)$. However, for clarity in the transfer function notation we will use "$\overset{s}{=}$" instead of only "$=$". In many applications, such as circuit simulation, or time dependent PDE control problems, n is quite large, while the number of inputs m and outputs p usually satis£es $m, p \ll n$. In these large-scale settings, the system dimension makes the computation infeasible due to memory, time limitations and ill-conditioning. One approach to overcoming this is through model reduction. The goal is to produce a low dimensional system that has similar response characteristics as the original system with far lower storage requirements and evaluation time. The resulting reduced model might be used to replace the original system as a component in a larger simulation or it might be used to develop a low dimensional controller suitable for real time applications.

The model reduction problem we are interested in can be stated as follows: Given the linear dynamical system $G(s)$ in (1.1), £nd a reduced order system $G_r(s)$

$$G_r(s): \left\{ \begin{array}{rcl} \dot{x}_r(t) & = & A_r x_r(t)+B_r u(t) \\ y_r(t) & = & C_r x_r(t)+D_r u(t) \end{array} \right. \quad \Leftrightarrow \quad G_r(s) := \left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array} \right] \qquad (1.2)$$

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where $A_r \in \mathbb{R}^{r \times r}$, $B_r \in \mathbb{R}^{r \times m}$, $C_r \in \mathbb{R}^{p \times r}$, $D_r \in \mathbb{R}^{p \times m}$, with $r << n$ such that the following properties are satis£ed:

1. The approximation error $\|y - y_r\|$ is *small*, and there exists a *global* error bound.
2. System properties, like *stability*, *passivity*, are preserved.
3. The procedure is *computationally ef£cient*.

One model reduction scheme that is well grounded in theory and most commonly used is the so-called Balanced Model Reduction £rst introduced by Mullis and Roberts [31] and later in the systems and control literature by Moore [30]. To apply balanced reduction, £rst the system is transformed to a basis where the states which are dif£cult to reach are simultaneously dif£cult to observe. This is achieved by simultaneously diagonalizing the reachability and the observability gramians, which are solutions to the reachability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states which have this property. We will call this *the Lyapunov balancing method*. When applied to stable systems, Lyapunov balanced reduction preserves stability [36] and provides a bound on the approximation error [13], i.e. satis£es 1. and 2. above. For small-to-medium scale problems, Lyapunov balancing can be implemented ef£ciently. However, for large-scale settings, *exact balancing* is expensive to implement because it requires dense matrix factorizations and results in a computational complexity of $\mathcal{O}(n^3)$ and a storage requirement of $\mathcal{O}(n^2)$; hence do not satisfy 3. above. In this case, *approximate balanced reduction* is an active research area which aims to obtain an approximately balanced system in a numerically ef£cient way; see, for example, [23], [5], [6], [34], [35] and the references therein.

Besides the Lyapunov balancing method, other types of balancing exist such as stochastic balancing, bounded real balancing, positive real balancing, LQG balancing and frequency weighted balancing. *The stochastic balancing method*[1] was £rst proposed by Desai and Pal [12] for balancing stochastic systems and later generalized by Green [19],[20]. The relative error bound for stochastic balancing is due to [19]. Unlike the Lyapunov balancing method, the stochastic balancing algorithm requires solving one Lyapunov and one Riccati equation. A closely related balancing method is *positive real balancing* [12] which is applied for model reduction of positive real (passive) systems, an important subclass of dynamical systems. The positive real balancing method can be viewed as the stochastic balancing method applied to the spectral factor of the given passive system and requires solving two positive real Riccati equations. Another method which also requires solving two Riccati equations, is *bounded real balancing* which is applied to the bounded real systems. This method, together with the absolute error bound, is £rst introduced by Opdenacker and Jonckheere in [33]. LQG balancing, also referred as the closed loop balancing £rst introduced by Jonckheere and Silverman [24], is mainly used for reduced order controller design and will not be included in this paper.

All the balancing techniques mentioned above try to approximate the full-order model $G(s)$ over all frequencies. However, in many applications one is only interested in a given frequency interval. In these cases *the frequency weighted balanced reduction* is used which tries to reduce the error between $G(s)$ and $G_r(s)$ over the speci£ed frequency range, i.e. the weighted error. Several ways of weighted balancing have been introduced in the literature. Lyapunov balanced reduction was extended to the frequency weighted balanced reduction by Enns [13]. The method allowed the use of both input and output weighting, but in case of a two sided weighting, stability is not guaranteed. To overcome the stability problem Lin and Chiu [28] proposed a new technique which uses only strictly proper weighting functions. Later their method was modi£ed by Sreeram *et al.* [37] to allow proper

[1] Originally stochastic balancing was introduced as a spectral factor based algorithm, i.e. given a positive real function G, the method approximates the spectral factor V of Φ where $\Phi = VV^{\sim} = G + G^{\sim}$ which results in solving two Riccati equations. Later the method is generalized and the stochastic balanced reduction is de£ned as approximating V given V, which results in solving one Lyapunov and one Riccati equation, see [43], [44],[40],[10]. In this note, by stochastic balancing, we mean the latter which only requires that the original model is square and invertible. We will discuss the former version of the stochastic balancing, which requires solving two Riccati equations, under the name positive real balancing. These issues will be clari£ed throughout the text.

weighting functions. In [27] and [37], error bounds for these techniques were introduced. On the other hand, recently Wang *et al.* [39] introduced a new frequency weighted balancing method as a modi£cation to Enns' method. The method guarantees stability and yields a simple error bound. In [43], Zhou proposed a self-weighted balanced reduction technique using Enns' method where the output weighting is the inverse of the transfer function $G(s)$. Stability results and relative and multiplicative error bounds were also introduced in [43].

All these frequency weighted balancing methods need input and output weights $W_i(s)$ and $W_o(s)$ which are usually not explicitly speci£ed, and try to £nd a reduced order model $G_r(s)$ which minimizes the weighted error $\|W_o(G - G_r)W_i\|_{\mathcal{H}_\infty}$. However, often, the original problem is to approximate $G(s)$ over a frequency interval $[\, w_1,\, w_2\,]$ and no input or output weights are given. Gawronski and Juang [16] introduced another type of weighted balanced reduction where for a given frequency band $[\, w_1,\, w_2\,]$, the construction of the weights are avoided simply by using the frequency domain representation of the reachability and observability gramians. Although the method works quite ef£ciently in practice, stability is not guaranteed and no error bound exists. Similarly to their band-limited frequency weighted balancing method, Gawronski and Juang [16] introduced also a time-limited balancing method where the gramians are computed over a £nite time interval $[\, t_1,\, t_2\,]$. The impulse response of the resulting reduced model is expected to match that of the original model over $[\, t_1,\, t_2\,]$. Even though it is not a frequency weighted method, this method will also be examined.

In this paper, we £rst present a survey of the balancing related model reduction methods with the corresponding error bounds whenever they exist. At this stage, we refer the reader to Ober's paper [32]. In addition, we introduce a multiplicative-type error bound for positive real balancing. Based on this error result, we propose a modi£ed positive real balanced truncation with an absolute error bound for a certain subclass of positive real systems. We then turn our attention to weighted balanced reduction and introduce a new algorithm. The method, which is a modi£cation of Gawronski and Juang algorithm, guarantees stability and yields a simple error bound.

In the sequel, we will assume that the full-order model $G(s)$ in (1.1) is asymptotically stable[2] and minimal[3].

The rest of the paper is organized as follows: Section 2 examines Lyapunov balancing followed by a study of stochastic balancing in Section 3. Then we review the bounded real and positive real balancing methods in Sections 4 and 5, respectively. Section 5.1 introduces a multiplicative-type error for positive real balancing followed by a modi£ed positive real balancing method developed in Section 5.2. Section 6 surveys the frequency weighted balanced reduction method and presents a new weighted balancing scheme. A comparison of proposed methods with the current methods is presented through numerical examples in Section 7. Section 8 contains conclusions.

2 Lyapunov Balancing Method

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{R}^{(n+p)\times(n+m)}$ be the to-be-reduced model as de£ned in (1.1). Closely related to this system are two continuous time Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0, \quad A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0. \tag{2.1}$$

Under the assumptions that $G(s)$ is asymptotically stable and minimal, the above equations have unique symmetric positive de£nite solutions $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{n\times n}$, called the *reachability* and *observability gramians*, respectively. The square roots of the eigenvalues of the product $\mathcal{PQ}$ are the so-called Hankel singular values $\sigma_i(G(s))$ of the system $G(s)$:

$$\sigma_i(G(s)) = \sqrt{\lambda_i(\mathcal{PQ})}.$$

It is easy to see that, $\sigma_i(G(s))$ are basis independent. In many cases, the eigenvalues of $\mathcal{P}$, $\mathcal{Q}$ as well as the Hankel singular values $\sigma_i(G(s))$ decay very rapidly; see [7] for details.

[2] $G(s)$ in (1.1) is called asymptotically stable if $\Re(\lambda_i(A)) < 0$, and is called stable if $\Re(\lambda_i(A)) \leq 0$ where $\Re(\lambda)$ denotes the real part of λ.

[3] $G(s)$ in (1.1) is called minimal if the pair (A, B) is reachable and the pair (C, A) is observable.

De£nition 2.1 *[30] The reachable, observable and stable system $G(s)$ is called Lyapunov-balanced if*

$$\mathcal{P} = \mathcal{Q} = \Sigma = \mathrm{diag}(\sigma_1 I_{m_1}, \cdots, \sigma_q I_{m_q}), \tag{2.2}$$

where $\sigma_1 > \sigma_2 > \cdots > \sigma_q > 0$, m_i, $i = 1, \cdots, q$ are the multiplicities of σ_i, and $m_1 + \cdots + m_q = n$.

The balanced basis has the property that the states which are dif£cult to reach are simultaneously dif£cult to observe. Hence, a reduced model is obtained by truncating the states which have this property, i.e. those which correspond to small Hankel singular values σ_i.

Theorem 2.1 *[36, 13] Let the asymptotically stable and minimal system $G(s)$ have the following Lyapunov balanced realization:*

$$G(s) = \left[\begin{array}{c|c} A_b & B_b \\ \hline C_b & D_b \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

with $\mathcal{P} = \mathcal{Q} = \mathrm{diag}(\Sigma_1, \Sigma_2)$ where

$$\Sigma_1 = \mathrm{diag}(\sigma_1 I_{m_1}, \cdots, \sigma_k I_{m_k}) \text{ and } \Sigma_2 = \mathrm{diag}(\sigma_{k+1} I_{m_{k+1}}, \cdots, \sigma_q I_{m_q}).$$

Then the reduced order model $G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ obtained by truncation is asymptotically stable, minimal and satis£es

$$\|G(s) - G_r(s)\|_{\mathcal{H}_\infty} \leq 2\,(\sigma_{k+1} + \cdots + \sigma_q). \tag{2.3}$$

Equality holds if $\Sigma_2 = \sigma_q I_{m_q}$.

Lyapunov balanced truncation as outlined above can be applied to any $G(s)$ which is asymptotically stable and minimal. For an application of Lyapunov balancing to unstable and non-minimal systems, see [44],[11], [26],[38] and the references therein.

3 Stochastic Balancing Method

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{R}^{(n+p)\times(n+m)}$ be asymptotically stable and minimal with two additional properties, namely **(i)** $G(s)$ is square, i.e. m=p, and **(ii)** $\det(D) \neq 0$. Let $W(s)$ be a minimal phase left spectral factor of $G(s)G^{\sim}(s)$, i.e., $W^{\sim}(s)W(s) \stackrel{s}{=} G(s)G^{\sim}(s)$ where $G^{\sim}(s) :\stackrel{s}{=} G^T(-s)$. A realization of $W(s)$ can be computed as

$$W(s) = \left[\begin{array}{c|c} A & B_W \\ \hline C_W & D^T \end{array}\right]$$

with

$$B_W := \mathcal{P}C^T + BD^T, \quad \text{and} \quad C_W := D^{-1}(C - B_W^T \mathcal{X})$$

where $\mathcal{P}$ is the reachability gramian of $G(s)$, i.e. $\mathcal{P}$ solves $A\mathcal{P} + \mathcal{P}A^T + BB^T = 0$ and $\mathcal{X}$ is the solution to the Riccati equation

$$A^T\mathcal{X} + \mathcal{X}A + (C - B_W^T\mathcal{X})^T(DD^T)^{-1}(C - B_W^T\mathcal{X}) = 0.$$

Balanced stochastic realization of $G(s)$ is obtained by balancing $\mathcal{P}$ and $\mathcal{X}$.

De£nition 3.1 *[20, 45] The asymptotically stable, minimal, square and non-singular system $G(s)$ is called stochastically balanced if*

$$\mathcal{P} = \mathcal{X} = \mathrm{diag}(\mu_1 I_{t_1}, \cdots, \mu_q I_{t_q}). \tag{3.1}$$

where $\mu_1 > \mu_2 > \cdots > \mu_q > 0$, t_i, $i = 1, \cdots, q$ are the multiplicities of μ_i, and $t_1 + \cdots + t_q = n$.

It turns out that μ_i are Hankel singular values of the stable part of the so-called *phase matrix* $(W^{\sim}(s))^{-1}G(s)$.

Theorem 3.1 *[19] Let the asymptotically stable, minimal, square and non-singular system $G(s)$ have the following stochastic balanced realization:*

$$G(s) = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

with $\det(D) \neq 0$ and $\mathcal{P}_s = \mathcal{X}_s = \mathrm{diag}(\Gamma_1, \Gamma_2)$ where

$$\Gamma_1 = \mathrm{diag}(\mu_1 I_{t_1}, \cdots, \mu_k I_{t_k}) \text{ and } \Gamma_2 = \mathrm{diag}(\mu_{k+1} I_{t_{k+1}}, \cdots, \mu_q I_{t_q}).$$

Then the reduced order model $G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ obtained by truncation is asymptotically stable, minimal and satis£es

$$\left\| (G(s))^{-1} (G(s) - G_r(s)) \right\|_{\mathcal{H}_\infty} \leq \prod_{i=k+1}^{q} \frac{1+\mu_i}{1-\mu_i} - 1 \tag{3.2}$$

$$\left\| (G_r(s))^{-1} (G(s) - G_r(s)) \right\|_{\mathcal{H}_\infty} \leq \prod_{i=k+1}^{q} \frac{1+\mu_i}{1-\mu_i} - 1 \tag{3.3}$$

In addition, if $G(s)$ is minimum phase, $G_r(s)$ is minimum phase as well.

