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A new compact finite difference scheme for solving the complex Ginzburg–Landau equation

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ABSTRACT

The complex Ginzburg–Landau equation is often encountered in physics and engineering applications, such as nonlinear transmission lines, solitons, and superconductivity. However, it remains a challenge to develop simple, stable and accurate finite difference schemes for solving the equation because of the nonlinear term. Most of the existing schemes are obtained based on the Crank–Nicolson method, which is fully implicit and must be solved iteratively for each time step. In this article, we present a fourth-order accurate iterative scheme, which leads to a tri-diagonal linear system in 1D cases. We prove that the present scheme is unconditionally stable. The scheme is then extended to 2D cases. Numerical errors and convergence rates of the solutions are tested by several examples.

1. Introduction

The complex Ginzburg–Landau equation is often encountered in physics and engineering applications, such as nonlinear transmission lines [1–5], solitons [6–10], and superconductivity [11–17]. However, it remains a challenge to develop a simple, stable and accurate finite difference scheme for solving this equation due to the nonlinear term, particularly in multi-dimensional cases. There are many numerical schemes have been proposed for solving this equation [11–29]. In particular, Tsertsvadze [27] proposed a second-order Crank–Nicolson type of finite difference scheme for a one-dimensional (1D) Ginzburg–Landau equation, and Sun and Zhu [28] proved its unconditional convergence in the l_∞ norm. Recently, Hu [29] developed several fourth-order compact finite difference schemes by coupling the Crank–Nicolson method with the fourth-order compact finite difference method [30]. The unconditional convergence in the l_∞ norm of these schemes was analyzed. However, these fourth-order accurate finite difference schemes are implicit, particularly the implicit approximation for the nonlinear term in some schemes, which must be solved iteratively for each time step. In this study, we present a fourth-order accurate iterative scheme which is also obtained based on the compact finite difference method and Crank–Nicolson method, but leads to a tri-diagonal linear system in 1D cases. We further prove the scheme to be unconditionally stable in the l_2 norm by using the discrete energy techniques [28,29,31–38]. The obtained scheme is then extended to 2D cases. Finally, the numerical errors and convergence rates of the solutions are tested by several examples.

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2. Finite difference schemes

We first consider the 1D complex Ginzburg–Landau equation with initial and periodic boundary conditions as follows:

$$\frac{\partial u}{\partial t} = (1+i\alpha)\frac{\partial^2 u}{\partial x^2} + Ru - (1+i\beta)\,|u|^{q-1}\,u, \quad x \in (0,L),\, t \in (0,T]; \tag{2.1a}$$

$$u(x,0) = u_0(x), \quad x \in [0,L]; \tag{2.1b}$$

$$u(x,t) = u(x+L,t), \quad t \in [0,T]; \tag{2.1c}$$

where α and β are real constants, $q \geq 3$ is a positive integer, $i = \sqrt{-1}$, the functions $u(x,t)$ and $u_0(x)$ are complex valued functions, R is a positive constant, and L is the period of $u(x,t)$ with respect to x.

To develop a finite difference scheme for solving the above Ginzburg–Landau problem, we first divide spatial interval $[0,L]$ into M subintervals where the grid size is $h = \frac{L}{M}$, and the time interval $[0,T]$ is divided into N subintervals with a time step $\tau = \frac{T}{N}$. We denote u_j^n to be the numerical approximation of $u(x_j,t_n)$, where $x_j = jh$, $t_n = n\tau$, $0 \leq j \leq M$ and $0 \leq n \leq N$. It can be seen from periodicity that $u_M^n = u_0^n$, $u_{M+1}^n = u_1^n$ and so on. Furthermore, we use the following first-order and second-order finite difference operators:

$$\delta_t u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}, \quad \delta_t u_j^{n+\frac{1}{2}} = \frac{u_j^{n+\frac{3}{2}} - u_j^{n+\frac{1}{2}}}{\tau}; \tag{2.2a}$$

$$\delta_x u_j^n = \begin{cases} \dfrac{u_{j+1}^n - u_j^n}{h}, & 0 \leq j \leq M-1, \\[2ex] \dfrac{u_{M+1}^n - u_M^n}{h} \equiv \dfrac{u_1^n - u_M^n}{h}, & j = M; \end{cases} \tag{2.2b}$$

$$\delta_x^2 u_j^n = \begin{cases} \dfrac{u_{-1}^n - 2u_0^n + u_1^n}{h^2} \equiv \dfrac{u_{M-1}^n - 2u_0^n + u_1^n}{h^2}, & j = 0, \\[2ex] \dfrac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}, & 1 \leq j \leq M-1, \\[2ex] \dfrac{u_{M+1}^n - 2u_M^n + u_{M-1}^n}{h^2} \equiv \dfrac{u_1^n - 2u_M^n + u_{M-1}^n}{h^2}, & j = M. \end{cases} \tag{2.2c}$$

For the discrete space, a fourth-order compact difference scheme [30] at all grid points, x_j, $0 \leq j \leq M$, can be written as:

$$\frac{1}{12}\left[\frac{\partial^2 u(x_{j-1},t)}{\partial x^2} + 10\frac{\partial^2 u(x_j,t)}{\partial x^2} + \frac{\partial^2 u(x_{j+1},t)}{\partial x^2}\right] = \frac{1}{h^2}[u(x_{j-1},t) - 2u(x_j,t) + u(x_{j+1},t)] + O(h^4), \tag{2.3}$$

which can be denoted using the second-order finite difference operator δ_x^2 as

$$\left(1 + \frac{h^2}{12}\delta_x^2\right)\frac{\partial^2 u(x_j,t)}{\partial x^2} = \delta_x^2 u(x_j,t) + O(h^4). \tag{2.4}$$

We now rewrite Eq. (2.1) as $(1+i\alpha)\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} - Ru + (1+i\beta)|u|^{q-1}u$ at (x_j,t), multiply it both sides by $1 + \frac{h^2}{12}\delta_x^2$, and then use Eq. (2.4). This gives

$$(1+i\alpha)\delta_x^2 u(x_j,t) = \left(1 + \frac{h^2}{12}\delta_x^2\right)\left[\frac{\partial u(x_j,t)}{\partial t} - Ru(x_j,t) + (1+i\beta)\,|u(x_j,t)|^{q-1}\,u(x_j,t)\right] + O(h^4). \tag{2.5}$$

Using the Crank–Nicolson technique to Eq. (2.5) with respect to time t at $t_{n+\frac{1}{2}}$, we obtain

$$\begin{aligned}\left(1 + \frac{h^2}{12}\delta_x^2\right)\frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\tau} &= (1+i\alpha)\frac{\delta_x^2 u(x_j,t_{n+1}) + \delta_x^2 u(x_j,t_n)}{2} + R\left(1 + \frac{h^2}{12}\delta_x^2\right)u\left(x_j,t_{n+\frac{1}{2}}\right) \\ &\quad - (1+i\beta)\left(1 + \frac{h^2}{12}\delta_x^2\right)\left|u\left(x_j,t_{n+\frac{1}{2}}\right)\right|^{q-1} u\left(x_j,t_{n+\frac{1}{2}}\right) + O(\tau^2 + h^4).\end{aligned} \tag{2.6a}$$

Note that $u(x_j, t_{n+\frac{1}{2}})$ must be known in order to obtain $u(x_j, t_{n+1})$ in Eq. (2.6). We may use a similar argument in obtaining Eq. (2.6) from Eq. (2.5) at t_{n+1} and obtain

$$\left(1+\frac{h^2}{12}\delta_x^2\right)\frac{u(x_j,t_{n+\frac{3}{2}})-u(x_j,t_{n+\frac{1}{2}})}{\tau} = (1+i\alpha)\frac{\delta_x^2 u(x_j,t_{n+\frac{3}{2}})+\delta_x^2 u(x_j,t_{n+\frac{1}{2}})}{2} + R\left(1+\frac{h^2}{12}\delta_x^2\right)u(x_j,t_{n+1}) - (1+i\beta)\left(1+\frac{h^2}{12}\delta_x^2\right)|u(x_j,t_{n+1})|^{q-1}u(x_j,t_{n+1}) + O(\tau^2+h^4). \tag{2.6b}$$

Thus, coupling Eq. (2.6) and then dropping the truncation error $O(\tau^2+h^4)$, we obtain an iterative fourth-order in space compact finite difference scheme for solving Eq. (2.1) as follows:

$$\left(1+\frac{h^2}{12}\delta_x^2\right)\delta_t u_j^n = (1+i\alpha)\delta_x^2\left(\frac{u_j^{n+1}+u_j^n}{2}\right) + R\left(1+\frac{h^2}{12}\delta_x^2\right)u_j^{n+\frac{1}{2}} - (1+i\beta)\left(1+\frac{h^2}{12}\delta_x^2\right)\left(\left|u_j^{n+\frac{1}{2}}\right|^{q-1}u_j^{n+\frac{1}{2}}\right), \tag{2.7a}$$

$$\left(1+\frac{h^2}{12}\delta_x^2\right)\delta_t u_j^{n+\frac{1}{2}} = (1+i\alpha)\delta_x^2\left(\frac{u_j^{n+\frac{3}{2}}+u_j^{n+\frac{1}{2}}}{2}\right) + R\left(1+\frac{h^2}{12}\delta_x^2\right)u_j^{n+1} - (1+i\beta)\left(1+\frac{h^2}{12}\delta_x^2\right)\left(\left|u_j^{n+1}\right|^{q-1}u_j^{n+1}\right), \tag{2.7b}$$

where $u_j^0 = u_0(x_j)$, and $0 \le j \le M, 0 \le n \le N-1$. To start the above iteration, $u_j^{\frac{1}{2}}, 0 \le j \le M$, must be known. This can be obtained using other methods, such as those proposed methods in [29] or the explicit pseudo-spectral methods [39] built in MATLAB. It should be pointed out that for each time step, one needs only to solve a tri-diagonal linear system for u_j^{n+1} from Eq. (2.7) once u_j^n and $u_j^{n+\frac{1}{2}}$ are known, and then substitute u_j^{n+1} into Eq. (2.7) to obtain $u_j^{n+\frac{3}{2}}$ by solving another tri-diagonal linear system for $u_j^{n+\frac{3}{2}}$. Thus, the computation is simple and fast.