Stochastic balanced truncation can be applied to all asymptotically stable dynamical systems which are square and nonsingular. For application of stochastic balancing to singular systems, see [40] and [18]. It was pointed out in [40] that stochastic balanced truncation yields a uniformly good approximant over whole frequency range instead of small absolute errors. Also, Zhou [43] showed that for minimal phase systems, stochastic balanced truncation is the same as self-weighted balanced truncation where the output weighting is given by $G^{-1}(s)$. This issue will be discussed in Section 6.3 in more detail.

4 Bounded Real Balancing

An important class of dynamical systems is the class of bounded real systems. These are systems which are stable and whose transfer function is bounded by one on the imaginary axis. This class of systems is used in parameterizing all stabilizing controllers of a system such that the closed-loop satis£es an $\mathcal{H}_\infty$ constraint [32, 17].

De£nition 4.1 *The asymptotically stable system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is called bounded real if*

$$I - D^T D \geq 0 \text{ and } I - G^{\sim}(jw)G(jw) \geq 0, \quad \text{for} \quad \forall w \in \mathbb{R}.$$

It is called strictly bounded real if the above inequalities are strict where $G(s) \overset{s}{=} C(sI-A)^{-1}B + D$.

Here, we will examine only strictly bounded real systems. Hence, in the sequel, by bounded real, we mean strictly bounded real.

De£ne $R_C := I - D^T D$. Then $G(s)$ is bounded real if and only if there exists a $\mathcal{Y} = \mathcal{Y}^T > 0$ such that

$$A^T\mathcal{Y} + \mathcal{Y}A + C^TC + (\mathcal{Y}B + C^TD)R_C^{-1}(\mathcal{Y}B + C^TD)^T = 0. \tag{4.1}$$

Any solution $\mathcal{Y}$ of (4.1) lies between two extremal solutions, i.e. $0 < \mathcal{Y}_{\min} \leq \mathcal{Y} \leq \mathcal{Y}_{\max}$. $\mathcal{Y}_{\min}$ is the unique solution to (4.1) such that $A + BR_C^{-1}(B^T\mathcal{Y} + D^TC)$ is asymptotically stable. De£ne $R_B := I - DD^T$. Then a dual Riccati equation

$$A\mathcal{Z} + \mathcal{Z}A^T + BB^T + (\mathcal{Z}C^T + BD^T)R_B^{-1}(\mathcal{Z}C^T + BD^T)^T = 0. \tag{4.2}$$

is obtained where $\mathcal{Z} = \mathcal{Z}^T > 0$. As in the case for (4.1), any solution $\mathcal{Z}$ of (4.2) lies between two extremal solutions, i.e. $0 < \mathcal{Z}_{\min} \leq \mathcal{Z} \leq \mathcal{Z}_{\max}$. (4.1) and (4.2) are called *the bounded real Riccati equations* of the system $G(s)$.

Lemma 4.1 *[32] If $\mathcal{Y} = \mathcal{Y}^T > 0$ is a solution to (4.1), then $\mathcal{Z} = \mathcal{Y}^{-1}$ is a solution to (4.2). Hence $\mathcal{Z}_{\min} = \mathcal{Y}_{\max}^{-1}$ and $\mathcal{Z}_{\max} = \mathcal{Y}_{\min}^{-1}$.*

A bounded real balanced representation [33] is obtained by balancing (i.e., simultaneously diagonalizing) $\mathcal{Y}_{\min}$ and $\mathcal{Y}_{\max}^{-1} = \mathcal{Z}_{\min}$.

De£nition 4.2 *[33] A bounded real system $G(s)$ is called bounded real balanced if*

$$\mathcal{Y}_{\min} = \mathcal{Z}_{\min} = \mathcal{Y}_{\max}^{-1} = \mathcal{Z}_{\max}^{-1} = \mathrm{diag}(\xi_1 I_{l_1}, \cdots, \xi_q I_{l_q})$$

where $1 \geq \xi_1 > \xi_2 > \cdots > \xi_q > 0$, l_i, $i = 1, \cdots, q$ are the multiplicities of ξ_i, and $l_1 + \cdots + l_q = n$.

We will call ξ_i *the bounded real singular values* of $G(s)$.

Theorem 4.1 *[33] Let the asymptotically stable, minimal, bounded-real system $G(s)$ have the following bounded real balanced realization:*

$$G(s) = \left[\begin{array}{c|c} A_{br} & B_{br} \\ \hline C_{br} & D_{br} \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

with $I - D^TD > 0$ and $\mathcal{Y}_{\min} = \mathcal{Z}_{\min} = \mathrm{diag}(\Xi_1, \Xi_2)$ where

$$\Xi_1 = \mathrm{diag}(\xi_1 I_{l_1}, \cdots, \xi_k I_{l_k}) \text{ and } \Xi_2 = \mathrm{diag}(\xi_{k+1} I_{l_{k+1}}, \cdots, \xi_q I_{l_q}).$$

Let a reduced order model $G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ be obtained by truncation. Also let $\bar{W}(s)$ and $\bar{V}(s)$ be the stable minimum phase spectral factors of $I - G^{\sim}(s)G(s)$ and $I - G(s)G^{\sim}(s)$, respectively, i.e., $\bar{W}^{\sim}(s)\bar{W}(s) \overset{s}{=} I - G^{\sim}(s)G(s)$ and $\bar{V}(s)\bar{V}^{\sim}(s) \overset{s}{=} I - G(s)G^{\sim}(s)$. Similarly de£ne $\bar{W}_r(s)$ and $\bar{V}_r(s)$ for $G_r(s)$. Then $G_r(s)$ is asymptotically stable, minimal, bounded real balanced and satis£es

$$\max\left\{\left\| \begin{array}{c} G(s) - G_r(s) \\ \bar{W}(s) - \bar{W}_r(s) \end{array} \right\|_{\mathcal{H}_\infty}, \left\| \begin{array}{c} G(s) - G_r(s) \\ \bar{V}(s) - \bar{V}_r(s) \end{array} \right\|_{\mathcal{H}_\infty}\right\} \leq 2\sum_{i=k+1}^{q} \xi_i. \tag{4.3}$$

(4.3) states that if $2\sum_{i=k+1}^{q} \xi_i$ is small, not only $G(s)$ and $G_r(s)$ are close, but also the reduced spectral factors $\bar{W}_r(s)$ and $\bar{V}_r(s)$ are guaranteed to be close, respectively, to the full order spectral factors $\bar{W}(s)$ and $\bar{V}(s)$.

5 Positive Real Balancing

Another important class of dynamical systems is the class of positive real (passive) systems. In a physical sense, positive realness means that the energy produced by the system can never exceed the energy received by it. Electric circuits are one class of positive real dynamical systems.

Definition 5.1 *The asymptotically stable system* $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{R}^{(n+p)\times(n+m)}$*, is called positive real if*

$$m = p, \quad D^T + D \geq 0 \text{ and } G^{\sim}(jw) + G(jw) \geq 0, \quad \text{for} \quad \forall w \in \mathbb{R},$$

and is called strictly positive real if the above inequalities are strict where $G(s) \overset{s}{=} C(sI - A)^{-1}B + D$.

In the sequel, we will examine only strictly positive real systems, and hence, positive real will mean strictly positive real.

$G(s)$ is positive real if and only if there exists a $\mathcal{K} = \mathcal{K}^T > 0$ such that

$$A^T\mathcal{K} + \mathcal{K}A + (\mathcal{K}B - C^T)(D + D^T)^{-1}(\mathcal{K}B - C^T)^T = 0. \tag{5.1}$$

As in the bounded real case, a dual Riccati equation

$$A\mathcal{L} + \mathcal{L}A^T + (\mathcal{L}C^T - B)(D + D^T)^{-1}(\mathcal{L}C^T - B)^T = 0, \tag{5.2}$$

is obtained where $\mathcal{L} = \mathcal{L}^T > 0$. (5.1) and (5.2) are the so-called *positive real Riccati equations* of $G(s)$.

Corollary 5.1 *[32] Any solutions* $\mathcal{K}$ *and* $\mathcal{L}$ *of, respectively, (5.1) and (5.2) lie between two extremal solutions, i.e.* $0 < \mathcal{K}_{\min} \leq \mathcal{K} \leq \mathcal{K}_{\max}$ *and* $0 < \mathcal{L}_{\min} \leq \mathcal{L} \leq \mathcal{L}_{\max}$*. If* $\mathcal{K} = \mathcal{K}^T > 0$ *is a solution to (5.1), then* $\mathcal{L} = \mathcal{K}^{-1}$ *is a solution to (5.2). Hence* $\mathcal{K}_{\min} = \mathcal{L}_{\max}^{-1}$ *and* $\mathcal{K}_{\max} = \mathcal{L}_{\min}^{-1}$.

Analogously to the bounded real case, a positive real balancing transformation is obtained by balancing the minimal solutions $\mathcal{K}_{\min}$ and $\mathcal{L}_{\min}$ to (5.1) and (5.2), respectively.

Definition 5.2 *[12, 32] A positive real system* $G(s)$ *is called positive real balanced if*

$$\mathcal{K}_{\min} = \mathcal{L}_{\min} = \mathcal{K}_{\max}^{-1} = \mathcal{L}_{\max}^{-1} = \operatorname{diag}(\pi_1 I_{s_1}, \cdots, \pi_q I_{s_q}),$$

where $1 \geq \pi_1 > \pi_2 > \cdots > \pi_q > 0$*,* s_i*,* $i = 1, \cdots, q$ *are the multiplicities of* π_i*, and* $s_1 + \cdots + s_q = n$*.*

We will call π_i *the positive real singular values* of $G(s)$.

The Moebius transformation, denoted by $\mathcal{M}$, of a square bounded-real system $H(s)$, is defined as

$$H(s) \overset{\mathcal{M}}{\longmapsto} G(s) \overset{s}{=} (I - H(s))^{-1}(I + H(s)). \tag{5.3}$$

It is well known that $G(s)$ in (5.3) is positive real. $\mathcal{M}$ is a bijection with inverse

$$G(s) \overset{\mathcal{M}^{-1}}{\longmapsto} H(s) \overset{s}{=} (G(s) - I)(G(s) + I)^{-1}. \tag{5.4}$$

If $G(s)$ is a positive real system, $H(s)$ in (5.4) is a square bounded real system. The following lemma lists the important properties of the Moebius transformation:

Lemma 5.1 *[32] Given the square bounded real system* $H(s) = \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right]$*, let* $G(s)$ *be obtained by the Moebius transformation applied on* $H(s)$ *as in (5.3). Then* $G(s)$ *is positive real and a state space realization for* $G(s)$ *is given by*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A_\psi + B_\psi (I - D_\psi)^{-1} C_\psi & \sqrt{2} B_\psi (I - D_\psi)^{-1} \\ \hline \sqrt{2}(I - D_\psi)^{-1} C_\psi & (I - D_\psi)^{-1}(I + D_\psi) \end{array}\right]. \tag{5.5}$$

Similarly, given a positive real system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$*, let* $H(s)$ *be obtained by applying* $\mathcal{M}^{-1}$ *on* $G(s)$ *as in (5.4). Then* $H(s)$ *is a square bounded real system with the state-space realization*

$$H(s) = \left[\begin{array}{c|c} A_\psi & B_\psi \\ \hline C_\psi & D_\psi \end{array}\right] = \left[\begin{array}{c|c} A - B(I + D)^{-1} C & \sqrt{2} B (I + D)^{-1} \\ \hline \sqrt{2}(I + D)^{-1} C & (D - I)(D + I)^{-1} \end{array}\right]. \tag{5.6}$$

Moreover, $\mathcal{K} = \mathcal{K}^T > 0$ *is a solution to the positive real Riccati equation*

$$A^T \mathcal{K} + \mathcal{K} A + (\mathcal{K} B - C^T)(D + D^T)^{-1} (\mathcal{K} B - C^T)^T = 0,$$

if and only if $\mathcal{K}$ *is a solution to the bounded real Riccati equation*

$$A_\psi^T \mathcal{K} + \mathcal{K} A_\psi + C \Psi^T C_\Psi + (\mathcal{K} B_\psi + C_\psi^T D_\psi)(I - D_\psi^T D_\psi)^{-1} (\mathcal{K} B_\psi + C_\psi^T D_\psi)^T = 0.$$

where A_ψ*,* B_ψ*,* C_ψ *and* D_ψ *are as in (5.6). Hence,* $H(s)$ *is bounded-real balanced with bounded real gramians* Ξ *if and only if* $G(s) = \mathcal{M}(H(s))$ *is positive real balanced with positive real gramians* $\Pi = \Xi$*.*

Theorem 5.1 *[12] Let the asymptotically stable, minimal, positive real system* $G(s)$ *have the following positive real balanced realization:*

$$G(s) = \left[\begin{array}{c|c} A_{pr} & B_{pr} \\ \hline C_{pr} & D_{pr} \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \tag{5.7}$$

with $D + D^T > 0$ *and* $\mathcal{K}_{\min} = \mathcal{L}_{\min} = \mathrm{diag}(\Pi_1, \Pi_2)$ *where*

$$\Pi_1 = \mathrm{diag}(\pi_1 I_{s_1}, \cdots, \pi_k I_{s_k}) \text{ and } \Pi_2 = \mathrm{diag}(\pi_{k+1} I_{s_{k+1}}, \cdots, \pi_q I_{s_q}).$$

Let the reduced order model $G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ *obtained by truncation. Then* $G_r(s)$ *is asymptotically stable, minimal and positive real balanced.*

It is clear that the error results of the stochastic balancing can be employed for positive-real balancing. However, in that case *the bounds will be in terms of the spectral factors of* $G(s)$*, not in terms of* $G(s)$; that is, we will have bounds on the error $\|V^{-1}(V - V_r)\|_\infty$ where $G + G^\sim = V^\sim V$ and $G_r + G_r^\sim = V_r^\sim V_r$. It is the goal of the next section to obtain such a bound in terms $G(s)$ and $G_r(s)$.

5.1 A multiplicative-type error bound for positive real balancing

In this section, we will introduce a multiplicative-type error bound for the positive real balanced reduction in terms of $G(s)$ and $G_r(s)$. The following theorem is the £rst step toward this goal:

Theorem 5.2 *Given the asymptotically stable positive real system* $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$*, the reduced order model* $G_r(s)$*, obtained by positive real balanced truncation as de£ned in Theorem 5.1, satis£es*

$$\|(D^T + G(s))^{-1} - (D^T + G_r(s))^{-1}\|_{\mathcal{H}_\infty} \leq 2\,\|R\|^2 \sum_{i=k+1}^{q} \pi_i, \tag{5.8}$$

where $\Pi = \mathrm{diag}(\pi_1 I_{s_1}, \cdots, \pi_q I_{s_q})$, $R^2 := (D + D^T)^{-1}$.

Proof: We will assume that $G(s)$ is in the positive real balanced basis as given in (5.7). Hence the following two Riccati equations hold:

$$A\Pi + \Pi A^T + (\Pi C^T - B)(D + D^T)^{-1}(\Pi C^T - B)^T = 0 \tag{5.9}$$

$$A^T\Pi + \Pi A + (\Pi B - C^T)(D + D^T)^{-1}(\Pi B - C^T)^T = 0 \tag{5.10}$$

It is easy to see that (5.9) and (5.10) can be written as

$$\underbrace{(A - BRRC)}_{=:\widehat{A}}\Pi + \Pi(A - BRRC)^T + \Pi C^T R \underbrace{RC}_{=:\widehat{C}}\Pi + \underbrace{BR}_{=:\widehat{B}} RB^T = 0 \tag{5.11}$$

$$(A - BRRC)^T\Pi + \Pi(A - BRRC) + \Pi BRRB^T\Pi + C^T RRC = 0 \tag{5.12}$$

It follows from De£nition 4.2 that the system $\widehat{G}(s) = \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline \widehat{C} & 0 \end{array}\right]$ is bounded real balanced with bounded real gramian Π. Partition $\widehat{G}(s)$ as

$$\widehat{A} = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} \widehat{C}_1 & \widehat{C}_2 \end{bmatrix} \quad \Pi = \begin{bmatrix} \Pi_1 & \\ & \Pi_2 \end{bmatrix},$$

where $\Pi_1 = \mathrm{diag}(\pi_1 I_{s_1}, \cdots, \pi_k I_{s_k})$ and $\Pi_2 = \mathrm{diag}(\pi_{k+1} I_{s_{k+1}}, \cdots, \pi_q I_{s_q})$, and de£ne the bounded real reduced system $\widehat{G}_r(s) := \left[\begin{array}{c|c} \widehat{A}_{11} & \widehat{B}_1 \\ \hline \widehat{C}_1 & 0 \end{array}\right]$. Then it follows from Theorem 4.1 that

$$\|\widehat{G}(s) - \widehat{G}_r(s)\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=k+1}^{q} \pi_i. \tag{5.13}$$

Since $\| \; R\,(\widehat{G}(s) - \widehat{G}_r(s)\,(-R) \; \|_{\mathcal{H}_\infty} \leq \; \|R\|^2\,\|\widehat{G}(s) - \widehat{G}_r(s)\|_{\mathcal{H}_\infty}$, (5.13) leads to

$$\|R(\widehat{G}(s) - \widehat{G}_r(s))(-R)\|_{\mathcal{H}_\infty} = \|\underbrace{R\widehat{G}(s)(-R)}_{\stackrel{s}{=}:\Theta(s)} - \underbrace{R\widehat{G}_r(s)(-R)}_{\stackrel{s}{=}:\Theta_r(s)}\|_{\mathcal{H}_\infty} \leq 2\,\|R\|^2 \sum_{i=k+1}^{q} \pi_i.$$

A realization for $\Theta(s)$ and $\Theta_r(s)$ can be obtained as

$$\Theta(s) = \left[\begin{array}{c|c} A - BR^2C & -BR^2 \\ \hline R^2C & 0 \end{array}\right] \text{ and } \Theta_r(s) = \left[\begin{array}{c|c} A_{11} - B_1R^2C_1 & -B_1R^2 \\ \hline R^2C_1 & 0 \end{array}\right].$$

Since $\|\Theta(s) - \Theta_r(s)\|_{\mathcal{H}_\infty} = \| \; (\Theta(s) + R^2) \; - \; (\Theta_r(s) + R^2) \; \|_{\mathcal{H}_\infty}$, we obtain

$$\left\| \; \left[\begin{array}{c|c} A - BR^2C & -BR^2 \\ \hline R^2C & R^2 \end{array}\right] - \left[\begin{array}{c|c} A_{11} - B_1R^2C_1 & -B_1R^2 \\ \hline R^2C_1 & R^2 \end{array}\right] \; \right\|_{\mathcal{H}_\infty} \leq 2\,\|R\|^2 \sum_{i=k+1}^{q} \pi_i.$$

It is clear that

$$\left[\begin{array}{c|c} A-BR^2C & -BR^2 \\ \hline R^2C & R^2 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & R^{-2} \end{array}\right]^{-1} \quad \text{and} \quad \left[\begin{array}{c|c} A_{11}-B_1R^2C_1 & -B_1R^2 \\ \hline R^2C_1 & R^2 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & R^{-2} \end{array}\right]^{-1}$$

Recall that $R^{-2} = D + D^T$. Hence, one obtains

$$\left\| \, (D^T + G(s))^{-1} \; - \; (D^T + G_r(s))^{-1} \, \right\|_{\mathcal{H}_\infty} \leq 2\, \|R\|^2 \sum_{i=k+1}^{q} \pi_i. \tag{5.14}$$

This completes the proof. ■

Remark 5.1 *We note that (5.14) is equivalent to*

$$\|(D^T + G(s))^{-1}(G(s) - G_r(s))(D^T + G_r(s))^{-1}\|_{\mathcal{H}_\infty} \leq 2\, \|R\|^2 \sum_{i=k+1}^{q} \pi_i. \tag{5.15}$$

(5.15) is indeed a frequency weighted bound of the error system $G(s) - G_r(s)$ where the input and the output weights are $(D^T + G(s))^{-1}$ and $(D^T + G_r(s))^{-1}$, respectively.

Theorem 5.2 leads to the following multiplicative-type error result for the positive real balanced reduction. Note that, as mentioned above, this bound is in terms of $G(s)$ and $G_r(s)$, not in terms of the spectral factors.

Lemma 5.2 *Given the asymptotically stable, minimal and positive real system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, let $G_r(s)$ be obtained by positive real balanced truncation as de£ned in Theorem 5.1. The following error bound holds:*

$$\left\| \left(D^T + G_r(s)\right)^{-1} \, \left(G(s) - G_r(s)\right) \right\|_{\mathcal{H}_\infty} \leq 2\, \|R\|^2 \, \|D^T + G(s)\|_{\mathcal{H}_\infty} \sum_{i=k+1}^{q} \pi_i \tag{5.16}$$

where $R^2 = (D + D^T)^{-1}$.

Proof: Directly follows from Theorem 5.2. ■

We state (5.16) as a multiplicative-type error bound rather than a multiplicative error bound because of having the term of $D^T + G_r(s)$ instead of $G_r(s)$ only. However, one can easily see that in terms of $G(s) + D^T$ and $G_r(s) + D^T$ it is a multiplicative error result, namely

$$\left\| \left(D^T + G_r(s)\right)^{-1} \, \left((G(s) + D^T) - (G_r(s) + D^T)\right) \right\|_{\mathcal{H}_\infty} \leq 2\, \|R\|^2 \, \|D^T + G(s)\|_{\mathcal{H}_\infty} \sum_{i=k+1}^{q} \pi_i. \tag{5.17}$$

5.2 A modi£ed positive real balancing method with an absolute error bound

In this section, we will introduce a modi£ed positive real balancing method for a certain subclass of positive real systems. Then based on Theorem 5.2, we will derive an absolute error bound for this reduction method.

De£nition 5.3 *Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be an asymptotically stable and minimal positive real system and $R^2 = (D + D^T)^{-1} > 0$. De£ne $F_G(s) := \left[\begin{array}{c|c} A-BR^2C & -BR^2 \\ \hline R^2C & 0 \end{array}\right]$. By $\mathcal{D}$ we denote the set of all positive real systems $G(s)$ such that $F_G(jw) + F_{\tilde{G}}^{\sim}(jw) > -R^2$, i.e.*

$$\mathcal{D} := \left\{ G(s) : G(s) \text{ is positive real and } F_G(jw) + F_{\tilde{G}}^{\sim}(jw) > -R^2 \right\}$$

Remark 5.2 **1.** *It is easy to see that* $F_G(s) + R^2 \overset{s}{=} (G(s) + D^T)^{-1}$ *and consequently* $F_G(s) + R^2$ *is positive real. In the above de£nition we require that* $F_G(s) + \frac{R^2}{2}$ *be positive real as well. Therefore, another way of stating De£nition 5.3 is that* $\mathcal{D}$ *is a family of positive real systems for which* $F_G(s) + \frac{R^2}{2}$ *is also positive real.*
2. *The condition* $F_G(jw) + F_G^{\sim}(jw) > -R^2$ *is not satis£ed for all positive real* $G(s)$*. For example take* $G(s) \overset{s}{=} 1 + \frac{1}{s+p}$ *where* p *is a positive number. The above condition is satis£ed for all* $p > 0.5$*. Simulations suggest that* $F_G(jw) + F_G^{\sim}(jw) > -R^2$ *is not a severe restriction.*

Throughout the next section, positive real systems will refer to systems belonging to the family $\mathcal{D}$.

5.2.1 Modi£ed positive real balanced truncation

Given the positive real system $G(s)$, de£ne the dynamical system $H(s)$ with the corresponding D-term D_H as

$$D_H^T + H(s) \overset{s}{=} (D^T + G(s))^{-1}. \tag{5.18}$$

Note that $D_H^T + H(s)$ is positive real even if $G(s)$ does not belong to $\mathcal{D}$. A state-space representation of $H(s)$ is easily computed as

$$H(s) = \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & D_H \end{array}\right] = \left[\begin{array}{c|c} A - BR^2C & -BR^2 \\ \hline R^2C & R^2/2 \end{array}\right]. \tag{5.19}$$

Since $G \in \mathcal{D}$, by De£nition 5.3, $H(s)$ is positive real. Then we apply the positive real balanced truncation of Section 5 to $H(s)$. Let $H(s)$ have positive real gramians

$$\bar{\mathcal{K}}_{min} = \bar{\mathcal{N}}_{min} = \bar{\Pi} = \text{diag}(\bar{\pi}_1 I_{s_1}, \cdots, \bar{\pi}_q I_{s_k}, \bar{\pi}_{k+1} I_{s_{k+1}}, \cdots, \bar{\pi}_q I_{s_q}).$$

Let $H_r(s)$ denote the reduced positive real system obtained by keeping £rst k positive real singular values $\bar{\pi}_i$ of $H(s)$. $H_r(s)$ is the intermediate reduced model. We then compute the £nal reduced order model $\bar{G}_r(s) = \left[\begin{array}{c|c} \bar{A}_r & \bar{B}_r \\ \hline \bar{C}_r & \bar{D}_r \end{array}\right]$ from $H_r(s)$ using the relationship

$$\bar{D}_r + \bar{G}_r(s) \overset{s}{=} (D_H^T + H_r(s))^{-1}. \tag{5.20}$$

It is easy to show that by construction $\bar{D}_r = D$. Now we state the main result of this section:

Theorem 5.3 *Given the positive real system* $G(s) \in \mathcal{D}$*, let* $\bar{G}_r(s)$ *be obtained by the modi£ed positive real balanced truncation method introduced above. Then* $\bar{G}_r(s)$ *is asymptotically stable, positive real and satis£es*

$$\| G(s) - \bar{G}_r(s) \|_{\mathcal{H}_\infty} \leq 2\|R^{-1}\|^2 \sum_{i=k+1}^{q} \bar{\pi}_i. \tag{5.21}$$

Proof: Asymptotic stability and positive realness follow by construction. We only need to prove the error bound. Since $H_r(s)$ is obtained from $H(s)$ by positive real balanced truncation, Theorem 5.2 yields

$$\|(D_H^T + H(s))^{-1} - (D_H^T + H_r(s))^{-1}\|_{\mathcal{H}_\infty} \leq 2\,\|R_H\|^2 \sum_{i=k+1}^{q} \bar{\pi}_i \tag{5.22}$$

where $R_H := (D_H + D_H^T)^{-2} = R^{-1}$. Then noticing that by construction, we have $(D_H^T + H(s))^{-1} \overset{s}{=} D^T + G(s)$ and $(D_H^T + H_r(s))^{-1} \overset{s}{=} D^T + G_r(s)$, the desired result (5.21) follows. ■

By Theorem 5.3, we are able to approximate a positive real system $G(s)$ by a reduced order positive real system with an absolute error bound on the $\mathcal{H}_\infty$ norm of the error if $G(s) \in \mathcal{D}$. This error result is analogous to the error result (2.3) of the Lyapunov balancing and (4.3) of the bounded real balancing methods. In Section 7, through a numerical example, we compare the above modi£ed positive real balanced truncation with the positive real balanced truncation of Section 5. Also, π_i and $\bar{\pi}_i$ will be compared in terms of their decay rates. The example illustrates that if $G(s) \in \mathcal{D}$, the proposed method is an alternative to positive real balancing.

6 Frequency Weighted Balanced Truncation

The balancing methods introduced above try to approximate the full order model $G(s)$ over all frequencies. However, in many applications one is only interested in a certain frequency range. This problem leads to the so-called frequency weighted balancing method. Given some input weighting $W_i(s)$ and output weighting $W_o(s)$, the problem becomes to compute a reduced order model so that the weighted error $\| W_o(s)(G(s) - G_r(s))W_i(s) \|_{\mathcal{H}_\infty}$ is small. The frequency weighted balanced reduction methods of Enns [13], Lin and Chiu [28], Wang *et al.* [39] and Zhou [43] are the most common approaches used to tackle this problem.

Although the frequency weighted reduction problem is stated as reducing $\| W_o(s)(G(s) - G_r(s))W_i(s) \|_{\mathcal{H}_\infty}$, we want to mention that the input and output weightings $W_i(s)$ and $W_o(s)$ are often £ctitious quantities *unless they are speci£ed by the user*. In many cases, the original problem is to approximate $G(s)$ over a frequency interval $[\ w_1,\ w_2\ ]$ and no input and output weighting is given. Then to use the frequency weighted methods mentioned, one has to construct weights to re¤ect this frequency range. Choosing the weights is a problem in itself. To remedy this situation, Gawronski and Juang [16] introduced another type of weighted balanced reduction method where for a given frequency band $[\ w_1,\ w_2\ ]$, the construction of the weights is avoided by using the frequency domain representation of the gramians. This method has not been recognized in the literature as much as the other methods. We will propose a frequency weighted balancing method as a modi£cation to Gawronski and Juang's method. With the modi£cation, we will guarantee asymptotic stability and provide a simple error bound. We note that our modi£cation to Gawronski and Juang's method is analogous to Wang's *et al.* modi£cation [39] to Enns' [13] and Lin and Chiu's [28] method. Below we will also review Gawronski and Juang's time limited balancing method.