The above finite difference scheme can be extended to 2D cases. Indeed, we consider the 2D complex Ginzburg–Landau equation as

$$\frac{\partial u}{\partial t} = (1+i\alpha)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + Ru - (1+i\beta)|u|^{q-1}u, \quad (x,y)\in(0,L_x)\times(0,L_y), t\in(0,T], \tag{2.8}$$

with initial and periodic boundary conditions, where L_x, L_y are periods of $u(x,y,t)$ in the x and y directions, respectively. Again, we divide $[0, L_x]$ into M_x subintervals, $[0, L_y]$ into M_y subintervals, and $[0, T]$ into N subintervals with mesh sizes h_x, h_y and τ, respectively. We denote $u_{j,k}^n$ to be the numerical approximation of $u(x_j, y_k, t_n)$, where $x_j = jh_x$, $y_k = kh_y$, $t_n = n\tau$, $0 \le j \le M_x$, $0 \le k \le M_y$, and $0 \le n \le N$.

Instead using Eq. (2.3), we now employ a fourth-order compact finite difference scheme for Poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ given in [40] as

$$\left[\frac{D_x^2}{h_x^2}+\frac{D_y^2}{h_y^2}+\frac{D_x^2D_y^2}{12h_x^2}+\frac{D_x^2D_y^2}{12h_y^2}\right]u(x_j,y_k) = \left(1+\frac{D_x^2}{12}+\frac{D_y^2}{12}\right)f(x_j,y_k) + O(h_x^4+h_y^4), \tag{2.9}$$

where the finite difference operators D_x^2, D_y^2 and $D_x^2D_y^2$ are defined as follows:

$$D_x^2 u_{j,k} = \begin{cases} u_{M_x-1,k} - 2u_{0,k} + u_{1,k}, & j=0, 0\le k\le M_y, \\ u_{j-1,k} - 2u_{j,k} + u_{j+1,k}, & 1\le j\le M_x-1, 0\le k\le M_y, \\ u_{1,k} - 2u_{M_x,k} + u_{M_x-1,k}, & j=M_x, 0\le k\le M_y; \end{cases} \tag{2.10a}$$

$$D_y^2 u_{j,k} = \begin{cases} u_{j,M_y-1} - 2u_{j,0} + u_{j,1}, & 0\le j\le M_x, k=0, \\ u_{j,k-1} - 2u_{j,k} + u_{j,k+1}, & 0\le j\le M_x, 1\le k\le M_y-1, \\ u_{j,1} - 2u_{j,M_y} + u_{j,M_y-1}, & 0\le j\le M_x, k=M_y; \end{cases} \tag{2.10b}$$

$$D_x^2D_y^2u_{j,k} = \begin{cases} (u_{1,1}+u_{M_x-1,1}+u_{M_x-1,M_y-1}+u_{1,M_y-1}) \\ \quad -2(u_{1,0}+u_{0,1}+u_{M_x-1,0}+u_{0,M_y-1})+4u_{0,0}, & j=0, k=0, \\ (u_{1,k+1}+u_{M_x-1,k+1}+u_{M_x-1,k-1}+u_{1,k-1}) \\ \quad -2(u_{1,k}+u_{0,k+1}+u_{M_x-1,k}+u_{0,k-1})+4u_{0,k}, & j=0, 1\le k\le M_y-1, \\ (u_{1,1}+u_{M_x-1,1}+u_{M_x-1,M_y-1}+u_{1,M_y-1}) \\ \quad -2(u_{1,M_y}+u_{0,1}+u_{M_x-1,M_y}+u_{0,M_y-1})+4u_{0,M_y}, & j=0, k=M_y, \\ (u_{j+1,1}+u_{j-1,1}+u_{j-1,M_y-1}+u_{j+1,M_y-1}) \\ \quad -2(u_{j+1,0}+u_{j,1}+u_{j-1,0}+u_{j,M_y-1})+4u_{j,0}, & 1\le j\le M_x-1, k=0, \\ (u_{j+1,k+1}+u_{j-1,k+1}+u_{j-1,k-1}+u_{j+1,k-1}) \\ \quad -2(u_{j+1,k}+u_{j,k+1}+u_{j-1,k}+u_{j,k-1})+4u_{j,k}, & 1\le j\le M_x-1, 1\le k\le M_y-1, \\ (u_{j+1,1}+u_{j-1,1}+u_{j-1,M_y-1}+u_{j+1,M_y-1}) \\ \quad -2(u_{j+1,M_y}+u_{j,1}+u_{j-1,M_y}+u_{j,M_y-1})+4u_{j,M_y}, & 1\le j\le M_x-1, k=M_y, \\ (u_{1,1}+u_{M_x-1,1}+u_{M_x-1,M_y-1}+u_{1,M_y-1}) \\ \quad -2(u_{1,0}+u_{M_x,1}+u_{M_x-1,0}+u_{M_x,M_y-1})+4u_{M_x,0}, & j=M_x, k=0, \\ (u_{1,k+1}+u_{M_x-1,k+1}+u_{M_x-1,k-1}+u_{1,k-1}) \\ \quad -2(u_{1,k}+u_{M_x,k+1}+u_{M_x-1,k}+u_{M_x,k-1})+4u_{M_x,k}, & j=M_x, 1\le k\le M_y-1, \\ (u_{1,1}+u_{M_x-1,1}+u_{M_x-1,M_y-1}+u_{1,M_y-1}) \\ \quad -2(u_{1,M_y}+u_{M_x,1}+u_{M_x-1,M_y}+u_{M_x,M_y-1})+4u_{M_x,M_y}, & j=M_x, k=M_y. \end{cases} \tag{2.10c}$$

We then rewrite Eq. (2.8) as $(1+i\alpha)(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}) = \frac{\partial u}{\partial t} - Ru + (1+i\beta)|u|^{q-1}u$ at (x_j, y_k, t) and then apply Eq. (2.9) to it. This gives

$$(1+i\alpha)\left[\frac{D_x^2}{h_x^2}+\frac{D_y^2}{h_y^2}+\frac{D_x^2D_y^2}{12h_x^2}+\frac{D_x^2D_y^2}{12h_y^2}\right]u(x_j,y_k,t) \\ = \left(1+\frac{D_x^2}{12}+\frac{D_y^2}{12}\right)\left[\frac{\partial u(x_j,y_k,t)}{\partial t} - Ru(x_j,y_k,t)+(1+i\beta)\,|u(x_j,y_k,t)|^{q-1}\,u(x_j,y_k,t)\right]+O(h_x^4+h_y^4). \tag{2.11}$$

Finally, we then use the Crank–Nicolson method similarly and obtain the iterative fourth-order in space compact finite difference iterative scheme for solving 2D complex Ginzburg–Landau equation in Eq. (2.8) as

$$(1+i\alpha)\left[\frac{D_x^2}{h_x^2}+\frac{D_y^2}{h_y^2}+\frac{D_x^2D_y^2}{12h_x^2}+\frac{D_x^2D_y^2}{12h_y^2}\right]\frac{u_{j,k}^{n+1}+u_{j,k}^{n}}{2} \\ = \left(1+\frac{D_x^2}{12}+\frac{D_y^2}{12}\right)\left[\frac{u_{j,k}^{n+1}-u_{j,k}^{n}}{\tau} - Ru_{j,k}^{n+\frac12}+(1+i\beta)\left|u_{j,k}^{n+\frac12}\right|^{q-1}u_{j,k}^{n+\frac12}\right], \tag{2.12a}$$

$$(1+i\alpha)\left[\frac{D_x^2}{h_x^2}+\frac{D_y^2}{h_y^2}+\frac{D_x^2D_y^2}{12h_x^2}+\frac{D_x^2D_y^2}{12h_y^2}\right]\frac{u_{j,k}^{n+\frac32}+u_{j,k}^{n+\frac12}}{2} \\ = \left(1+\frac{D_x^2}{12}+\frac{D_y^2}{12}\right)\left[\frac{u_{j,k}^{n+\frac32}-u_{j,k}^{n+\frac12}}{\tau} - Ru_{j,k}^{n+1}+(1+i\beta)\left|u_{j,k}^{n+1}\right|^{q-1}u_{j,k}^{n+1}\right], \tag{2.12b}$$

where $u_{j,k}^0 = u_0(x_j, y_k)$, and $0\le j\le M_x, 0\le k\le M_y, 0\le n\le N-1$. It should be noted that Eq. (2.12) can be written as linear systems, which avoid any nonlinear iterations. Moreover, the scheme can be solved using an ADI method. For instance, Eq. (2.12) can be solved using a Peaceman–Rachford ADI scheme [41,42] as follows:

$$\left[1+\frac{D_x^2}{12}-(1+i\alpha)\,\frac{\tau}{2}\frac{D_x^2}{h_x^2}\right]u_{j,k}^{*} = \left[1+\frac{D_y^2}{12}+(1+i\alpha)\,\frac{\tau}{2}\frac{D_y^2}{h_y^2}\right]u_{j,k}^{n}+\frac{\tau}{2}\left(1+\frac{D_y^2}{12}\right)\left[Ru_{j,k}^{n+\frac12}-(1+i\beta)\left|u_{j,k}^{n+\frac12}\right|^{q-1}u_{j,k}^{n+\frac12}\right], \tag{2.13a}$$

$$\left[1+\frac{D_y^2}{12}-(1+i\alpha)\,\frac{\tau}{2}\frac{D_y^2}{h_y^2}\right]u_{j,k}^{n+1} = \left[1+\frac{D_x^2}{12}+(1+i\alpha)\,\frac{\tau}{2}\frac{D_x^2}{h_x^2}\right]u_{j,k}^{*}+\frac{\tau}{2}\left(1+\frac{D_y^2}{12}\right)\left[Ru_{j,k}^{n+\frac12}-(1+i\beta)\left|u_{j,k}^{n+\frac12}\right|^{q-1}u_{j,k}^{n+\frac12}\right], \tag{2.13b}$$

where $u_{j,k}^{*}$ is an intermediate mesh function, which is equivalent to Eq. (2.12) if we ignore those terms whose truncation errors are $O(\tau^2)$ and $O(h_x^2h_y^2)$. Similarly, we can construct an ADI scheme for Eq. (2.12).