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $W_i(s) = \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$, and $W_o(s) = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right]$ be the state space representations of the original model $G(s)$, the input weight $W_i(s)$ and the output weight $W_o(s)$. Assuming there is no pole-zero cancellation, the minimal state-space realizations of $G(s)W_i(s)$ and $W_o(s)G(s)$ are given by

$$G(s)W_i(s) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A & BC_i & BD_i \\ 0 & A_i & B_i \\ \hline C & 0 & DD_i \end{array}\right] \tag{6.1}$$

and

$$W_o(s)G(s) = \left[\begin{array}{c|c} \bar{A}_o & \bar{B}_o \\ \hline \bar{C}_o & \bar{D}_o \end{array}\right] = \left[\begin{array}{cc|c} A & 0 & B \\ B_oC & A_o & 0 \\ \hline D_oC & C_o & D_oD \end{array}\right]. \tag{6.2}$$

Let

$$\bar{\mathcal{P}} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{12}^T & \mathcal{P}_{22} \end{bmatrix}, \text{ and } \bar{\mathcal{Q}} = \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{12}^T & \mathcal{Q}_{22} \end{bmatrix}, \tag{6.3}$$

be the solutions to the following Lyapunov equations:

$$\bar{A}_i\bar{\mathcal{P}} + \bar{\mathcal{P}}\bar{A}_i^T + \bar{B}_i\bar{B}_i^T = 0 \text{ and } \bar{A}_o^T\bar{\mathcal{Q}} + \bar{\mathcal{Q}}\bar{A}_o + \bar{C}_o^T\bar{C}_o = 0.$$

6.1 Enns' frequency weighted method [13]

This method is based on the simultaneous diagonalization of $\mathcal{P}_{11}$ and $\mathcal{Q}_{11}$. Assume that $\mathcal{P}_{11}$ and $\mathcal{Q}_{11}$ are the frequency weighted balanced gramians with

$$\mathcal{P}_{11} = \mathcal{Q}_{11} = \mathrm{diag}(\sigma_1 I_{n_1}, \cdots, \sigma_k I_{n_k}, \sigma_{k+1} I_{n_{k+1}}, \cdots, \sigma_q I_{n_q})$$

where n_i are the multiplicities of σ_i with $n_1 + \cdots + n_q = n$. In this balanced basis, let $G(s)$ and the reduced model $G_r(s)$ be given by

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \qquad G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] \tag{6.4}$$

where $G_r(s)$ corresponds to largest k weighted singular values σ_i.

Theorem 6.1 *[27] Given the asymptotically stable and minimal system $G(s)$, let $G_r(s)$ be obtained by Enns' frequency weighted balanced truncation method as above. Assume that $G_r(s)$ is asymptotically stable, which is guaranteed if $W_i = I$ or $W_o = I$. Then*

$$\| W_o(s)(G(s) - G_r(s))W_i(s) \|_{\mathcal{H}_\infty} \leq 2 \sum_{i=k+1}^{q} \sqrt{\sigma_k^2 + (\alpha_k + \beta_k)\sigma_k^{3/2} + \alpha_k \beta_k \sigma_k}, \tag{6.5}$$

where α_k and β_k denote the $\mathcal{H}_\infty$ norms of transfer function which depends on $W_o(s)$, $W_i(s)$ and $G_{r_j}(s)$, $j = 1, \cdots, k$.

The computation of this upper bound is quite complex and requires evaluating many $\mathcal{H}_\infty$ norms. For details on the computation of α_k and β_k, see the original source [27].

6.2 Lin and Chiu's frequency weighted balancing method [28]

Unlike Enns' method where balancing is based on $\mathcal{P}_{11}$ and $\mathcal{Q}_{11}$, Lin and Chiu's frequency weighted balancing method is based on the simultaneous diagonalization of $\tilde{\mathcal{P}} := \mathcal{P}_{11} - \mathcal{P}_{12}\mathcal{P}_{22}^{-1}P_{12}^T$ and $\tilde{\mathcal{Q}} := \mathcal{Q}_{11} - \mathcal{Q}_{12}^T\mathcal{Q}_{22}^{-1}\mathcal{Q}_{12}^T$. Let $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{Q}}$ be balanced as

$$\tilde{\mathcal{P}} = \tilde{\mathcal{Q}} = \mathrm{diag}(\tilde{\sigma}_1 I_{n_1}, \cdots, \tilde{\sigma}_k I_{n_k}, \tilde{\sigma}_{k+1} I_{n_{k+1}}, \cdots, \tilde{\sigma}_q I_{n_q}).$$

The reduced order model $G_r(s)$ is obtained by truncation as in (6.4).

Theorem 6.2 *[28, 37] Given asymptotically stable and minimal $G(s)$, let $G_r(s)$ be obtained by Lin and Chiu's frequency weighted balanced truncation as above. Then $G_r(s)$ is stable and satis£es*

$$\| W_o(s)(G(s) - G_r(s))W_i(s) \|_{\mathcal{H}_\infty} \leq 2 \sum_{i=k+1}^{q} \sqrt{(\tilde{\sigma}_k^2 + \alpha_k + \lambda_k)(\tilde{\sigma}_k + \beta_k + \omega_k)} \tag{6.6}$$

where α_k, β_k, λ_k and ω_k denote the $\mathcal{H}_\infty$ norms of transfer function which depend on $W_o(s)$, $W_i(s)$ and $G_{r_j}(s)$, $j = 1, \cdots, k$.

The computation of the upper bound is complex as in Enns' method, see [28].

6.3 Zhou's self-weighted frequency weighted balancing method [43]

Zhou's method is applicable to any asymptotically stable $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ for which an asymptotically stable right inverse $G^{+}(s)$ exists and D is full-rank. However, for simplicity, we will only discuss the case where $G(s)$ is square, nonsingular, i.e. $\det(D) \neq 0$, and G^{-1} is asymptotically stable, i.e. $G(s)$ is minimum phase. Zhou's method is a special case of Enns' method where

$$W_i(s) = I \qquad \text{and} \qquad W_o(s) = G^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right].$$

Then Zhou's frequency weighted gramians $\mathcal{P}_{11}$ and $\mathcal{Q}_{11}$ in (6.3) are the solutions to

$$A\mathcal{P}_{11} + \mathcal{P}_{11}A^T + BB^T = 0 \quad \text{and} \quad \mathcal{Q}_{11}(A - BD^{-1}C) + (A - BD^{-1}C)^T\mathcal{Q}_{11} + (D^{-1}C)^T(D^{-1}C) = 0.$$

The self-weighted balanced realization is obtained by simultaneously diagonalizing $\mathcal{P}_{11}$ and $\mathcal{Q}_{11}$, i.e.,

$$\mathcal{P}_{11} = \mathcal{Q}_{11} = \operatorname{diag}(\sigma_1 I_{n_1}, \cdots, \sigma_k I_{n_k}, \sigma_{k+1} I_{n_{k+1}}, \cdots, \sigma_q I_{n_q}). \tag{6.7}$$

Theorem 6.3 *[43] Let $G(s)$ be an asymptotically stable, square, non-singular and minimum phase system. Also, let $G_r(s)$ be obtained by Zhou's frequency weighted balanced truncation method. Then $G_r(s)$ is asymptotically stable, minimum phase and satis£es*

$$\left\|(G(s))^{-1}(G(s) - G_r(s))\right\|_{\mathcal{H}_\infty} \leq \prod_{i=k+1}^{q} (1 + 2\sigma_i\sqrt{1+\sigma_i^2} + 2\sigma_i^2) - 1 \tag{6.8}$$

$$\left\|(G_r(s))^{-1}(G(s) - G_r(s))\right\|_{\mathcal{H}_\infty} \leq \prod_{i=k+1}^{q} (1 + 2\sigma_i\sqrt{1+\sigma_i^2} + 2\sigma_i^2) - 1 \tag{6.9}$$

Moreover, it was shown in [43] that if $G(s)$ is square, asymptotically stable, nonsingular and minimum phase as in the above theorem, then balancing $\mathcal{P}_{11}$ and $\mathcal{Q}_{11}$ is equivalent to balancing the gramians $\mathcal{P}$ and $\mathcal{X}$ in the stochastic balancing case. Therefore, the following result holds:

Corollary 6.1 *Let $G(s)$ be a square, asymptotically stable, nonsingular and minimum phase system. Then the self-weighted balanced realization of $G(s)$ is also stochastically balanced. Hence $G_r(s)$, obtained by Zhou's method, is stochastically balanced, minimum phase and asymptotically stable. In this case, σ_i in (6.7) and μ_i in (3.1) are related by*

$$\mu_i = \frac{\sigma_i}{\sqrt{1+\sigma_i^2}}$$

The above corollary states that, if $G(s)$ is minimum phase, stochastic balancing can be obtained by solving two Lyapunov equations avoiding the Riccati equation.

6.4 Wang's *et al.* frequency weighted balancing method [39]

Given the setup in (6.1), (6.2) and (6.3), de£ne

$$X_B := BC_i\mathcal{P}_{12} + \mathcal{P}_{12}^T C i^T B^T + BD_i D_i^T B^T \;\text{ and }\; X_C := \mathcal{Q}_{12}B_oC + C^T B_o^T \mathcal{Q}_{12}^T + C^T D_o^T D_o C.$$

Let $X_B = USU^T$ and $X_C = VHV^T$ be the eigenvalue decompositions of X_B and X_C where $UU^T = I$, $VV^T = I$, $S = \operatorname{diag}(s_1, \cdots, s_n)$, $H = \operatorname{diag}(h_1, \cdots, h_n)$ with $\mid s_1 \mid \geq \cdots \mid s_n \mid \geq 0$ and $\mid h_1 \mid \geq \cdots \mid h_n \geq 0$. Let $\operatorname{rank}(X_B) := \iota$ and $\operatorname{rank}(X_C) := \nu$ and de£ne

$$\bar{B} := U\operatorname{diag}(\mid s_1 \mid^{1/2}, \cdots, \mid s_\iota \mid^{1/2}, \cdots, 0, \cdots, 0) \text{ and } \bar{C} := \operatorname{diag}(\mid h_1 \mid^{1/2}, \cdots, \mid h_\nu \mid^{1/2}, \cdots, 0, \cdots, 0)V^T.$$

The frequency weighted gramians $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ for Wang's *et al.* approach are the solutions to the following two Lyapunov equations:

$$A\bar{\mathcal{P}} + \bar{\mathcal{P}}A^T + \bar{B}\bar{B}^T = 0 \quad \text{and} \quad A^T\bar{\mathcal{Q}} + \bar{\mathcal{Q}}A + \bar{C}^T\bar{C} = 0.$$

Let $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ be balanced with

$$\bar{\mathcal{P}} = \bar{\mathcal{Q}} = \operatorname{diag}(\bar{\sigma}_1 I_{n_1}, \cdots, \bar{\sigma}_k I_{n_k}, \bar{\sigma}_{k+1} I_{n_{k+1}}, \cdots, \bar{\sigma}_q I_{n_q}).$$

Then Wang's *et. al* reduced model $G_r(s)$ is obtained by truncation as in (6.4). The following result holds.

Theorem 6.4 *[39] Given $G(s)$, let $G_r(s)$ be obtained by Wang's et. al frequency weighted balanced truncation as above. Then $G_r(s)$ is stable. In addition if*

$$\operatorname{rank}([\, B \;\; \bar{B} \,]) = \operatorname{rank}(\bar{B}) \quad \text{and} \quad \operatorname{rank}([\, C^T \;\; \bar{C}^T \,]) = \operatorname{rank}(\bar{C}^T)$$

then $G_r(s)$ is asymptotically stable and satis£es

$$\| \, W_o(s)(G(s) - G_r(s))W_i(s) \, \|_{\mathcal{H}_\infty} \leq 2\|W_o(s)L\|_{\mathcal{H}_\infty} \|KW_i(s)\|_{\mathcal{H}_\infty} \sum_{i=k+1}^{q} \bar{\sigma}_i, \tag{6.10}$$

where $K := \operatorname{diag}(\mid s_1 \mid^{-1/2}, \cdots, \mid s_\iota \mid^{-1/2}, \cdots, 0, \cdots, 0)U^T B$ and $L := CV \operatorname{diag}(\mid h_1 \mid^{-1/2}, \cdots, \mid h_\nu \mid^{-1/2}, \cdots, 0, \cdots, 0)$.

The assumptions $\operatorname{rank}([\, B \;\; \bar{B} \,]) = \operatorname{rank}(\bar{B})$ and $\operatorname{rank}([\, C^T \;\; \bar{C}^T \,]) = \operatorname{rank}(\bar{C}^T)$ are not always satis£ed. This will be analyzed further in Section 6.6 where we make a similar assumption. It is clear that the error bound for Wang's *et al.* approach is simpler than those of Enns' and Lin and Chiu's methods.

6.5 Gawronski and Juang's frequency weighted balanced reduction method [16]

Using Parseval's relationship it follows that in the frequency domain, the reachability and observability gramians $\mathcal{P}$ and $\mathcal{Q}$, are given by

$$\mathcal{P} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(w)BB^T H^*(w)dw \quad \text{and} \quad \mathcal{Q} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H^*(w)C^T CH(w)dw, \tag{6.11}$$

where $H_w := (jwI - A)^{-1}$ and $H^*(w) := (-jwI - A^*)^{-1}$. For a given frequency band $\Omega = [\, w_1, \; w_2 \,]$, Gawronski and Juang suggested to choose the frequency weighted gramians as

$$\mathcal{P}_\Omega := \mathcal{P}(w_2) - \mathcal{P}(w_1) \quad \text{and} \quad \mathcal{Q}_\Omega := \mathcal{Q}(w_2) - \mathcal{Q}(w_1) \tag{6.12}$$

where

$$\mathcal{P}(w) = \frac{1}{2\pi}\int_{-w}^{+w} H(w)BB^T H^*(w)dw \quad \text{and} \quad \mathcal{Q}(w) = \frac{1}{2\pi}\int_{-w}^{+w} H^*(w)C^T CH(w)dw. \tag{6.13}$$

Note that $\mathcal{P}(w)$ and $\mathcal{Q}(w)$ are both positive de£nite. From $BB^T = -A\mathcal{P} - \mathcal{P}A^T = (jwI - A)\mathcal{P} + \mathcal{P}(jwI - A)^*$, one obtains

$$\mathcal{P}(w) = \frac{1}{2\pi}\int_{-w}^{+w} (\mathcal{P}H^*(w) + H(w)\mathcal{P})dw.$$

The £nal equation yields

$$\mathcal{P}(w) = \mathcal{P}S^*(w) + S(w)\mathcal{P} \tag{6.14}$$

where

$$S(w) := \frac{1}{2\pi}\int_{-w}^{+w} H(w)dw = \frac{j}{2\pi}\ln((jwI + A)(-jwI + A)^{-1}) \tag{6.15}$$

A similar argument leads to

$$\mathcal{Q}(w) = S^*(w)\mathcal{Q} + \mathcal{Q}S(w).$$

From the de£nitions of $S(w)$ and $\mathcal{P}$ in (6.11) and (6.15), and the fact that $H(w_1)H(w_2) = H(w_2)H(w_1)$ for any $w_1,\ w_2 \in \mathbb{R}$, follows that

$$S(w)\mathcal{P} = \frac{1}{4\pi^2}\int_{-w}^{+w}\int_{-\infty}^{+\infty} H(w)H(\phi)BB^T H^*(\phi)d\phi dw = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\phi)S(w)BB^T H^*(\phi)d\phi. \tag{6.16}$$

Plugging this into (6.14) gives

$$\mathcal{P}(w) = \mathcal{P}S^*(w) + S(w)\mathcal{P} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\phi)W_c(w)H^*(\phi)d\phi, \tag{6.17}$$

where $W_c(w) := S(w)BB^T + BB^T S^*(w)$. Since A is asymptotically stable, $\mathcal{P}(w)$ is the solution to the Lyapunov equation

$$A\mathcal{P}(w) + \mathcal{P}(w)A^T + W_c(w) = 0.$$

Therefore, the weighted gramian $\mathcal{P}_\Omega$ in (6.12) is obtained by solving

$$A\mathcal{P}_\Omega + \mathcal{P}_\Omega A^T + W_c(\Omega) = 0, \tag{6.18}$$

where $W_c(\Omega) := W_c(w_2) - W_c(w_1)$. A similar argument yields

$$A^T\mathcal{Q}_\Omega + \mathcal{Q}_\Omega A^T + W_o(\Omega) = 0, \tag{6.19}$$

where $W_o(\Omega) := W_o(w_2) - W_o(w_1)$, and $W_o(w) := S^*(w)C^TC + C^TCS^(w)$. Hence the computations of $\mathcal{P}_\Omega$ and $\mathcal{Q}_\Omega$ require evaluating the logarithm in $S(w)$ in addition to solving two Lyapunov equations. For small-to-medium scale problems for which an exact balanced realization can be computed, $S(w)$ can be ef£ciently computed as well. However, for large-scale problems, this issue is still under investigation. But we note that computing an exact solution to a Lyapunov equation in large-scale settings is an ill-conditioned problem itself. Therefore $S(w)$ can be computed whenever a balanced realization can be computed.

Gawronski and Juang's frequency weighted method is obtained by balancing (simultaneously diagonalizing) $\mathcal{P}_\Omega$ and $\mathcal{Q}_\Omega$, i.e., £nding a basis so that

$$\mathcal{P}_\Omega = \mathcal{Q}_\Omega = \mathrm{diag}(\sigma_{n_1}I_{n_1}, \cdots, \sigma_{n_q}I_{n_q}), \tag{6.20}$$

where n_i are the multiplicities of each singular value σ_i and $n_1 + \cdots + n_q = n$. Then the reduced order model is obtained by truncation. However, since $W_c(\Omega)$ and $W_o(\Omega)$ are not guaranteed to be positive de£nite, stability of the reduced model cannot be guaranteed.

As seen from the above discussion, the construction of input and output weights is avoided by de£ning the gramians over the speci£ed frequency range. A comparison with the other methods will be presented in Section 7.

6.6 A modi£ed frequency weighted balanced truncation method

In this section, we will introduce a modi£cation to Gawronski and Juang's and obtain a frequency balancing method which guarantees stability and provides a simple error result.

Given the set-up in Section 6.5, let $W_c(\Omega)$ and $W_c(\Omega)$ have the following EVD:

$$W_c(\Omega) := M\Lambda M^T = M\mathrm{diag}(\lambda_1, \cdots, \lambda_n)M^T \tag{6.21}$$

$$W_o(\Omega) := N\Delta N^T = N\mathrm{diag}(\delta_1, \cdots, \delta_n)N^T, \tag{6.22}$$

where $MM^T = NN^T = I_n$ with $|\lambda_1| \geq \cdots |\lambda_n| \geq 0$ and $|\delta_1| \geq \cdots |\delta_n \geq 0$. Since both $W_c(\Omega)$ and $W_o(\Omega)$ are symmetric, such decompositions exist. Let ρ and ϱ denote the ranks of, respectively, $W_c(\Omega)$ and $W_o(\Omega)$. Based on these de£nitions, let

$$\begin{aligned} \hat{B} &:= M\mathrm{diag}(|\lambda_1|^{1/2}, \cdots, |\lambda_\rho|^{1/2}, \cdots, 0, \cdots, 0) \quad \text{and} &(6.23)\\ \hat{C} &:= \mathrm{diag}(|\delta_1|^{1/2}, \cdots, |\delta_\varrho|^{1/2}, \cdots, 0, \cdots, 0)N^T. &(6.24)\end{aligned}$$

We now de£ne the modi£ed frequency weighted gramians $\bar{\mathcal{P}}_\Omega$ and s $\bar{\mathcal{Q}}_\Omega$ as the solutions to

$$A\bar{\mathcal{P}}_\Omega + \bar{\mathcal{P}}_\Omega A^T + \hat{B}\hat{B}^T = 0 \quad \text{and} \quad \bar{\mathcal{Q}}_\Omega A + A^T\bar{\mathcal{Q}}_\Omega + \hat{C}^T\hat{C} = 0. \tag{6.25}$$

Then the modi£ed frequency weighted balancing is obtained by simultaneously diagonalizing $\bar{\mathcal{P}}_\Omega$ and $\bar{\mathcal{Q}}_\Omega$, i.e. in the balanced basis, we have

$$\bar{\mathcal{P}}_\Omega = \bar{\mathcal{Q}}_\Omega = \mathrm{diag}(\bar{\sigma}_{\tau_1} I_{\tau_1}, \cdots, \bar{\sigma}_{\tau_q} I_{\tau_q})$$

where $\bar{\sigma}_i$ are the modi£ed frequency-weighted singular values, τ_i are the multiplicities of $\bar{\sigma}_i$ and $\tau_1 + \cdots + \tau_q = n$.

Theorem 6.5 *Let the asymptotically stable, minimal system $G(s)$ have the following modi£ed frequency weighted balanced realization:*

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \text{ with } \bar{\mathcal{P}}_\Omega = \bar{\mathcal{Q}}_\Omega = \mathrm{diag}(\bar{\sigma}_{\tau_1} I_{\tau_1}, \cdots, \bar{\sigma}_{\tau_k} I_{\tau_k}, \bar{\sigma}_{\tau_{k+1}} I_{\tau_{k+1}}, \cdots, \bar{\sigma}_{\tau_q} I_{\tau_q}).$$

Let $G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ be obtained by truncation. Then $G_r(s)$ is balanced and stable. If, in addition,

$$\mathrm{rank}([\,B \;\; \hat{B}\,]) = \mathrm{rank}(\hat{B}) \quad \text{and} \quad \mathrm{rank}([\,C^T \;\; \hat{C}^T\,]) = \mathrm{rank}(\hat{C}^T) \tag{6.26}$$

then $G_r(s)$ is asymptotically stable, minimal and satis£es

$$\| G(s) - G_r(s) \|_{\mathcal{H}_\infty} \leq 2\|J_B\|\|J_C\| \sum_{i=k+1}^{q} \bar{\sigma}_i, \tag{6.27}$$

where $J_B := \mathrm{diag}(|\lambda_1|^{-1/2}, \cdots, |\lambda_\rho|^{-1/2}, \cdots, 0, \cdots, 0)M^T B$ and $J_C := CN\mathrm{diag}(|\delta_1|^{-1/2}, \cdots, |\delta_\varrho|^{-1/2}, \cdots, 0, \cdots, 0)$.

Proof: The £rst part of the theorem is clear. By assumption (6.26), there exist J_B and J_c such that $B = \hat{B}J_B$ and $C = J_C\hat{C}$. Asymptotic stability follows from the fact that the reachability of the pair (A, B) implies the reachability of the pair $(A, \hat{B})$ and the observability of the pair (C, A) implies the observability of the pair $(\hat{C}, A)$. To prove the error bound, we proceed as follows:

$$\begin{aligned} \|G(s) - G_r(s)\|_{\mathcal{H}_\infty} &= \|C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1\|_{\mathcal{H}_\infty} &(6.28)\\ &= \|J_C\left(\hat{C}(sI - A)^{-1}\hat{B} - \hat{C}_1(sI - A_{11})^{-1}\hat{B}_1\right)J_B\|_{\mathcal{H}_\infty} &(6.29)\\ &\leq 2\|J_B\|\|J_C\| \sum_{i=k+1}^{q} \bar{\sigma}_i. &(6.30)\end{aligned}$$

This completes the proof of the Theorem.

Remark 6.1 **(1) Discussion on the assumption (6.26):** *Here we follow the steps of [39]. De£ne* $\mathcal{G}(Z) := BZ + Z^T B^T$*. Let* $\mathcal{G}(Z) = M\Lambda M^T$ *be the EVD of* $\mathcal{G}(Z)$*. Denote* $\hat{B} = M \mid \Lambda \mid^{1/2}$*. It was shown in [39] that for almost all* $Z \in \mathbb{C}^{r_1 \times n}$*,* $\mathrm{rank}([\,B\ \ \hat{B}\,]) = \mathrm{rank}(\hat{B})$*. Notice that in our setup,* $Z = B^T(S(w_2) - S(w_1))^*$*. Hence we expect that our approach will apply in most of the cases. Indeed, during our simulations, the assumption has always been satis£ed.*
(2) Multiple Frequency bands *Assume that we want to match* $G(s)$ *over two frequency bands, namely* $[\,w_1,\,w_2\,]$ *and* $[\,w_3,\,w_4\,]$ *where* $w_1 < w_2 < w_3 < w_4$*. Then the weighted reachability gramian is given by*

$$\mathcal{P}_\Omega = \underbrace{\mathcal{P}(w_2) - \mathcal{P}(w_1)}_{:=\mathcal{P}_{12}} + \underbrace{\mathcal{P}(w_4) - \mathcal{P}(w_3)}_{:=\mathcal{P}_{34}}.$$

Since $\mathcal{P}_{12}$ *is the solution to* $A\mathcal{P}_{12} + \mathcal{P}_{12}A^T + W_c(\Omega_{12}) = 0$ *and* $\mathcal{P}_{34}$ *is the solution to* $A\mathcal{P}_{34} + \mathcal{P}_{34}A^T + W_c(\Omega_{34}) = 0$ *where* $W_c(\Omega_{12}) = W_c(w_2) - W_c(w_1)$ *and* $W_c(\Omega_{34}) = W_c(w_4) - W_c(w_3)$*,* $\mathcal{P}_\Omega$ *can be obtained as the solution to*

$$A\mathcal{P}_\Omega + \mathcal{P}_\Omega A^T + W_c(\Omega) = 0,$$

where $W_c(\Omega) = W_c(\Omega_{12}) + W_c(\Omega_{34})$*. Hence the method allows the usage of multiple frequency bands without an increase in the number of Lyapunov equations to be solved.*
(3) *Although the modi£ed method is a frequency weighted balancing algorithm, the above upper bound (6.30) in Theorem 6.5 is an* $\mathcal{H}_\infty$ *bound for the whole frequency range. Therefore, it might be pessimistic in some cases. If one wants to get a tighter result, i.e. an upper bound for the error over the interval* $[\,w_1,\,w_2\,]$*, one can use*

$$\|W(s)(G(s) - G_r(s))W(s)\|_{\mathcal{H}_\infty} \leq 2\|W(s)J_C\|_{\mathcal{H}_\infty}\|J_B W(s)\|_{\mathcal{H}_\infty}\sum_{i=1}^{q}\bar{\sigma}_i,$$

where $W(s)$ *is a perfect band-pass £lter with amplitude 1 over the frequency interval* $[\,w_1,\,w_2\,]$*. We note that this error bound has the same structure as the error bound (6.10) of Wang's et al. method.*

6.7 Connection to Enns' Method

In this subsection we discuss the relationship between the frequency weighted balancing methods of Enns and Gawronski and Juang. Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be the given model. Then the state equation is given by

$$\dot{x}(t) = Ax(t) + Bu(t).$$

De£ne $X_u(s)$ as the transfer function from the input $u(t)$ to the state $x(t)$. It readily follows that $X_u(s)$ is given by $X_u(s) \overset{s}{=} (sI - A)^{-1}B$. Hence the reachability gramian $\mathcal{P}$ in the frequency domain is given by

$$\mathcal{P} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_u(jw)X_u(jw)^* dw = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(jwI - A)^{-1}BB^T(jwI - A)^{-*}dw.$$

Now assume that there is an input weighting with impulse response $w_i(t)$ and the transfer function $W_i(s)$. The new state equation is given by

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B(w_i \star u)(t)$$

where $(\star)$ denotes the convolution operator. Hence the input weighted input-to-state (from $u(t)$ to $\bar{x}(t)$) transfer function $\bar{X}_u(s)$ is given by

$$\bar{X}_u(s) \overset{s}{=} (sI - A)^{-1}BW_i(s).$$

Then the reachability gramian $\bar{\mathcal{P}}$ for the weighted system is computed as

$$\bar{\mathcal{P}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{X}_u(jw)\bar{X}_u(jw)^* dw. = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(jwI-A)^{-1}BW_i(jw)W_i(jw)^*B^T(jwI-A)^{-*}dw.$$

Now let $W_i(s)$ be a bandpass £lter over the frequency interval $[\ w_1,\ w_2\ ]$ with an amplitude 1. Then we obtain

$$\bar{\mathcal{P}} = \mathcal{P}_\Omega = \frac{1}{2\pi}\int_{-w_2}^{-w_1}(jwI-A)^{-1}BB^T(jwI-A)^{-*}dw + \frac{1}{2\pi}\int_{w_1}^{w_2}(jwI-A)^{-1}BB^T(jwI-A)^{-*}dw.$$

This discussion reveals the connection between Enns' and Gawronski and Juang's frequency weighted balancing methods. The latter is obtained from the former by choosing the $W_i(s)$ and $W_o(s)$ as the perfect bandpass £lters over the frequency range of interest. However, the realizations of weights are never computed. We note that an in£nite dimensional realization will be needed to obtain perfect band-pass £lters. Hence, in Enns' method, these band-pass £lters are approximated by low order bandpass £lters. The resulting Lyapunov equations have dimension $n+n_i$ where n_i is the order of $W_i(s)$. In Section 7, we will show that as the order of the weightings increases, i.e., as they get closer to perfect bandpass £lters, the two methods show similar behavior. Moreover, since our modi£cation to Gawronski and Juang's method is analogous to Wang's modi£cation to Enns' method, we expect that our modi£ed method will yield close approximants to those of Wang's method as the order of the weights is increased. The simulations in Section 7.2 show this to be the case.