3. Stability analysis

For stability, we would like to show that if solution A and solution B are different periodic solutions obtained by Eq. (2.7) based on two different initial conditions, then the difference between these two solutions is controlled by their initial difference. We will employ the discrete energy method [28,29,31–38] to analyze the stability of the present scheme in Eq. (2.7). To this end, we will first introduce the vector and matrix notations, obtain some lemmas, show the numerical solution to be bounded (as seen in Theorem 1), and finally use these properties and results to show the stability (as seen in Theorem 2).

We now introduce the vector and matrix notations as

$$\begin{aligned}
\mathbf{u}^n &= \left(u_0^n, u_1^n, \ldots, u_{M-1}^n\right)^T, \\
\mathbf{u}^{n+\frac{1}{2}} &= \left(u_0^{n+\frac{1}{2}}, u_1^{n+\frac{1}{2}}, \ldots, u_{M-1}^{n+\frac{1}{2}}\right)^T, \\
|\mathbf{u}^n|^{q-1}\mathbf{u}^n &= \left(|u_0^n|^{q-1}\, u_0^n, \ldots, |u_{M-1}^n|^{q-1}\, u_{M-1}^n\right)^T, \\
\left|\mathbf{u}^{n+\frac{1}{2}}\right|^{q-1}\mathbf{u}^{n+\frac{1}{2}} &= \left(\left|u_0^{n+\frac{1}{2}}\right|^{q-1} u_0^{n+\frac{1}{2}}, \ldots, \left|u_{M-1}^{n+\frac{1}{2}}\right|^{q-1} u_{M-1}^{n+\frac{1}{2}}\right)^T,
\end{aligned}$$

where $(\ldots)^T$ is the transpose of the vector (...), and

$$\mathbf{H} = \frac{1}{12}\begin{bmatrix} 10 & 1 & 0 & \cdots & 1 \\ 1 & 10 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 1 & 10 & 1 \\ 1 & 0 & \cdots & 1 & 10 \end{bmatrix}_{M\times M}.$$

It should be pointed out that because of the periodic mesh function, $u_0^n = u_M^n$, we do not include u_M^n in the vectors.

Thus, the present scheme in Eq. (2.7) can be written into vector form as

$$\delta_t \mathbf{u}^n = (1+i\alpha)\mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2}\right) + R\mathbf{u}^{n+\frac{1}{2}} - (1+i\beta)\left(\left|\mathbf{u}^{n+\frac{1}{2}}\right|^{q-1}\mathbf{u}^{n+\frac{1}{2}}\right), \tag{3.1a}$$

$$\delta_t \mathbf{u}^{n+\frac{1}{2}} = (1+i\alpha)\mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{u}^{n+\frac{3}{2}}+\mathbf{u}^{n+\frac{1}{2}}}{2}\right) + R\mathbf{u}^{n+1} - (1+i\beta)\left(|\mathbf{u}^{n+1}|^{q-1}\mathbf{u}^{n+1}\right). \tag{3.1b}$$

It can be seen that if $\mathbf{H} = \mathbf{I}$, an identity matrix, then Eq. (3.1) will reduce to a second-order Crank–Nicolson scheme.

We define the inner product of two vectors and norms as

$$\langle \mathbf{u}^n, \mathbf{v}^n\rangle = \sum_{j=0}^{M-1} u_j^n \overline{v_j^n} = (\mathbf{u}^n)^T\cdot\overline{\mathbf{v}^n}, \quad \|\mathbf{u}^n\| \equiv \|\mathbf{u}^n\|_2 = \left[h\left(\sum_{j=0}^{M-1}\left|u_j^n\right|^2\right)\right]^{\frac{1}{2}} = \langle \mathbf{u}^n, h\mathbf{u}^n\rangle^{\frac{1}{2}}, \tag{3.2a}$$

$$\|\mathbf{u}^n\|_\infty = \max_{0\le j\le M-1}\left|u_j^n\right|, \quad \|\mathbf{u}^n\|_p = \left[h\left(\sum_{j=0}^{M-1}\left|u_j^n\right|^p\right)\right]^{\frac{1}{p}}, \quad p \ge 1, \tag{3.2b}$$

where $\overline{v_j^n}$ stands for the conjugate of complex v_j^n. To analyze the stability of the present scheme in Eq. (3.1), we need the following six lemmas.

Lemma 1 ([36,37])**.** *For any periodic mesh functions* $\mathbf{u}^n, \mathbf{v}^n$, *it holds that*

$$\left\langle \delta_x^2\mathbf{u}^n, \mathbf{v}^n\right\rangle = -\langle \delta_x\mathbf{u}^n, \delta_x\mathbf{v}^n\rangle. \tag{3.3}$$

Lemma 2. *For any periodic mesh function* $\mathbf{u}^n$, *it holds that*

$$\mathrm{Re}\left\langle \mathbf{H}^{-1}\delta_x^2\mathbf{u}^n, h\mathbf{u}^n\right\rangle = \left\langle \mathbf{H}^{-1}\delta_x^2\mathbf{u}^n, h\mathbf{u}^n\right\rangle = -\|\mathbf{Q}\delta_x\mathbf{u}^n\|^2, \tag{3.4a}$$

$$\mathrm{Re}\left\langle \mathbf{H}^{-1}\delta_x^2\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2}, \delta_t h\mathbf{u}^n\right\rangle = -\tfrac{1}{2\tau}\left(\|\mathbf{Q}\delta_x\mathbf{u}^{n+1}\|^2 - \|\mathbf{Q}\delta_x\mathbf{u}^n\|^2\right), \tag{3.4b}$$

where $\mathbf{Q} = Chol(\mathbf{H}^{-1})$, *the Cholesky factorization, and Re denotes the real part of complex value.*

Proof. The proof is similar to that in [31]. Since $\mathbf{H}$ is a real, symmetric and positive definite matrix, $\mathbf{H}^{-1}$ is also a real, symmetric and positive definite matrix. By the Cholesky factorization, there exists a real upper-triangular matrix $\mathbf{Q}$ such that $\mathbf{H}^{-1} = \mathbf{Q}^T\mathbf{Q}$. Hence, we obtain from Lemma 1 that

$$\begin{aligned}\left\langle \mathbf{H}^{-1}\delta_x^2\mathbf{u}^n, h\mathbf{u}^n\right\rangle &= \left\langle \delta_x^2\mathbf{H}^{-1}\mathbf{u}^n, h\mathbf{u}^n\right\rangle \\ &= -\langle \delta_x\mathbf{H}^{-1}\mathbf{u}^n, h\delta_x\mathbf{u}^n\rangle \\ &= -\langle \mathbf{H}^{-1}\delta_x\mathbf{u}^n, h\delta_x\mathbf{u}^n\rangle \\ &= -\langle \mathbf{Q}^T\mathbf{Q}\delta_x\mathbf{u}^n, h\delta_x\mathbf{u}^n\rangle \\ &= -\langle \mathbf{Q}\delta_x\mathbf{u}^n, h\mathbf{Q}\delta_x\mathbf{u}^n\rangle \\ &= -\|\mathbf{Q}\delta_x\mathbf{u}^n\|^2. \end{aligned} \tag{3.5}$$

Here, we have used two facts that (1)

$$\begin{aligned}\mathbf{H}^{-1}\delta_x\mathbf{u}^n &= \mathbf{H}^{-1}\frac{(u_1^n, u_2^n, \ldots, u_M^n)^T - (u_0^n, u_1^n, \ldots, u_{M-1}^n)^T}{h} \\ &= \frac{\mathbf{H}^{-1}(u_1^n, u_2^n, \ldots, u_M^n)^T - \mathbf{H}^{-1}(u_0^n, u_1^n, \ldots, u_{M-1}^n)^T}{h} \\ &= \delta_x\mathbf{H}^{-1}\mathbf{u}^n, \end{aligned} \tag{3.6}$$

and similarly, $\mathbf{H}^{-1}\delta_x^2\mathbf{u}^n = \delta_x^2\mathbf{H}^{-1}\mathbf{u}^n$; and (2) $\langle \mathbf{Q}\mathbf{u}^n, \mathbf{Q}\mathbf{u}^n\rangle = (\mathbf{Q}\mathbf{u}^n)^T\cdot\overline{\mathbf{Q}\mathbf{u}^n} = (\mathbf{u}^n)^T\mathbf{Q}^T\mathbf{Q}\cdot\overline{\mathbf{u}^n} = (\mathbf{Q}^T\mathbf{Q}\mathbf{u}^n)^T\cdot\overline{\mathbf{u}^n} = \langle \mathbf{Q}^T\mathbf{Q}\mathbf{u}^n, \mathbf{u}^n\rangle$. Thus, we obtain from Eq. (3.5) that Re $\langle \mathbf{H}^{-1}\delta_x^2\mathbf{u}^n, h\mathbf{u}^n\rangle = \langle \mathbf{H}^{-1}\delta_x^2\mathbf{u}^n, h\mathbf{u}^n\rangle = -\|\mathbf{Q}\delta_x\mathbf{u}^n\|^2$ and hence Eq. (3.4) holds.