6.8 Gawronski and Juang's balanced reduction method using time-limited gramians [16]

Next, we review the time-limited balanced reduction method of Gawronski and Juang [16] since it follows a very similar approach to their frequency weighted balanced reduction method. In the time domain, the reachability gramian $\mathcal{P}$ and observability gramian $\mathcal{Q}$ are given by

$$\mathcal{P} = \int_0^\infty e^{A\tau}BB^Te^{A^T\tau}d\tau \quad \text{and} \quad \mathcal{Q} = \int_0^\infty e^{A^T\tau}C^TCe^{A\tau}d\tau.$$

For a £nite time interval $T = [\ t_1,\ t_2\ ]$, the time limited gramians are de£ned as

$$\mathcal{P}_T = \int_{t_1}^{t_2} e^{A\tau}BB^Te^{A^T\tau}d\tau \quad \text{and} \quad \mathcal{Q}_T = \int_{t_1}^{t_2} e^{A^T\tau}C^TCe^{A\tau}d\tau. \tag{6.31}$$

Let $\theta_c(t) := \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau$. It follows that (see [25, 16])

$$\theta_c(t) = \mathcal{P} - S_c(t)\mathcal{P}S_c(t)^T \ \text{ where } \ S_c(t) := e^{At}.$$

From the de£nition of $\mathcal{P}_T$ in (6.31), one obtains

$$\begin{aligned}
\mathcal{P}_T &= \theta_c(t_2)-\theta_c(t_1) = S_c(t_1)\mathcal{P}S_c(t_1)^T - S_c(t_2)\mathcal{P}S_c(t_2)^T && (6.32)\\
&= \int_0^\infty e^{A\tau}\underbrace{\left(e^{At_1}BB^Te^{A^Tt_1} - e^{At_2}BB^Te^{A^Tt_2}\right)}_{:=V_c(T)}e^{A^T\tau}d\tau && (6.33)\\
&= \int_0^\infty e^{A\tau}V_c(T)e^{A^T\tau}d\tau. && (6.34)
\end{aligned}$$

A similar argument yields

$$\mathcal{Q}_T = \int_0^\infty e^{A^T\tau}V_c(T)e^{A\tau}d\tau. \tag{6.35}$$

where $V_o(T) := e^{A^T t_1} C^T C e^{At_1} - e^{A^T t_2} C^T C e^{At_2}$. Hence $\mathcal{P}_T$ and $\mathcal{Q}_T$ are the solutions to the following Lyapunov equations:

$$A\mathcal{P}_T + \mathcal{P}_T A^T + V_c(T) = 0 \text{ and } A^T \mathcal{Q}_T + \mathcal{Q}_T A + V_o(T) = 0.$$

It was suggested in [16] that time-limited balanced realization is obtained by balancing the time-limited gramians $\mathcal{P}_T$ and $\mathcal{Q}_T$, i.e.

$$\mathcal{P}_T = \mathcal{Q}_T = \operatorname{diag}(\sigma_{n_1} I_{n_1}, \cdots, \sigma_{n_q} I_{n_q}).$$

A reduced model is then obtained by truncation. The impulse response of the reduced model is thus expected to match that of the full order model in the time interval $T = [\ t_1,\ t_2\ ]$, see [16]. However, as in the frequency weighted case, the reduced model is not guaranteed to be stable. Below, we will modify the time limited gramians and the corresponding model reduction scheme as we did in the frequency weighted case to guarantee stability. We follow the same steps as in Section 6.6.

6.8.1 A modi£ed time-limited balanced truncation method

Given the set-up above, let

$$V_c(T) := \bar{M}\bar{\Lambda}\bar{M}^T = \bar{M}\operatorname{diag}(\bar{\lambda}_1, \cdots, \bar{\lambda}_n)\bar{M}^T \text{ and } V_o(T) := \bar{N}\bar{\Delta}\bar{N}^T = \bar{N}\operatorname{diag}(\bar{\delta}_1, \cdots, \bar{\delta}_n)\bar{N}^T \tag{6.36}$$

be the EVD of $V_c(T)$ and $V_o(T)$ with $\bar{M}\bar{M}^T = \bar{N}\bar{N}^T = I_n$, $|\ \bar{\lambda}_1\ | \geq \cdots |\ \bar{\lambda}_n\ | \geq 0$ and $|\ \bar{\delta}_1\ | \geq \cdots |\ \bar{\delta}_n\ | \geq 0$. De£ne $\bar{\rho} := \operatorname{rank}(V_o(T))$ and $\bar{\varrho} := \operatorname{rank}(V_c(T))$. Let

$$\begin{aligned} B_T &:= M\operatorname{diag}(|\ \bar{\lambda}_1\ |^{1/2}, \cdots, |\ \bar{\lambda}_{\bar{\rho}}\ |^{1/2}, \cdots, 0, \cdots, 0) \text{ and} & (6.37)\\ C_T &:= \operatorname{diag}(|\ \bar{\delta}_1\ |^{1/2}, \cdots, |\ \bar{\delta}_{\bar{\varrho}}\ |^{1/2}, \cdots, 0, \cdots, 0)N^T. & (6.38)\end{aligned}$$

The modi£ed time-limited gramians $\bar{\mathcal{P}}_T$ and $\bar{\mathcal{Q}}_T$ are obtained as the solutions to

$$A\bar{\mathcal{P}}_T + \bar{\mathcal{P}}_T A^T + B_T B_T^T = 0 \quad \text{and} \quad \bar{\mathcal{Q}}_T A + A^T \bar{\mathcal{Q}}_T + C_T^T C_T = 0. \tag{6.39}$$

Then we balance $\bar{\mathcal{P}}_T$ and $\bar{\mathcal{Q}}_T$, i.e., £nd a basis such that

$$\bar{\mathcal{P}}_T = \bar{\mathcal{Q}}_T = \operatorname{diag}(\bar{\sigma}_{\tau_1} I_{\tau_1}, \cdots, \bar{\sigma}_{\tau_q} I_{\tau_q}),$$

where $\bar{\sigma}_i$ are the modi£ed singular values, τ_i are the multiplicities of σ_i and $\tau_1 + \cdots + \tau_q = n$.

Corollary 6.2 *Let the asymptotically stable, minimal system $G(s)$ have the following modi£ed time-limited balanced realization:*

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \text{ with } \bar{\mathcal{P}}_T = \bar{\mathcal{Q}}_T = \operatorname{diag}(\bar{\sigma}_{\tau_1} I_{\tau_1}, \cdots, \bar{\sigma}_{\tau_k} I_{\tau_k}, \bar{\sigma}_{\tau_{k+1}} I_{\tau_{k+1}}, \cdots, \bar{\sigma}_{\tau_q} I_{\tau_q}).$$

Let $G_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ be obtained by truncation. Then $G_r(s)$ is balanced, stable and minimal. If, in addition,

$$\operatorname{rank}([\ B\ \ B_T\ ]) = \operatorname{rank}(B_T) \quad \text{and} \quad \operatorname{rank}([\ C^T\ \ C_T^T\ ]) = \operatorname{rank}(C_T^T), \tag{6.40}$$

$G_r(s)$ is asymptotically stable, minimal and satis£es

$$\|\ G(s) - G_r(s)\ \|_{\mathcal{H}_\infty} \leq 2\|\bar{J}_B\|\|\bar{J}_C\| \sum_{i=k+1}^{q} \bar{\sigma}_i, \tag{6.41}$$

where $\bar{J}_B := \operatorname{diag}(|\ \bar{\lambda}_1\ |^{-1/2}, \cdots, |\ \bar{\lambda}_{\bar{\rho}}\ |^{-1/2}, 0, \cdots, 0)M^T B$ and $\bar{J}_C := CN\operatorname{diag}(|\ \bar{\delta}_1\ |^{-1/2}, \cdots, |\ \bar{\delta}_{\bar{\varrho}}\ |^{-1/2}, 0, \cdots, 0)$.

Proof: Follows similarly to the proof of Theorem 6.5.

6.9 Numerical issues in computing the balanced truncation

The various balancing transformations and the corresponding balanced reduction schemes discussed require balancing of the whole system $G(s)$ followed by truncation. This is numerically inef£cient and ill-conditioned for large-scale settings. Instead, below we will propose another implementation of balanced reduction which directly obtains a reduced balanced system without balancing the whole $G(s)$.

Let $\mathcal{P}$ and $\mathcal{Q}$ denote the gramians corresponding to the underlying balancing method. For all of the balancing methods studied above, $\mathcal{P}$ and $\mathcal{Q}$ and can be written as $\mathcal{P} = UU^T$ and $\mathcal{Q} = LL^T$ since both $\mathcal{P}$ and $\mathcal{Q}$ are symmetric positive de£nite matrices. U and L are called **square roots** of the gramians $\mathcal{P}$ and $\mathcal{Q}$ respectively. Let $U^T L = ZSY^T$ be the singular value decomposition (SVD). It is easy to show that the singular values of $U^T L$ are the corresponding singular values of $G(s)$, hence we have $U^T L = Z\Sigma Y^T$ where $\Sigma = \mathrm{diag}(\sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \ldots, \sigma_q I_{m_q})$, q is the number of distinct singular values with $\sigma_{m_i} > \sigma_{m_{i+1}} > 0$, m_i's is the multiplicity of σ_i, and $m_1 + m_2 + \cdots + m_q = n$. Let $\Sigma_1 = diag(\sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \ldots, \sigma_k I m_k)$, $k < q$, $r := m_1 + \cdots + m_k$ and de£ne

$$W_1 := LY_1\Sigma_1^{-1/2} \text{ and } V_1 := UZ_1\Sigma_1^{-1/2},$$

where Z_1 and Y_1 are composed of the leading r columns of Z and Y respectively. It is easy to check that $W_1^T V_1 = I_r$ and hence that $V_1 W_1^T$ is an oblique projector. We obtain a reduced model of order r by setting

$$A_r = W_1^T A V_1, \;\; B_r = W_1^T B, \;\; C_r = CV_1.$$

Noting that $\mathcal{P}W_1 = V_1 S_1$ and $\mathcal{Q}V_1 = W_1 S_1$ yields that the reduced model is balanced (of the appropriate type) for any $k \leq q$. The formulas above provide a numerically stable scheme for computing the reduced order model based on a numerically stable scheme for computing the square roots U and L directly in upper triangular and lower triangular form respectively. It is important to truncate Z, Σ, Y to Z_1, Σ_1, Y_1 prior to forming W_1 or V_1. $\mathcal{P}$ and $\mathcal{Q}$ are often found to have *numerically* low-rank compared to n. In most cases, the eigenvalues of $\mathcal{P}, \mathcal{Q}$ as well as the singular values $\sigma_i(G(s))$ decay rapidly. For a discussion on decay rates, see [7]. Therefore, it is also important to avoid formulas involving the inverses of L or U as these matrices are typically ill-conditioned due to this rapid decay of the eigenvalues of the gramians.

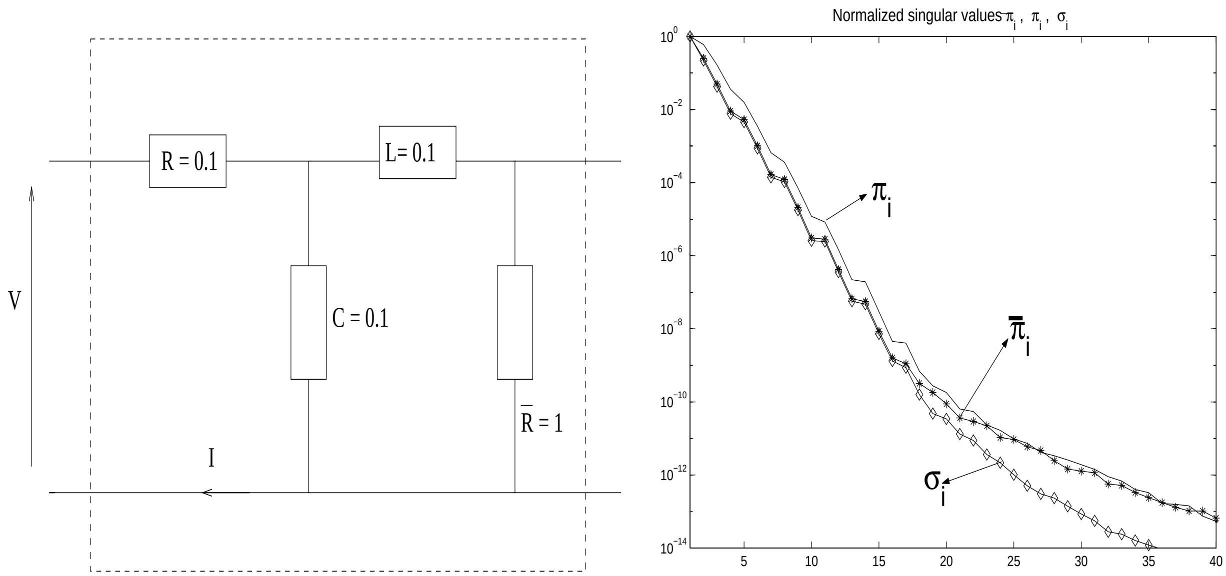
Due to this low-rank phenomenon, in large-scale settings, U and L are approximated by their low-rank versions and approximate balancing is applied. We refer the reader to [23], [5], [6], [34], [35], [41] and the references therein for further information on these issues.