Since

$$\begin{aligned}&\left\langle \mathbf{H}^{-1}\delta_x^2\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2}, \delta_t h\mathbf{u}^n\right\rangle \\ &= \frac{1}{2\tau}\left\langle \delta_x^2\mathbf{H}^{-1}(\mathbf{u}^{n+1}+\mathbf{u}^n), h(\mathbf{u}^{n+1}-\mathbf{u}^n)\right\rangle \\ &= -\frac{1}{2\tau}\langle \delta_x\mathbf{H}^{-1}(\mathbf{u}^{n+1}+\mathbf{u}^n), h\delta_x(\mathbf{u}^{n+1}-\mathbf{u}^n)\rangle \\ &= -\frac{1}{2\tau}\langle \mathbf{H}^{-1}\delta_x(\mathbf{u}^{n+1}+\mathbf{u}^n), h\delta_x(\mathbf{u}^{n+1}-\mathbf{u}^n)\rangle \\ &= -\frac{1}{2\tau}\langle \mathbf{Q}^T\mathbf{Q}\delta_x(\mathbf{u}^{n+1}+\mathbf{u}^n), h\delta_x(\mathbf{u}^{n+1}-\mathbf{u}^n)\rangle \\ &= -\frac{1}{2\tau}\langle \mathbf{Q}\delta_x(\mathbf{u}^{n+1}+\mathbf{u}^n), h\mathbf{Q}\delta_x(\mathbf{u}^{n+1}-\mathbf{u}^n)\rangle \\ &= -\frac{1}{2\tau}\langle \mathbf{Q}\delta_x\mathbf{u}^{n+1}+\mathbf{Q}\delta_x\mathbf{u}^n, h\mathbf{Q}\delta_x\mathbf{u}^{n+1}-h\mathbf{Q}\delta_x\mathbf{u}^n\rangle \\ &= -\frac{1}{2\tau}[\langle \mathbf{Q}\delta_x\mathbf{u}^{n+1}, h\mathbf{Q}\delta_x\mathbf{u}^{n+1}\rangle - \langle \mathbf{Q}\delta_x\mathbf{u}^n, h\mathbf{Q}\delta_x\mathbf{u}^n\rangle] - \frac{1}{2\tau}[\langle \mathbf{Q}\delta_x\mathbf{u}^n, h\mathbf{Q}\delta_x\mathbf{u}^{n+1}\rangle - \langle \mathbf{Q}\delta_x\mathbf{u}^{n+1}, h\mathbf{Q}\delta_x\mathbf{u}^n\rangle] \\ &= -\frac{1}{2\tau}[\|\mathbf{Q}\delta_x\mathbf{u}^{n+1}\|^2 - \|\mathbf{Q}\delta_x\mathbf{u}^n\|^2] - \frac{1}{2\tau}[\langle \mathbf{Q}\delta_x\mathbf{u}^n, h\mathbf{Q}\delta_x\mathbf{u}^{n+1}\rangle - \langle \mathbf{Q}\delta_x\mathbf{u}^{n+1}, h\mathbf{Q}\delta_x\mathbf{u}^n\rangle] \end{aligned} \tag{3.7}$$

and the fact that Re$(\langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{v}, \mathbf{u}\rangle) = 0$ for any vectors $\mathbf{u}$ and $\mathbf{v}$, we obtain

$$\begin{aligned}&\mathrm{Re}\left\langle \mathbf{H}^{-1}\delta_x^2\frac{\mathbf{u}^{n+1}+\mathbf{u}^n}{2}, \delta_t h\mathbf{u}^n\right\rangle \\ &= -\frac{1}{2\tau}[\|\mathbf{Q}\delta_x\mathbf{u}^{n+1}\|^2 - \|\mathbf{Q}\delta_x\mathbf{u}^n\|^2] - \frac{1}{2\tau}[\mathrm{Re}(\langle \mathbf{Q}\delta_x\mathbf{u}^n, h\mathbf{Q}\delta_x\mathbf{u}^{n+1}\rangle - \langle \mathbf{Q}\delta_x\mathbf{u}^{n+1}, h\mathbf{Q}\delta_x\mathbf{u}^n\rangle)] \\ &= -\frac{1}{2\tau}(\|\mathbf{Q}\delta_x\mathbf{u}^{n+1}\|^2 - \|\mathbf{Q}\delta_x\mathbf{u}^n\|^2), \end{aligned} \tag{3.8}$$

and hence Eq. (3.4) holds. □

Lemma 3. *The following inequalities hold*

$$\mathrm{Re}\left(\frac{z_1}{z_2}\right) \le \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \mathrm{Re}(z_1\cdot z_2) \le |z_1\cdot z_2| = |z_1||z_2|; \tag{3.9a}$$

$$||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|; \tag{3.9b}$$

$$||z_1|^n z_1 - |z_2|^n z_2| \le \left(|z_1| + |z_2|\right)^n |z_1 - z_2|, \quad n \ge 0; \tag{3.9c}$$

$$|\langle \mathbf{u}^n, h\mathbf{v}^n \rangle| \le \|\mathbf{u}^n\| \, \|\mathbf{v}^n\| \le \frac{\|\mathbf{u}^n\|^2}{2} + \frac{\|\mathbf{v}^n\|^2}{2}; \tag{3.9d}$$

$$\left\| \frac{\mathbf{u}^n + \mathbf{v}^n}{2} \right\|^2 \le \frac{\|\mathbf{u}^n\|^2}{2} + \frac{\|\mathbf{v}^n\|^2}{2}, \tag{3.9e}$$

where z_1 and z_2 are complex numbers.

Proof. We prove only for Eq. (3.9) and omit the detailed proofs for others since they are straightforward. It can be seen that Eq. (3.9) holds if $|z_1| = |z_2|$. For the case $|z_1| > |z_2|$, we obtain

$$\begin{aligned}
\left| \frac{|z_1|^n z_1 - |z_2|^n z_2}{z_1 - z_2} \right| &= \left| \frac{|z_1|^n (z_1 - z_2) + \left(|z_1|^n - |z_2|^n\right) z_2}{z_1 - z_2} \right| \\
&\le \left| \frac{|z_1|^n (z_1 - z_2)}{z_1 - z_2} \right| + \left| \frac{\left(|z_1|^n - |z_2|^n\right) z_2}{z_1 - z_2} \right| \\
&= \frac{|z_1|^n |z_1 - z_2|}{|z_1 - z_2|} + \frac{\left||z_1|^n - |z_2|^n\right| |z_2|}{|z_1 - z_2|} \\
&\le |z_1|^n + \frac{\left(|z_1|^n - |z_2|^n\right) |z_2|}{|z_1| - |z_2|} \\
&= |z_1|^n + |z_2| \left(\sum_{k=0}^{n-1} |z_1|^k |z_2|^{n-k-1} \right) \\
&= \left(\sum_{k=0}^{n} |z_1|^k |z_2|^{n-k} \right) \\
&\le (|z_1| + |z_2|)^n,
\end{aligned} \tag{3.10}$$

and hence Eq. (3.9) holds. Similarly, we can prove that Eq. (3.9) holds for the case $|z_1| < |z_2|$. □

Lemma 4 ([42,43])**.** *For any $\mathbf{u}^n$ and integer $p \ge 2$, it holds that*

$$\|\mathbf{u}^n\|_p \le \varepsilon \left(\|\mathbf{u}^n\|^{1-\alpha} \|\delta_x \mathbf{u}^n\|^{\alpha} + \|\mathbf{u}^n\| \right) \tag{3.11}$$

with $\alpha = \frac{1}{2} - \frac{1}{p}$, where ε is a constant independent of p and h.

Lemma 5. *For any $x \ge 0$, $y \ge 0$ and integer $n \ge 1$, it holds that*

$$(x + y)^n \le 2^{n-1}(x^n + y^n). \tag{3.12}$$

Proof. We use the mathematical induction to prove it. It is obviously that Eq. (3.12) is true for $n = 1$. Assume that Eq. (3.12) holds for n up to k. Then, we have $(x+y)^k (x+y) \le 2^{k-1}\left(x^k + y^k\right)(x+y)$. Moreover, it can be seen that

$$\begin{aligned}
&2^{k-1}(x^k + y^k)(x+y) - 2^k(x^{k+1} + y^{k+1}) \\
&\quad = 2^{k-1}(x^{k+1} + y^{k+1} + x^k y + x y^k) - 2^k(x^{k+1} + y^{k+1}) \\
&\quad = 2^{k-1}[(x^k y + x y^k) - (x^{k+1} + y^{k+1})] \\
&\quad = 2^{k-1}[x^k(y - x) + y^k(x - y)] \\
&\quad = 2^{k-1}[(y - x)(x^k - y^k)] \\
&\quad = 2^{k-1}\left[(y - x)(x - y) \sum_{l=0}^{k-1} x^l y^{k-l-1} \right] \\
&\quad = 2^{k-1}\left[-(x - y)^2 \sum_{l=0}^{k-1} x^l y^{k-l-1} \right] \\
&\quad \le 0,
\end{aligned} \tag{3.13}$$

implying that $(x+y)^{k+1} \le 2^{k-1}(x^k + y^k)(x+y) \le 2^k(x^{k+1} + y^{k+1})$. By the induction, we conclude that Eq. (3.12) holds for any integer $n \ge 1$. □

Lemma 6. *For $\mathbf{H}^{-1} = \mathbf{Q}^T\mathbf{Q}$, where $\mathbf{Q}$ is a real upper-triangular matrix, then there exists two positive constants C_a and C_b such that*

$$C_a\,\|\mathbf{u}^n\| \le \|\mathbf{Q}\mathbf{u}^n\| \le C_b\,\|\mathbf{u}^n\| \tag{3.14}$$

Proof. Note that $\mathbf{H}^{-1}$ is a real, symmetric and positive definite matrix. By Schur's Lemma [44], there exists a unitary matrix $\mathbf{U}$ (i.e., $\mathbf{U}^{-1} = \bar{\mathbf{U}}^T$) such that $\mathbf{U}^{-1}\mathbf{H}^{-1}\mathbf{U} = \mathbf{D}$, where $\mathbf{D}$ is a real diagonal matrix and the diagonal elements are the eigenvalues of $\mathbf{H}^{-1}$. Thus, we have $\mathbf{H}^{-1} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$ and

$$\begin{aligned}\|\mathbf{Q}\mathbf{u}^n\|^2 &= h\langle \mathbf{Q}\mathbf{u}^n, \mathbf{Q}\mathbf{u}^n\rangle\\ &= h\langle \mathbf{Q}^T\mathbf{Q}\mathbf{u}^n, \mathbf{u}^n\rangle\\ &= h\langle \mathbf{H}^{-1}\mathbf{u}^n, \mathbf{u}^n\rangle\\ &= h\langle \mathbf{U}\mathbf{D}\mathbf{U}^{-1}\mathbf{u}^n, \mathbf{u}^n\rangle\\ &= h\langle \mathbf{D}\mathbf{U}^{-1}\mathbf{u}^n, \mathbf{U}^{-1}\mathbf{u}^n\rangle\\ &= h\langle \mathbf{D}\mathbf{w}^n, \mathbf{w}^n\rangle = h\sum_{j=0}^{M-1} d_j|w_j|^2,\end{aligned} \tag{3.15}$$

implying that $d_{\min}\|\mathbf{w}^n\|^2 \le \|\mathbf{Q}\mathbf{u}^n\|^2 \le d_{\max}\|\mathbf{w}^n\|^2$, where $\mathbf{w}^n = \mathbf{U}^{-1}\mathbf{u}^n$, d_j is the eigenvalue of $\mathbf{H}^{-1}$. Since $\mathbf{U}$ is a unitary matrix, we have $\|\mathbf{w}^n\|^2 = \|\mathbf{U}\mathbf{u}^n\|^2 = \|\mathbf{u}^n\|^2$, and hence $d_{\min}\|\mathbf{u}^n\|^2 \le \|\mathbf{Q}\mathbf{u}^n\|^2 \le d_{\max}\|\mathbf{u}^n\|^2$. By the definition of matrix $\mathbf{H}$, we obtain that the eigenvalues of $\mathbf{H}$ are $\lambda_j = \frac{1}{12}(10 + 2\cos\frac{j\pi}{M})$, $j = 0, \ldots, M-1$, indicating that $\frac{2}{3} \le \lambda_j \le 1$. Since the eigenvalues of $\mathbf{H}$ and $\mathbf{H}^{-1}$ are reciprocal, we obtain that $d_{\min} \ge 1$ and $d_{\max} \le \frac{3}{2}$. Hence, we may choose $C_a = 1$ and $C_b = \frac{3}{2}$. □

Theorem 1. *For the scheme in Eq. (3.1), its solution satisfies these priori estimates as follows:*

$$\|\mathbf{u}^n\| \le C_\alpha, \quad \left\|\mathbf{u}^{n+\frac{1}{2}}\right\| \le C_\alpha; \tag{3.16a}$$

$$\|\delta_x\mathbf{u}^n\| \le C_\beta, \quad \left\|\delta_x\mathbf{u}^{n+\frac{1}{2}}\right\| \le C_\beta; \tag{3.16b}$$

$$\|\mathbf{u}^n\|_\infty \le C_\gamma, \quad \left\|\mathbf{u}^{n+\frac{1}{2}}\right\|_\infty \le C_\gamma. \tag{3.16c}$$

where $0 \le n \le \frac{T}{\tau}$, $C_\alpha, C_\beta, C_\gamma$ are constants which are independent on both h and τ.

Proof. We use the mathematical induction method to prove it. For $n = 0$, from the initial condition we should have $\|\mathbf{u}^0\| \le C_\alpha$, $\|\delta_x\mathbf{u}^0\| \le C_\beta$, and $\|\mathbf{u}^0\|_\infty \le C_\gamma$. Since $\mathbf{u}^{\frac{1}{2}}$ must be obtained using another method, we may choose a stable numerical method to obtain the solution at $t_{\frac{1}{2}}$. Thus, we should have $\|\mathbf{u}^{\frac{1}{2}}\| \le C_\alpha$, $\|\delta_x\mathbf{u}^{\frac{1}{2}}\| \le C_\beta$, $\|\mathbf{u}^{\frac{1}{2}}\|_\infty \le C_\gamma$. Assume that Eq. (3.16) holds for n up to $m - 1 \le \frac{T}{\tau} - 1$. We would like to show Eq. (3.16) to be true for $n = m$.

To estimate $\|\mathbf{u}^m\|$ and $\|\mathbf{u}^{m+\frac{1}{2}}\|$, we first take an inner product of Eq. (3.1) at $n = m - 1$ with $h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}$. This gives

$$\begin{aligned}\left\langle \delta_t\mathbf{u}^{m-1}, h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle &= (1+i\alpha)\left\langle \mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right), h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle + R\left\langle \mathbf{u}^{m-\frac{1}{2}}, h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle\\ &\quad - (1+i\beta)\left\langle |\mathbf{u}^{m-\frac{1}{2}}|^{q-1}\mathbf{u}^{m-\frac{1}{2}}, h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle.\end{aligned} \tag{3.17}$$

Taking the real part of the above equation and using Lemmas 2 and 3, one may obtain

$$\begin{aligned}\frac{1}{2\tau}(\|\mathbf{u}^m\|^2 - \|\mathbf{u}^{m-1}\|^2) &= \mathrm{Re}\left\langle \delta_t\mathbf{u}^{m-1}, h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle\\ &= \mathrm{Re}\left((1+i\alpha)\left\langle \mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right), h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle\right) + R\,\mathrm{Re}\left\langle \mathbf{u}^{m-\frac{1}{2}}, h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle\\ &\quad + \mathrm{Re}\left(-(1+i\beta)\left\langle \left|\mathbf{u}^{m-\frac{1}{2}}\right|^{q-1}\mathbf{u}^{m-\frac{1}{2}}, h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle\right)\\ &\le -\mathrm{Re}\left((1+i\alpha)\left\|\mathbf{Q}\delta_x\left(\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right)\right\|^2\right) + R\left|\left\langle \mathbf{u}^{m-\frac{1}{2}}, h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle\right|\\ &\quad + |(1+i\beta)|\left|\left\langle \left|\mathbf{u}^{m-\frac{1}{2}}\right|^{q-1}\mathbf{u}^{m-\frac{1}{2}}, h\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\rangle\right|\\ &\le R\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\| + \sqrt{1+\beta^2}\left\|\left|\mathbf{u}^{m-\frac{1}{2}}\right|^{q-1}\mathbf{u}^{m-\frac{1}{2}}\right\|\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\|\end{aligned}$$

$$
\begin{aligned}
&= R\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\| + \sqrt{1+\beta^2}\left(h\sum_{j=0}^{M-1}\left|\left|u_j^{m-\frac{1}{2}}\right|^{q-1}u_j^{m-\frac{1}{2}}\right|^2\right)^{\frac{1}{2}}\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\|\\
&= R\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\| + \sqrt{1+\beta^2}\left(h\sum_{j=0}^{M-1}\left|u_j^{m-\frac{1}{2}}\right|^{2q}\right)^{\frac{1}{2}}\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\|\\
&= R\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\| + \sqrt{1+\beta^2}\left(h\sum_{j=0}^{M-1}\left|u_j^{m-\frac{1}{2}}\right|^{2q}\right)^{\frac{1}{2q}\cdot q}\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\|\\
&= R\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\| + \sqrt{1+\beta^2}\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{q}\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\|\\
&\le \frac{R}{2}\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^2+\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\|^2\right) + \frac{\sqrt{1+\beta^2}}{2}\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{2q}+\left\|\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right\|^2\right)\\
&\le \frac{R}{2}\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^2+\frac{1}{2}\|\mathbf{u}^m\|^2+\frac{1}{2}\|\mathbf{u}^{m-1}\|^2\right) + \frac{\sqrt{1+\beta^2}}{2}\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{2q}+\frac{1}{2}\|\mathbf{u}^m\|^2+\frac{1}{2}\|\mathbf{u}^{m-1}\|^2\right)\\
&\le \frac{c_1}{2}\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{2q}+\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^2+\|\mathbf{u}^m\|^2+\|\mathbf{u}^{m-1}\|^2\right),
\end{aligned}
\tag{3.18}
$$

where $c_1=\max\{R,\sqrt{1+\beta^2}\}$. Here, we have used the fact that $-\mathrm{Re}((1+i\alpha)\|\mathbf{Q}\delta_x(\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2})\|^2) = -\|\mathbf{Q}\delta_x(\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2})\|^2 \le 0$. By Lemma 4 with $p=2q$ and Lemma 5 as well as the assumption $\|\mathbf{u}^{m-\frac{1}{2}}\|\le C_\alpha$,$\|\delta_x\mathbf{u}^{m-\frac{1}{2}}\|\le C_\beta$, we have

$$
\begin{aligned}
\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{2q} &\le \left[\varepsilon\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^{\frac{1}{2}+\frac{1}{2q}}\left\|\delta_x\mathbf{u}^{m-\frac{1}{2}}\right\|^{\frac{1}{2}-\frac{1}{2q}}+\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|\right)\right]^{2q}\\
&\le \varepsilon^{2q}2^{2q-1}\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^{q+1}\left\|\delta_x\mathbf{u}^{m-\frac{1}{2}}\right\|^{q-1}+\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^{2q}\right)\\
&\le \varepsilon^{2q}2^{2q-1}(C_\alpha^{q+1}C_\beta^{q-1}+C_\alpha^{2q}) \equiv c_2.
\end{aligned}
\tag{3.19}
$$

Substituting Eq. (3.19) into Eq. (3.18) and then using the assumption $\|\mathbf{u}^{m-1}\|\le C_\alpha$, $\|\mathbf{u}^{m-\frac{1}{2}}\|\le C_\alpha$, we obtain

$$
\begin{aligned}
\frac{1}{\tau}(\|\mathbf{u}^m\|^2-\|\mathbf{u}^{m-1}\|^2) &\le c_1\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{2q}+\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^2+\|\mathbf{u}^m\|^2+\|\mathbf{u}^{m-1}\|^2\right)\\
&\le c_1(c_2+C_\alpha^2+C_\alpha^2+\|\mathbf{u}^m\|^2)\\
&= c_3+c_1\|\mathbf{u}^m\|^2,
\end{aligned}
\tag{3.20}
$$

where $c_3=c_1c_2+2c_1C_\alpha^2$. Thus, we obtain

$$
\begin{aligned}
\|\mathbf{u}^m\|^2 &\le \frac{1}{1-c_1\tau}[c_3\tau+\|\mathbf{u}^{m-1}\|^2]\\
&\le 2[c_3+C_\alpha^2],
\end{aligned}
\tag{3.21}
$$

if τ is small enough such that $1-c_1\tau\ge\frac{1}{2}$ and $\tau\le 1$, implying that $\|\mathbf{u}^m\|$ is bounded independently on h and τ. Once $\|\mathbf{u}^m\|$ is bounded, we can obtain from Eq. (3.1) that $\|\mathbf{u}^{m+\frac{1}{2}}\|$ is also bounded using a similar argument.