7 Examples

7.1 An example on positive-real balancing

Consider a circuit, $G(s)$ consisting of 50 sections interconnected in cascade; each section is as shown in Figure 1. The input is the voltage V applied to the £rst section; the output is the current I of the £rst section. The order of the overall system is $n = 100$. We apply 3 methods, namely **(i)** Positive real balanced reduction (**PRBR**) **(ii)** Modi£ed positive real balanced reduction (**MPRBR**) and **(iii)** Lyapunov balanced reduction (**LBR**); and reduce the order to $k = 10$. We note that $G(s)$ belongs to the family $\mathcal{D}$, hence allowing the usage of **MPRBR**. We £rst compute the Hankel singular values σ_i, the positive real singular values π_i and the modi£ed positive real singular values $\bar{\pi}_i$ of $G(s)$. The largest 40 of the normalized[4] singular values are shown in Figure 1-b. As the £gure illustrates, even though the computation of these singular values are different, they all show a very similar decay behavior. Hence each of these 3 sets of singular values reveals that the decay rate is fast, consequently $G(s)$ is easy to approximate. Finally, we note that the decay rates of π_i and $\bar{\pi}_i$ are almost the same.

[4]For better comparison, the highest singular values, i.e. σ_1, π_1 and $\bar{\pi}_1$ are normalized to 1.



(a) (b)

Figure 1: (a) One section of the circuit (b) Normalized singular values σ_i, π_i and $\bar{\pi}_i$

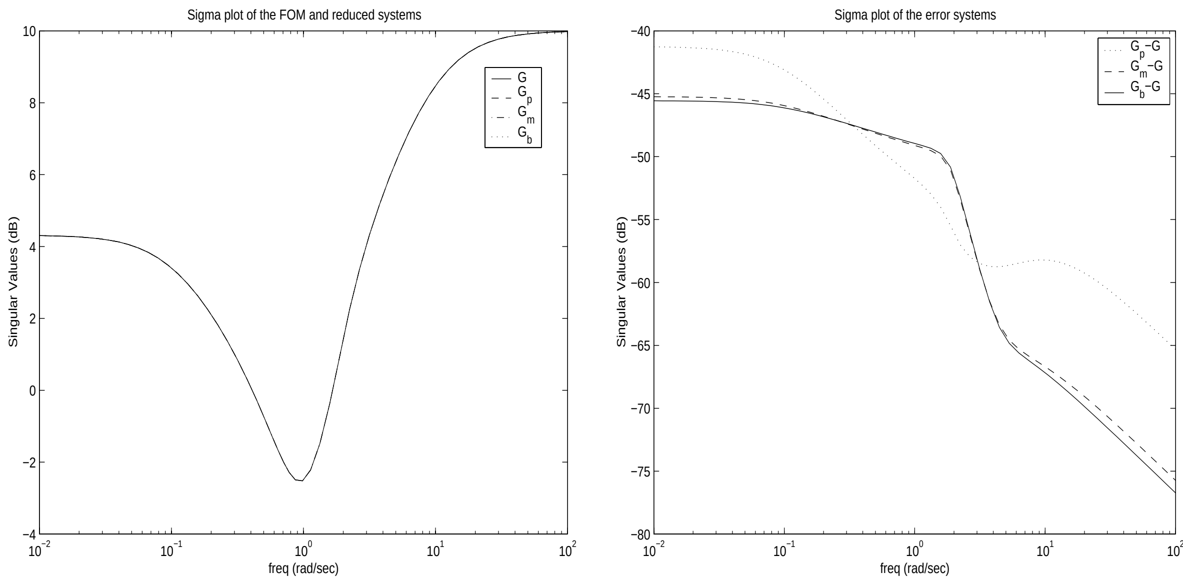
The sigma plots of the reduced and error systems are depicted in Figure 2-a and 2-b respectively. Let $G_b(s)$, $G_p(s)$ and $G_m(s)$ denote the reduced models obtained by, respectively, **LBR**, **PRBR** and **MPRBR**. Figure 2-a shows that all the reduced models approximate $G(s)$ well. In order to compare them better, we examine the error plot Figure 2-b. Figure 2-b reveals that $G_m(s)$ and $G_b(s)$ are very close to each other and they are both slightly better than $G_p(s)$. The error norms and corresponding upper bounds are tabulated in Table 1. As Table 1 illustrates, the error bound (5.16) for $G_p(s) + D$ and the absolute error bound (5.21) for $G_m(s)$ are tight like the upper bound (2.3) for $G_b(s)$. The results indicate that when $G(s) \in \mathcal{D}$, **MPRBR** is a promising alternative to **PRBR**.

	Exact error	Upper bound
$\|G(s) - G_b(s)\|_{\mathcal{H}_\infty}$	2.7×10^{-5}	2.9×10^{-5}
$\|G(s) - G_m(s)\|_{\mathcal{H}_\infty}$	3×10^{-5}	3.5×10^{-5}
$\|G(s) - G_p(s)\|_{\mathcal{H}_\infty}$	7.4×10^{-5}	
$\|(D^T + G_p(s))^{-1}(G(s) - G_p(s))\|_{\mathcal{H}_\infty}$	5.9×10^{-6}	1.4×10^{-5}
$\|(D^T + G(s))^{-1} - (D^T + G_p(s))^{-1}\|_{\mathcal{H}_\infty}$	$4.6. \times 10^{-7}$	7.2×10^{-7}

Table 1: Error norms and the corresponding upper bounds for the Circuit Example

7.2 An example of weighted balanced reduction

The full order model (FOM) describes the dynamics of a portable CD player and is single-input single-output of order 120. The sigma plot of the FOM is shown in Figure 3-(a). First we choose $w_1 = 10$ and $w_2 = 1 \times 10^3$ to match the maximum peak of the sigma plot. We reduce the order to $k = 15$ by applying **(i)** Gawronski and Juang's frequency weighted balanced truncation (**GFBT**), **(ii)** our modi£ed version of Gawronski and Juang's frequency weighted balanced truncation (**MGFBT**) and **(iii)** the unweighted Lyapunov balanced truncation (**LBT**). Let $G_f(s)$, $G_{mf}(s)$ and $G_b(s)$ denote the reduced order models obtained by using **GFBT**, **MGFBT** and **LBT**, respectively. The sigma plots of the reduced and error systems are depicted in Figure 3-(a) and Figure 3-(b). As Figure 3-(b) shows $G_f(s)$ and $G_{mf}(s)$ outperform $G_b(s)$ in the chosen frequency interval, by matching the peak of $G(s)$ better than $G_b(s)$. Furthermore, $G_{mf}(s)$ and $G(s)$ behave very similarly. Hence, for this example, our



(a) (b)

Figure 2: σ_{max} plot of the (a) reduced and (b) error systems of the circuit example

modi£cation to **GFBT** did not have a negative impact on the quality of approximation in the speci£ed region, on the other hand it added asymptotic stability and resulted in an error bound. The $\mathcal{H}_\infty$ errors and corresponding error norms are tabulated in Table 2.

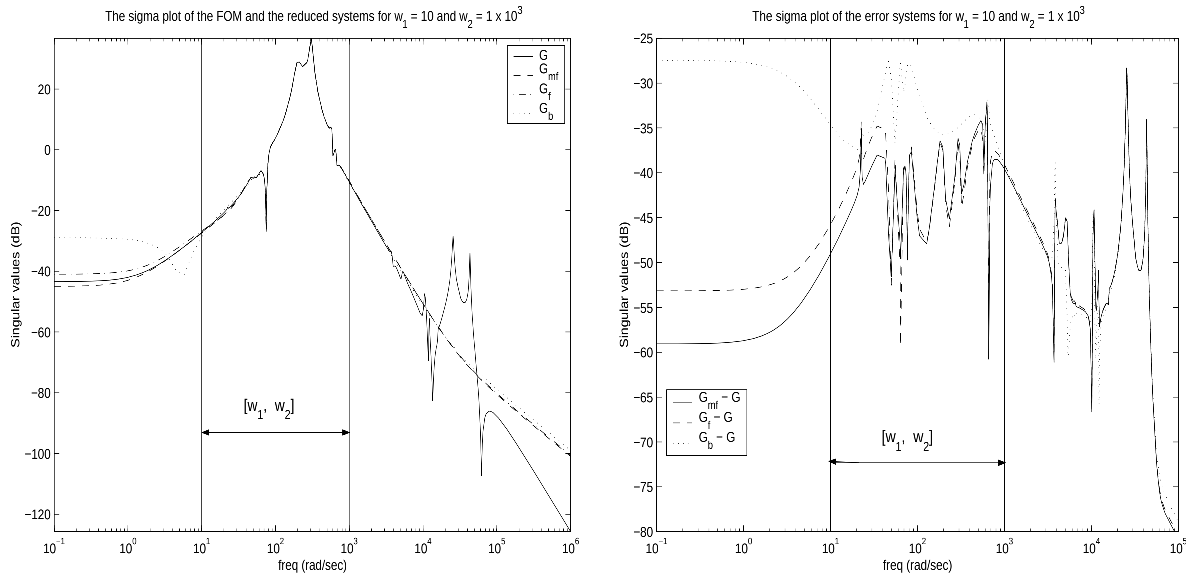
	Exact error	Upper bound
$\|\|G(s) - G_b(s)\|\|_{\mathcal{H}_\infty}$	4.23×10^{-2}	2.36×10^{-1}
$\|\|G(s) - G_{mf}(s)\|\|_{\mathcal{H}_\infty}$	3.84×10^{-2}	3.40×10^{-1}
$\|\|G(s) - G_f(s)\|\|_{\mathcal{H}_\infty}$	3.85×10^{-2}	

Table 2: Error norms and the corresponding upper bounds for the CD Player Example for $w_1 = 10$ and $w_2 = 1 \times 10^3$

Figures 3-(a) and 3-(b) reveal that all the reduced models miss the ripples of $G(s)$ between the frequencies 10^4 and 10^5 rad/sec. To match this part of the sigma plot, we choose $w_1 = 5 \times 10^3$ and $w_2 = 1 \times 10^5$. Figures 4-(a) and 4-(b) show the sigma plots of the resulting reduced systems and error systems. As expected, $G_f(s)$ and $G_{mf}(s)$ match $G(s)$ around the speci£ed interval and reproduce the ripples of the sigma plots. If we look at the error plots Figures 4-(a), we see that over the selected frequency interval even though $G_{mf}(s)$ matches $G(s)$ quite well, $G_f(s)$ behaves better than $G_{mf}(s)$. This is due to the fact that the modi£ed gramians are no longer the exact frequency-limited gramians. With the modi£cation and the guaranteed stability, $G_{mf}(s)$ performs slightly worse than $G_f(s)$ over $[\ w_1,\ w_2\ ]$, however the over all response is better. We note that while $G_{mf}(s)$ matches the peak of the sigma plot over $[\ 10, 10^3\ ]$ rad/sec, $G_f(s)$ is far from $G(s)$ over this range. The conclusion is that there is a trade-off between guaranteed stability and performance in the speci£ed frequency interval. The same observation is valid for Enns' frequency weighted balanced truncation and Wang's *et al.* modi£cation with guaranteed stability.

The $\mathcal{H}_\infty$ norms and the upper bounds of the error systems are presented in Table 3. As the table shows, the upper bound for $||G(s) - G_{mf}(s)||_{\mathcal{H}_\infty}$ is pessimistic for this example. As explained in Remark 6.1 **(3)**, this is because of the fact that although **MFBT** is a frequency weighted method, the bound is valid for the whole frequency range.

Next, we examine the issues of Section 6.7, i.e. **(i)** the relationship between Gawronski and Juang's frequency weighted balancing method, **GFBT**, and Enns' frequency weighted balancing method (**EFBT**), and also



(a) (b)

Figure 3: σ_{max} plot of the (a) reduced and (b) error systems of the CD player example for $w_1 = 10$ and $w_2 = 1 \times 10^3$

	Exact error	Upper bound
$\|\|G(s) - G_b(s)\|\|_{\mathcal{H}_\infty}$	4.23×10^{-2}	2.36×10^{-1}
$\|\|G(s) - G_{mf}(s)\|\|_{\mathcal{H}_\infty}$	1.45×10^{0}	1.70×10^{1}
$\|\|G(s) - G_f(s)\|\|_{\mathcal{H}_\infty}$	6.83×10^{1}	

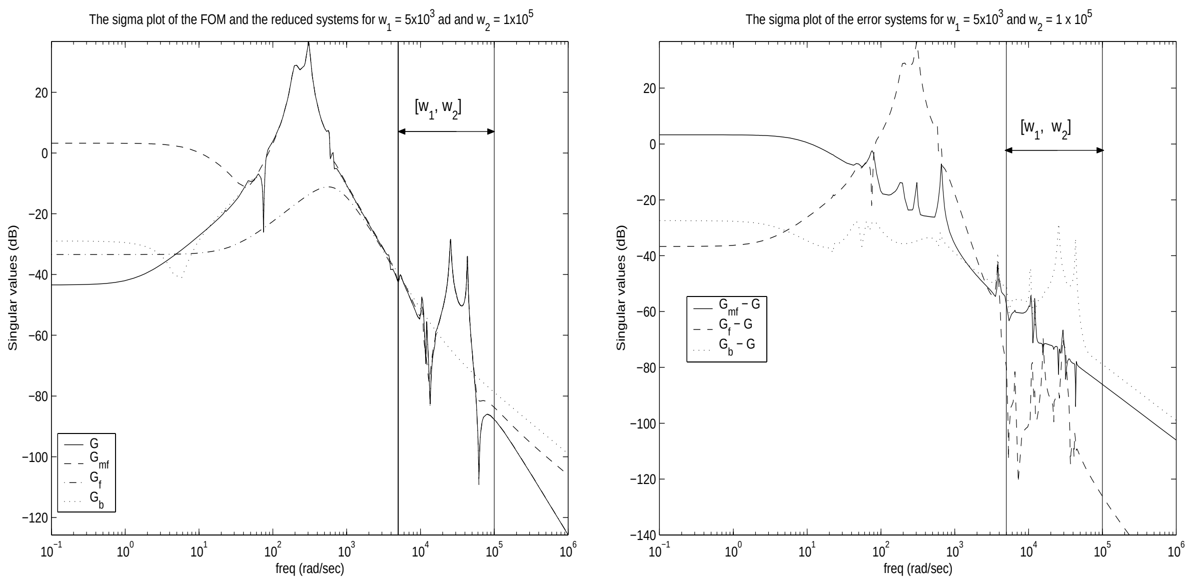
Table 3: Error norms and the corresponding upper bounds for the CD Player Example for $w_1 = 5 \times 10^3$ and $w_2 = 1 \times 10^5$

(ii) the relationship between our modi£ed version of Gawronski and Juang's method, **MGFBT**, and Wang's method (**WFBT**). We choose $w_1 = 10$ and $w_2 = 10^3$ and apply these four methods to reduce to order $k = 15$. For **EFBT** and **WFBT**, we take $W_i(s) = W_o(s)$ as Butterworth band-pass £lters over the frequency band $[\, w_1,\, w_2 \,]$. We keep increasing the order of the Butterworth £lter, denoted by n_b and compare **GFBT** with **EFBT** and **MGFBT** with **WFBT** as n_b increases. The n_b values we choose are: $4,\ 6,\ 10, 20,\ 40,\ 80,\ 100$. Table 4 tabulates the numerical results. In this table, the following notation is used:

$$\mathcal{E}_1 := \frac{\|\mathcal{P}_\Omega - \mathcal{P}_{11}\|}{\|\mathcal{P}_\Omega\|},\ \ \mathcal{E}_2 := \frac{\|\mathcal{Q}_\Omega - \mathcal{Q}_{11}\|}{\|\mathcal{Q}_\Omega\|},\ \ \mathcal{E}_3 := \frac{\|G_f(s) - G_e(s)\|_{\mathcal{H}_\infty}}{\|G_f(s)\|_{\mathcal{H}_\infty}},$$

$$\mathcal{E}_4 := \frac{\|\bar{\mathcal{P}}_\Omega - \bar{\mathcal{P}}\|}{\|\bar{\mathcal{P}}_\Omega\|},\quad \mathcal{E}_5 := \frac{\|\bar{\mathcal{Q}}_\Omega - \bar{\mathcal{Q}}\|}{\|\bar{\mathcal{Q}}_\Omega\|},\quad \mathcal{E}_6 := \frac{\|G_{mf}(s) - G_w(s)\|_{\mathcal{H}_\infty}}{\|G_{mf}(s)\|_{\mathcal{H}_\infty}}.$$

where $\mathcal{P}_\Omega$ and $\mathcal{Q}_\Omega$ are the gramians of **GFBT**, $\mathcal{P}_{11}$ and $\mathcal{Q}_{11}$ are the gramians of **EFBT**, $\bar{\mathcal{P}}_\Omega$ and $\bar{\mathcal{Q}}_\Omega$ are the gramians of **MGFBT**, and $\bar{\mathcal{P}}$ and $\bar{\mathcal{P}}$ are the gramians of **WFBT**; and $G_f(s)$, $G_{mf}(s)$, $G_e(s)$ and $G_w(s)$ are the reduced models obtained by **GFBT**, **MGFBT**, **EFBT**, and **WFBT** respectively. Note that all the error quantities are chosen as relative errors. Table 4 clearly illustrates that as n_b increases, i.e. as $W_i(s)$ and $W_o(s)$ become exact band-pass £lters, **EFBT** converges to **GFBT**, and **WFBT** converges to **MGFBT**. For $n_b = 100$, the corresponding reduced systems are very close. This shows that using frequency limited gramians, we apply frequency weighted balancing with weights being ideal band-pass £lters without computing them.



(a) (b)

Figure 4: σ_{max} plot of the (a) reduced and (b) error systems of the CD player example for $w_1 = 5 \times 10^3$ and $w_2 = 1 \times 10^5$

	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$	$\mathcal{E}_5$	$\mathcal{E}_6$
$n_b = 4$	8.84×10^{-3}	3.27×10^{-2}	4.93×10^{-5}	3.72×10^{-3}	1.44×10^{-2}	3.42×10^{-5}
$n_b = 6$	1.90×10^{-3}	6.74×10^{-3}	2.45×10^{-5}	1.16×10^{-3}	1.80×10^{-3}	1.46×10^{-5}
$n_b = 10$	4.74×10^{-4}	5.16×10^{-4}	9.00×10^{-6}	5.25×10^{-4}	2.04×10^{-3}	6.87×10^{-6}
$n_b = 20$	1.11×10^{-4}	6.93×10^{-5}	2.15×10^{-6}	1.29×10^{-4}	5.04×10^{-4}	1.87×10^{-6}
$n_b = 40$	2.75×10^{-5}	1.67×10^{-5}	5.29×10^{-7}	3.21×10^{-5}	1.23×10^{-4}	4.39×10^{-7}
$n_b = 80$	1.22×10^{-5}	7.4×10^{-6}	2.32×10^{-7}	1.42×10^{-5}	5.49×10^{-5}	1.92×10^{-7}
$n_b = 100$	4.39×10^{-6}	2.66×10^{-6}	8.27×10^{-8}	5.12×10^{-6}	1.97×10^{-5}	6.90×10^{-8}

Table 4: The relative errors between **GFBT**, **MGFBT**, **EFBT**, and **WFBT** related quantities

8 Conclusions

In this paper, we have presented a survey of balancing-related model reduction schemes and their corresponding error norms, and also introduced some new results. For positive-real balancing, we introduced an error bound. Also, for a certain subclass of positive real system, we proposed a modi£ed positive-real balancing scheme with an absolute error bound. Moreover, a new frequency-weighted balanced reduction method with guaranteed stability and $\mathcal{H}_\infty$ bound on the error system is developed. Two numerical examples have illustrated the ef£ciency of the proposed algorithms.

References

[1] A.C. Antoulas, *Lectures on the approximation of linear dynamical systems*, Advances in Design and Control, Draft, SIAM, Philadelphia, 2004.

[2] A.C. Antoulas, “Approximation of linear operators in the 2-norm”, Special Issue of LAA (Linear Algebra and Applications) on *Challenges in Matrix Theory*, **278**: 309-316, (1998).

[3] A.C. Antoulas, “Approximation of linear dynamical systems”, in the Wiley Encyclopedia of Electrical and Electronics Engineering, edited by J.G. Webster, volume **11**: 403-422 (1999).

[4] A. C. Antoulas, D. C. Sorensen, and S. Gugercin, *A survey of model reduction methods for large scale systems*, Contemporary Mathematics, AMS Publications, 280: 193-219, 2001.

[5] A.C. Antoulas and D.C. Sorensen, *Projection methods for balanced model reduction*, to appear in Linear Algebra and Its Applications, 2004.

[6] A.C. Antoulas and D.C. Sorensen, *The Sylvester equation and approximate balanced reduction*, Fourth Special Issue on Linear Systems and Control, Edited by V. Blondel, D. Hinrichsen, J. Rosenthal, and P.M. van Dooren, Linear Algebra and Its Applications, **351-352**: 671-700, 2002.

[7] A.C. Antoulas, D.C. Sorensen, and Y.K. Zhou, *On the decay rate of Hankel singular values and related issues*, Systems and Control Letters, vol 46 (5), pp. 323-342, 2002.

[8] B.D.O Anderson and S, Vongpanitlerd, *Network analysis and Synthesis*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[9] V. Belevitch, *Classical Network Theory*, San Fransisco, CA: Holdem-Day, 1968.

[10] P. Benner, E. S. Quintana-Orti, and G. Quintana-Orti, *Ef£cient Numerical Algorithms for Balanced Stochastic Truncation*, International Journal of Applied Mathematics and Computer Science, Special Issue: Numerical Analysis and Systems Theory, Editor Stephen L. CAMPBELL, Vol. 11, No. 5, 2001.

[11] T. Chiu, *Model reduction by the low-frequency approximation balancing method for unstable systems*, IEEE trans. Automat. Control, Vol. AC-41(7), pp. 995-997, 1996.

[12] U.B. Desai and D. Pal, *A transformation approach to stochastic model reduction*, IEEE Trans. Automat. Contr., vol AC-29, 1984.

[13] D. Enns, *Model reduction with balanced realizations: An error bound and a frequency weighted generalization*, in the Proceedings of the 23^{rd} IEEE Conf. Decision and Control, 1984.

[14] K. Gallivan, P. Van Dooren, and E. Grimme, *A rational Lanczos algorithm for model reduction*, Numerical Algorithms, 2(1-2):33-63, April 1996.

[15] K. Gallivan, P. Van Dooren, and E. Grimme, *On some recent developments in projection-based model reduction*, in ENUMATH 97 (Heidelberg), World Sci. Publishing, River Edge, NJ, 1998, pp. 98–113, 1998.

[16] W. Gawronski and J.-N. Juang, *Model reduction in limited time and frequency intervals*, Int. J. Systems Sci., vol. 21, No. 2, pp. 349-376, 1990.

[17] K. Glover and J.C. Doyle, *State-space formulae for all stabilizing controllers that satisfy an $\mathcal{H}_\infty$-norm bound and relations to risk sensitivity*, Systems Control Lett., vol. 11, pp. 167-172, 1988.

[18] K. Glover, *Multiplicative approximation of linear multivariable systems with $\mathcal{L}_\infty$ error bounds*, in the Proceedings of the ACC’86, pp. 1705-1709, 1986.

[19] M. Green, *A relative error bound for balanced stochastic truncation*, IEEE Trans. Automat. Contr., vol. AC-33, No. 10, pp. 961-965.

[20] M. Green, *Balanced stochastic realizations*, Journal of Linear Algebra and its Applications, Vol. 98, pp. 211-247, 1988.

[21] E.J. Grimme, *Krylov Projection Methods for Model Reduction*, Ph.D. Thesis, ECE Dept., U. of Illinois, Urbana-Champaign,(1997).

[22] S. Gugercin and A. C. Antoulas, *A comparative study of 7 model reduction algorithms*, Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, December 2000.

[23] S. Gugercin, D.C. Sorensen and A.C. Antoulas, *A modi£ed low-rank Smith method for large-scale Lyapunov equations*, Numerical Algorithms, Vol. 32, Issue 1, pp. 27-55, January 2003.

[24] E.A. Jonckheere and L.M. Silverman, *A new of invariants for linear system - application to reduced order compensator design*, IEEE Trans. Automat. Contr.. vol. AC-28, No. 10, pp. 953-964, 1983.

[25] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1980.

[26] C. Kenney and G. Hewer, *Necessary and suf£cient conditions for balancing unstable systems*, IEEE Trans. Automat. Contr., vol. AC-32(2), pp. 157-160, 1987.

[27] S.W. Kim, B.D.O. Anderson and A.G. Madievski, *Error bound for the transfer function order reduction using frequency weighted balanced truncation*, Syst. Contr. Lett., vol. 24, pp. 183-192, 1995.

[28] C.-A Lin and T.-Y Chiu, *Model reduction via frequency weighted balanced realization*, Control Theory and Advanced Technol., vol. 8, pp. 341-351, 1992.

[29] Y. Liu and B. D. O. Anderson, Singular Perturbation Approximation of Balanced Systems, *Int. J. Control*, **50**: 1379-1405, 1989.

[30] B. C. Moore, *Principal Component Analysis in Linear System: Controllability, Observability and Model Reduction*, IEEE Transactions on Automatic Control, AC-26:17-32, 1981.

[31] C. T. Mullis and R. A. Roberts, *Synthesis of minimum roundoff noise £xed point digital £lters*, IEEE Trans. on Circuits and Systems, **CAS-23:**, pp: 551-562, 1976.

[32] R. Ober, *Balanced parameterization of classes of linear systems*, SIAM J. Cont. App., 1991.

[33] P.C. Opdenacker and E.A. Jonckheere, *A contraction mapping preserving balanced reduction scheme and its in£nity norm error bounds*, IEEE Trans. Circuits and Systems, 1988.

[34] T. Penzl, *A cyclic low-rank Smith method for large sparse Lyapunov equations*, SIAM J. Sci. Comput., Vol. 21, No. 4, pp: 1401-1418, 2000.

[35] T. Penzl, *Algorithms for model reduction of large dynamical systems*, Technical Report SFB393/99-40, Sonderforschungsbereich 393 Numerische Simulation auf massiv parallelen Rechern, TU Chemnitz, 09107, FRG, 1999. Available from `http://www.tu-chemnitz.de/sfb393/sfb99pr.html`.

[36] L. Pernebo and L.M. Silverman, *Model reduction via balanced state space representation,* IEEE Trans. Automat. Contr., vol. AC-27. No. 2, pp. 382-382, 1982.

[37] V. Sreeram, B.D.O Anderson and A.G. Madievski, *Frequency weighted balanced reduction technique: A generalization and an error bound,* in Proc. 34^{th} IEEE Conf. Decision and Control, 1995.

[38] C.P. Therapos, *Balancing transformation for unstable nonminimal linear systems*, IEEE Trans. Automat. Contr., vol. 34(4), pp. 455-457, 1989.

[39] G. Wang, V. Sreeram and W.Q. Liu, *A new frequency weighted balanced truncation method and an error bound*, IEEE Trans. Automat. Contr., Vol. 44, No. 9, pp. 1734-1737, 1999.

[40] A. Varga, *On stochastic balancing related model reduction*, in the Proceeding of the 39^{th} IEEE Conference on Decision and Control, 2000.

[41] A. Varga and B.D.O Anderson, *Accuracy enhancing methods for the frequency-weighted balancing related model reduction*, in Proc. 40^{th} IEEE Conf. Decision and Control, 2001.

[42] P. Van Dooren, K. Gallivan and A. Vandendorpe *Sylvester equations and projection-based model reduction*, submitted to Journal CAM, Special Issues, 2001.

[43] K. Zhou, *Frequency-weighted $\mathcal{L}_\infty$ norm and optimal Hankel norm model reduction*, IEEE Trans. Automat. Contr., vol. 40, No. 10, pp. 1687-1699, 1995.

[44] K. Zhou, G, Salomon and E. Wu, *Balanced realization and model reduction for unstable systems*, Int. J. Robust Nonlinear Control, **9**, pp. 183-198, 1999.

[45] K. Zhou with J.C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1996.