To estimate $\|\delta_x\mathbf{u}^m\|^2$ and $\|\delta_x\mathbf{u}^{m+\frac{1}{2}}\|^2$, we first take the inner product of Eq. (3.1) at $n=m-1$ with $h\delta_t\mathbf{u}^{m-1}/(1+i\alpha)$. This gives

$$
\begin{aligned}
\frac{1}{(1+i\alpha)}\langle\delta_t\mathbf{u}^{m-1},h\delta_t\mathbf{u}^{m-1}\rangle &= \left\langle\mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right),h\delta_t\mathbf{u}^{m-1}\right\rangle\\
&\quad+\frac{R}{(1+i\alpha)}\left\langle\mathbf{u}^{m-\frac{1}{2}},h\delta_t\mathbf{u}^{m-1}\right\rangle-\frac{(1+i\beta)}{(1+i\alpha)}\left\langle\left|\mathbf{u}^{m-\frac{1}{2}}\right|^{q-1}\mathbf{u}^{m-\frac{1}{2}},h\delta_t\mathbf{u}^{m-1}\right\rangle.
\end{aligned}
\tag{3.22}
$$

Taking the real part of the above equation and then using Lemmas 2 and 4, we obtain

$$\begin{aligned}
&\frac{1}{(1+\alpha^2)}\|\delta_t\mathbf{u}^{m-1}\|^2\\
&=\frac{1}{(1+\alpha^2)}\langle\delta_t\mathbf{u}^{m-1},h\delta_t\mathbf{u}^{m-1}\rangle\\
&=\mathrm{Re}\left(\frac{1}{(1+i\alpha)}\langle\delta_t\mathbf{u}^{m-1},h\delta_t\mathbf{u}^{m-1}\rangle\right)\\
&=\mathrm{Re}\left(\left\langle\mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{u}^m+\mathbf{u}^{m-1}}{2}\right),h\delta_t\mathbf{u}^{m-1}\right\rangle\right)+\mathrm{Re}\left(\frac{R}{(1+i\alpha)}\left\langle\mathbf{u}^{m-\frac{1}{2}},h\delta_t\mathbf{u}^{m-1}\right\rangle\right)\\
&\quad+\mathrm{Re}\left(-\frac{(1+i\beta)}{(1+i\alpha)}\left\langle\left|\mathbf{u}^{m-\frac{1}{2}}\right|^{q-1}\mathbf{u}^{m-\frac{1}{2}},h\delta_t\mathbf{u}^{m-1}\right\rangle\right)\\
&\le-\frac{1}{2\tau}(\|\mathbf{Q}\delta_x\mathbf{u}^m\|^2-\|\mathbf{Q}\delta_x\mathbf{u}^{m-1}\|^2)+\left|\frac{R}{(1+i\alpha)}\right|\left|\left\langle\mathbf{u}^{m-\frac{1}{2}},h\delta_t\mathbf{u}^{m-1}\right\rangle\right|+\left|\frac{(1+i\beta)}{(1+i\alpha)}\right|\left|\left\langle\left|\mathbf{u}^{m-\frac{1}{2}}\right|^{q-1}\mathbf{u}^{m-\frac{1}{2}},h\delta_t\mathbf{u}^{m-1}\right\rangle\right|\\
&\le-\frac{1}{2\tau}(\|\mathbf{Q}\delta_x\mathbf{u}^m\|^2-\|\mathbf{Q}\delta_x\mathbf{u}^{m-1}\|^2)+\frac{R}{\sqrt{1+\alpha^2}}\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|\|\delta_t\mathbf{u}^{m-1}\|+\frac{\sqrt{1+\beta^2}}{\sqrt{1+\alpha^2}}\left\|\left|\mathbf{u}^{m-\frac{1}{2}}\right|^{q-1}\mathbf{u}^{m-\frac{1}{2}}\right\|\|\delta_t\mathbf{u}^{m-1}\|\\
&\le-\frac{1}{2\tau}(\|\mathbf{Q}\delta_x\mathbf{u}^m\|^2-\|\mathbf{Q}\delta_x\mathbf{u}^{m-1}\|^2)+\frac{R}{\sqrt{1+\alpha^2}}\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|\|\delta_t\mathbf{u}^{m-1}\|+\frac{\sqrt{1+\beta^2}}{\sqrt{1+\alpha^2}}\left(\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{2q}\right)^{\frac{1}{2}}\|\delta_t\mathbf{u}^{m-1}\|\\
&\le-\frac{1}{2\tau}(\|\mathbf{Q}\delta_x\mathbf{u}^m\|^2-\|\mathbf{Q}\delta_x\mathbf{u}^{m-1}\|^2)+\frac{R^2}{2}\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^2+\frac{1}{2(1+\alpha^2)}\|\delta_t\mathbf{u}^{m-1}\|^2\\
&\quad+\frac{(1+\beta^2)}{2}\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{2q}+\frac{1}{2(1+\alpha^2)}\|\delta_t\mathbf{u}^{m-1}\|^2. \qquad (3.23)
\end{aligned}$$

Thus, moving the term $\frac{1}{2\tau}(\|\mathbf{Q}\delta_x\mathbf{u}^m\|^2-\|\mathbf{Q}\delta_x\mathbf{u}^{m-1}\|^2)$ to the left-hand-side in Eq. (3.23) and dropping the term $\frac{1}{(1+\alpha^2)}\|\delta_t\mathbf{u}^{m-1}\|^2$, we simplify Eq. (3.23) to

$$\begin{aligned}
\frac{1}{\tau}(\|\mathbf{Q}\delta_x\mathbf{u}^m\|^2-\|\mathbf{Q}\delta_x\mathbf{u}^{m-1}\|^2)&\le R^2\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|^2+(1+\beta^2)\left\|\mathbf{u}^{m-\frac{1}{2}}\right\|_{2q}^{2q}\\
&\le R^2C_\alpha+(1+\beta^2)c_2, \qquad (3.24)
\end{aligned}$$

implying that $\|\mathbf{Q}\delta_x\mathbf{u}^m\|^2\le\|\mathbf{Q}\delta_x\mathbf{u}^{m-1}\|^2+R^2C_\alpha+(1+\beta^2)c_2$ if $\tau\le1$. By Lemma 6, we obtain

$$\begin{aligned}
\|\delta_x\mathbf{u}^m\|^2&\le\frac{1}{C_a}\|\mathbf{Q}\delta_x\mathbf{u}^m\|^2\\
&\le\frac{1}{C_a}(\|\mathbf{Q}\delta_x\mathbf{u}^{m-1}\|^2+R^2C_\alpha+(1+\beta^2)c_2)\\
&\le\frac{1}{C_a}(C_b^2\|\delta_x\mathbf{u}^{m-1}\|^2+R^2C_\alpha+(1+\beta^2)c_2)\\
&\le\frac{1}{C_a}(C_b^2C_\beta^2+R^2C_\alpha+(1+\beta^2)c_2)\equiv c_5. \qquad (3.25)
\end{aligned}$$

Using a similar argument, we may obtain from Eq. (3.1) that $\left\|\delta_x\mathbf{u}^{m+\frac{1}{2}}\right\|^2\le c_5$.

Finally, we have by Lemmas 4 and 5 that

$$\begin{aligned}
\|\mathbf{u}^m\|_\infty^2&\le\left[\varepsilon\left(\|\mathbf{u}^m\|^{\frac{1}{2}}\|\delta_x\mathbf{u}^m\|^{\frac{1}{2}}+\|\mathbf{u}^m\|\right)\right]^2\le2\varepsilon^2(\|\mathbf{u}^m\|\|\delta_x\mathbf{u}^m\|+\|\mathbf{u}^m\|^2)\\
&\le2\varepsilon^2\left(\frac{\|\mathbf{u}^m\|^2+\|\delta_x\mathbf{u}^m\|^2}{2}+\|\mathbf{u}^m\|^2\right)\le2\varepsilon^2\left(\frac{C_\alpha^2+C_\beta^2}{2}+C_\alpha^2\right)\equiv c_6. \qquad (3.26)
\end{aligned}$$

Using a similar argument, we also obtain $\|\mathbf{u}^{m+\frac{1}{2}}\|_\infty\le c_6$. Thus, by the mathematical induction, Eq. (3.16) holds for any n, and hence we have completed the proof. □

Theorem 2. *Assume that $\mathbf{u_a}$ and $\mathbf{u_b}$ are two different periodic solutions obtained by Eq. (3.1) based on two different initial conditions. Letting $\mathbf{e}^n=(\mathbf{u_a})^n-(\mathbf{u_b})^n$ and $\mathbf{e}^{n+\frac{1}{2}}=(\mathbf{u_a})^{n+\frac{1}{2}}-(\mathbf{u_b})^{n+\frac{1}{2}}$, then $\mathbf{e}^n$ and $\mathbf{e}^{n+\frac{1}{2}}$ satisfy*

$$\left\|\mathbf{e}^{n+\frac{3}{2}}\right\|^2+\|\mathbf{e}^{n+1}\|^2\le C_\zeta\left(\left\|\mathbf{e}^{\frac{1}{2}}\right\|^2+\|\mathbf{e}^0\|^2\right), \qquad (3.27)$$

for any n and small τ, where C_ζ is a constant, implying that the scheme is unconditionally stable.

Proof. It can be seen from Eq. (3.1) that for any $n > 0$, $\mathbf{e}^n$ and $\mathbf{e}^{n+\frac{1}{2}}$ satisfy

$$\delta_t \mathbf{e}^n = (1+i\alpha)\mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right) + R\mathbf{e}^{n+\frac{1}{2}} - (1+i\beta)\left(\left|(\mathbf{u_a})^{n+\frac{1}{2}}\right|^{q-1}(\mathbf{u_a})^{n+\frac{1}{2}} - \left|(\mathbf{u_b})^{n+\frac{1}{2}}\right|^{q-1}(\mathbf{u_b})^{n+\frac{1}{2}}\right), \tag{3.28a}$$

$$\delta_t \mathbf{e}^{n+\frac{1}{2}} = (1+i\alpha)\mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{e}^{n+\frac{3}{2}}+\mathbf{e}^{n+\frac{1}{2}}}{2}\right) + R\mathbf{e}^{n+1} - (1+i\beta)(|(\mathbf{u_a}^{n+1})|^{q-1}(\mathbf{u_a})^{n+1} - |(\mathbf{u_b})^{n+1}|^{q-1}(\mathbf{u_b})^{n+1}), \tag{3.28b}$$

where $\mathbf{e}^0 = (\mathbf{u_a})^0 - (\mathbf{u_b})^0$ and $\mathbf{e}^{\frac{1}{2}} = (\mathbf{u_a})^{\frac{1}{2}} - (\mathbf{u_b})^{\frac{1}{2}}$. Taking the inner products of Eq. (3.28) with $h\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}$ and $h\frac{\mathbf{e}^{n+\frac{3}{2}}+\mathbf{e}^{n+\frac{1}{2}}}{2}$, respectively, we obtain

$$\left\langle \delta_t\mathbf{e}^n, h\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right\rangle = (1+i\alpha)\left\langle \mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right), h\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right\rangle + R\left\langle \mathbf{e}^{n+\frac{1}{2}}, h\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right\rangle$$
$$- (1+i\beta)\left\langle \left|(\mathbf{u_a})^{n+\frac{1}{2}}\right|^{q-1}(\mathbf{u_a})^{n+\frac{1}{2}} - \left|(\mathbf{u_b})^{n+\frac{1}{2}}\right|^{q-1}(\mathbf{u_b})^{n+\frac{1}{2}}, h\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right\rangle, \tag{3.29a}$$

$$\left\langle \delta_t\mathbf{e}^{n+\frac{1}{2}}, h\frac{\mathbf{e}^{n+\frac{3}{2}}+\mathbf{e}^{n+\frac{1}{2}}}{2}\right\rangle = (1+i\alpha)\left\langle \mathbf{H}^{-1}\delta_x^2\left(\frac{\mathbf{e}^{n+\frac{3}{2}}+\mathbf{e}^{n+\frac{1}{2}}}{2}\right), h\frac{\mathbf{e}^{n+\frac{3}{2}}+\mathbf{e}^{n+\frac{1}{2}}}{2}\right\rangle + R\left\langle \mathbf{e}^{n+1}, h\frac{\mathbf{e}^{n+\frac{3}{2}}+\mathbf{e}^{n+\frac{1}{2}}}{2}\right\rangle$$
$$- (1+i\beta)\left\langle \left|(\mathbf{u_a})^{n+1}\right|^{q-1}(\mathbf{u_a})^{n+1} - \left|(\mathbf{u_b})^{n+1}\right|^{q-1}(\mathbf{u_b})^{n+1}, h\frac{\mathbf{e}^{n+\frac{3}{2}}+\mathbf{e}^{n+\frac{1}{2}}}{2}\right\rangle. \tag{3.29b}$$

Since $\|(\mathbf{u_a})^{n+\frac{1}{2}}\|_\infty \le C_\gamma$, $\|(\mathbf{u_b})^{n+\frac{1}{2}}\|_\infty \le C_\gamma$ by Theorem 1, we may simplify the jth component of the nonlinear vector in Eq. (3.29) based on Eq. (3.9) as

$$\begin{aligned}
&\left|\left(\left|(\mathbf{u_a})^{n+\frac{1}{2}}\right|^{q-1}(\mathbf{u_a})^{n+\frac{1}{2}}\right)_j - \left(\left|(\mathbf{u_b})^{n+\frac{1}{2}}\right|^{q-1}(\mathbf{u_b})^{n+\frac{1}{2}}\right)_j\right| \\
&\le \left(\left|(\mathbf{u_a})_j^{n+\frac{1}{2}}\right| + \left|(\mathbf{u_b})_j^{n+\frac{1}{2}}\right|\right)^{q-1}\left|(\mathbf{u_a})_j^{n+\frac{1}{2}} - (\mathbf{u_b})_j^{n+\frac{1}{2}}\right| \\
&\le (2C_\gamma)^{q-1}\left|\left(\mathbf{e}^{n+\frac{1}{2}}\right)_j\right| \\
&= c_7\left|\left(\mathbf{e}^{n+\frac{1}{2}}\right)_j\right|,
\end{aligned} \tag{3.30}$$

where $c_7 = (2C_\gamma)^{q-1}$. Similarly, we have

$$\left|\left(\left|(\mathbf{u_a})^{n+1}\right|^{q-1}(\mathbf{u_a})^{n+1}\right)_j - \left(\left|(\mathbf{u_b})^{n+1}\right|^{q-1}(\mathbf{u_b})^{n+1}\right)_j\right| \le c_7\left|\left(\mathbf{e}^{n+1}\right)_j\right|. \tag{3.31}$$

Taking the real part of Eq. (3.29), using Lemmas 2 and 3, and then using Eq. (3.30), we obtain

$$\begin{aligned}
&\frac{1}{2\tau}(\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2) \\
&\le -\left\|\mathbf{Q}\delta_x\left(\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right)\right\|^2 + R\left\|\mathbf{e}^{n+\frac{1}{2}}\right\|\left\|\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right\| + c_7\sqrt{1+\beta^2}\left\|\mathbf{e}^{n+\frac{1}{2}}\right\|\left\|\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right\| \\
&\le \left(c_7\sqrt{1+\beta^2}+R\right)\left\|\mathbf{e}^{n+\frac{1}{2}}\right\|\left\|\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right\| \\
&\le \frac{c_7\sqrt{1+\beta^2}+R}{2}\left[\left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2 + \left\|\frac{\mathbf{e}^{n+1}+\mathbf{e}^n}{2}\right\|^2\right] \\
&\le \frac{c_7\sqrt{1+\beta^2}+R}{2}\left[\left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2 + \frac{1}{2}\left(\|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2\right)\right] \\
&\le \frac{c_7\sqrt{1+\beta^2}+R}{2}\left[\left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2 + \|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2\right].
\end{aligned} \tag{3.32}$$

which can be simplified to

$$\frac{1}{\tau}(\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2) \le c_8\left[\left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2 + \left\|\mathbf{e}^{n+1}\right\|^2 + \|\mathbf{e}^n\|^2\right], \tag{3.33}$$

where $c_8 = \left(\sqrt{1+\beta^2}c_7 + R\right)$. Using a similar argument for Eq. (3.29), we obtain

$$\frac{1}{\tau}\left(\left\|\mathbf{e}^{n+\frac{3}{2}}\right\|^2 - \left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2\right) \le c_8\left[\left\|\mathbf{e}^{n+\frac{3}{2}}\right\|^2 + \left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2 + \|\mathbf{e}^{n+1}\|^2\right]. \tag{3.34}$$

Summing Eqs. (3.33) and (3.34) together gives

$$\begin{aligned}
&\frac{1}{\tau}\left(\left\|\mathbf{e}^{n+\frac{3}{2}}\right\|^2 + \|\mathbf{e}^{n+1}\|^2 - \left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2 - \|\mathbf{e}^n\|^2\right) \\
&\le c_8\left(\left\|\mathbf{e}^{n+\frac{3}{2}}\right\|^2 + 2\left\|\mathbf{e}^{n+1}\right\|^2 + 2\left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2 + \|\mathbf{e}^n\|^2\right) \\
&\le c_9\left(\left\|\mathbf{e}^{n+\frac{3}{2}}\right\|^2 + \left\|\mathbf{e}^{n+1}\right\|^2 + \left\|\mathbf{e}^{n+\frac{1}{2}}\right\|^2 + \|\mathbf{e}^n\|^2\right),
\end{aligned} \tag{3.35}$$

where $c_9 = 2c_8$, implying that

$$\frac{1}{\tau}(E^{n+1} - E^n) \le c_9(E^{n+1} + E^n), \tag{3.36}$$

where $E^n = \|\mathbf{e}^{n+\frac{1}{2}}\|^2 + \|\mathbf{e}^n\|^2$. Thus, we obtain from Eq. (3.36) that

$$\begin{aligned}
E^{n+1} &\le \left(\frac{1+\tau c_9}{1-\tau c_9}\right)E^n \\
&\le \cdots \\
&\le \left(\frac{1+\tau c_9}{1-\tau c_9}\right)^{n+1}E^0 \\
&\le e^{2\tau(n+1)c_9}E^0 \\
&\le c_{10}E^0,
\end{aligned} \tag{3.37}$$

where $c_{10} = e^{2Tc_9}$ and $(n+1)\tau \le T$ if τ is sufficiently small such that $1 - \tau c_9 \ge \frac{1}{2}$. Hence, Eq. (3.27) has been obtained and the proof of Theorem 2 is completed. □

Furthermore, the stability analysis for the 2D scheme is similar to the above analysis, but much more complicated. We omit the detailed derivations here because of the limitation of pages.

4. Numerical examples

To test the accuracy of our numerical schemes, we first considered a 1D complex Ginzburg–Landau equation with initial and periodic boundary conditions as follows:

$$\frac{\partial u}{\partial t} = (1+i\alpha)\frac{\partial^2 u}{\partial x^2} + Ru - (1+i\beta)|u|^2u, \quad x \in \left(0, \frac{2\sqrt{2}\pi}{\sqrt{R}}\right), t \in (0,1]; \tag{4.1a}$$

$$u(x,0) = \frac{\sqrt{2R}}{2}e^{i(\frac{\sqrt{2R}}{2}x)}, \quad x \in \left[0, \frac{2\sqrt{2}\pi}{\sqrt{R}}\right]; \tag{4.1b}$$

$$u(0,t) = u\left(\frac{2\sqrt{2}\pi}{\sqrt{R}}, t\right) = \frac{\sqrt{2R}}{2}e^{i(-\frac{(\alpha+\beta)R}{2}t)}, \tag{4.1c}$$

where the exact solution is $u(x,t) = \frac{\sqrt{2R}}{2}e^{i(\frac{\sqrt{2R}}{2}x - \frac{(\alpha+\beta)R}{2}t)}$.

We employed our present scheme in Eq. (2.7) to solve the above problem, in which the Thomas algorithm [45] was used for solving the obtained tri-diagonal linear systems. The maximum of l_∞-norm errors of the numerical solutions, as compared with the analytical solution, were computed for $0 \le t \le 1$ based on the formula

$$e_\infty(\tau,h) = \max_{0\le n\tau\le 1}\left(\max_{0\le j\le M}\left|u_j^n - u(x_j,t_n)\right|\right). \tag{4.2}$$

To obtain the convergence rate with respect to the spatial variable, we may assume that $e_\infty(\tau,h) = O(\tau^p + h^q)$. Thus, $e_\infty(2^{q/p}\tau, 2h) = O[(2^{q/p}\tau)^p + (2h)^q] = 2^qO(\tau^p + h^q)$. Consequently, $e_\infty(2^{q/p}\tau, 2h)/e_\infty(\tau,h) = 2^q$ and hence $q = \log_2[e_\infty(2^{q/p}\tau, 2h)/e_\infty(\tau,h)]$

Table 1
Maximum error and convergence rate in the first example ($R = 1, \alpha = \beta = 1$, and $\mathbf{u}^{\frac{1}{2}}$ was obtained using Eq. (4.3)).

(h, τ)	$e_\infty(h, \tau)$	Rate
$(2\sqrt{2}\pi/5, 1/5)$	0.00485726115701	–
$(2\sqrt{2}\pi/10, 1/20)$	2.946157986007180e-004	4.0432
$(2\sqrt{2}\pi/20, 1/80)$	1.820597872425982e-005	4.0164
$(2\sqrt{2}\pi/40, 1/320)$	1.139391147327777e-006	3.9981

Table 2
Maximum error and convergence rate in the first example ($R = 2, \alpha = \beta = 1$, and $\mathbf{u}^{\frac{1}{2}}$ was obtained using Eq. (4.3)).

(h, τ)	$e_\infty(h, \tau)$	Rate
$(2\pi/5, 1/5)$	0.01610696193415	–
$(2\pi/10, 1/20)$	0.00106759559593	3.9152
$(2\pi/20, 1/80)$	6.710783491111773e-005	3.9917
$(2\pi/40, 1/320)$	4.230900881952949e-006	3.9874

Table 3
Maximum error and convergence rate in the first example ($R = 1, \alpha = \beta = 1$, and $\mathbf{u}^{\frac{1}{2}}$ was obtained using the analytical solution).

(h, τ)	$e_\infty(h, \tau)$	Rate
$(2\sqrt{2}\pi/5, 1/5)$	0.00489551131950	–
$(2\sqrt{2}\pi/10, 1/20)$	2.943305960467438e-004	4.0560
$(2\sqrt{2}\pi/20, 1/80)$	1.819984324406890e-005	4.0154
$(2\sqrt{2}\pi/40, 1/320)$	1.139197836597871e-006	3.9980

Table 4
Maximum error and convergence rate in the first example ($R = 2, \alpha = \beta = 1$, and $\mathbf{u}^{\frac{1}{2}}$ was obtained using the analytical solution).

(h, τ)	$e_\infty(h, \tau)$	Rate
$(2\pi/5, 1/5)$	0.01550766547724	–
$(2\pi/10, 1/20)$	0.00105786656594	3.8738
$(2\pi/20, 1/80)$	6.696133491033216e-005	3.9817
$(2\pi/40, 1/320)$	4.228439981965586e-006	3.9851

is the convergence rate with respect to the spatial variable. For our present scheme, we expect to have $p = 2$ and $q = 4$.

Since the present scheme needs those values at time levels $n = 0$ and $n = \frac{1}{2}$, the values at the time level $n = \frac{1}{2}$ must be obtained using other methods. Here, we used the fully implicit method developed in [29] as

$$\left(1 + \frac{h^2}{12}\delta_x^2\right)\frac{u_j^{\frac{1}{2}} - u_j^0}{\tau/2} = (1 + i\alpha)\delta_x^2\left(\frac{u_j^{\frac{1}{2}} + u_j^0}{2}\right) + R\left(1 + \frac{h^2}{12}\delta_x^2\right)\left(\frac{u_j^{\frac{1}{2}} + u_j^0}{2}\right)$$
$$-(1 + i\beta)(1 + \frac{h^2}{12}\delta_x^2)\frac{|u_j^{\frac{1}{2}}|^{q-1} + |u_j^0|^{q-1}}{2}\frac{u_j^{\frac{1}{2}} + u_j^0}{2}. \tag{4.3}$$

Note that the analytical solution is known in this example, we also used the exact values at the time level $n = \frac{1}{2}$ to check if our scheme is fourth-order. In our computation, we chose two different values of R. Results are shown in Tables 1–4. It can be seen from these tables that the convergence rate of the scheme is approximately fourth-order with respect to the spatial variable, which coincides with the theoretical analysis. Fig. 1 shows the solution profiles obtained based on three different meshes as compared with the analytical solutions. Results do not show much of a difference between the analytical solution and the numerical solution.

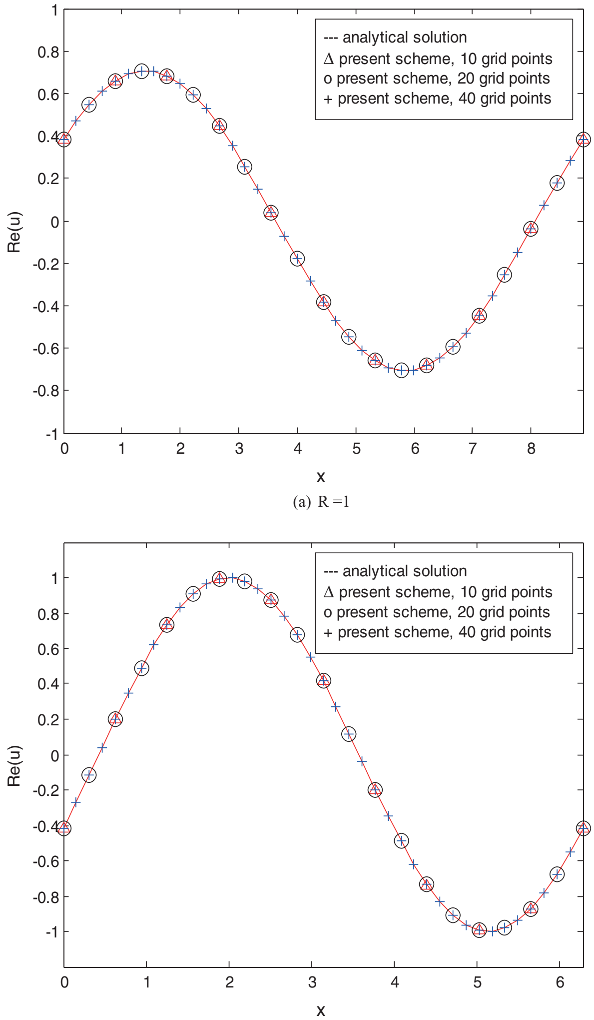

Fig. 1. Profiles of the real part of the numerical solution at $t = 1$ obtained using three different meshes and corresponding $\tau = 1/20, 1/80, 1/320$, respectively, as compared with the analytical solution in the first example.

Table 5
Maximum error and convergence rate in the second example ($R = 1, \alpha = \beta = 1$, and $\mathbf{u}^{\frac{1}{2}}$ was obtained using the analytical solution).

(h_x, h_y, τ)	$e_\infty\,(h_x, h_y, \tau)$	Rate
$\left(2\sqrt{2}\pi/5, 2\sqrt{2}\pi/5, 1/5\right)$	0.00110542709763	–
$\left(2\sqrt{2}\pi/10, 2\sqrt{2}\pi/10, 1/20\right)$	6.406423411185432*e*-005	4.1089
$\left(2\sqrt{2}\pi/20, 2\sqrt{2}\pi/20, 1/80\right)$	3.996877803767565*e*-006	4.0026
$\left(2\sqrt{2}\pi/40, 2\sqrt{2}\pi/40, 1/320\right)$	2.482725014645710*e*-007	4.0089

We then considered a 2D complex Ginzburg–Landau equation with initial and periodic boundary conditions as

$$\frac{\partial u}{\partial t} = (1 + i\alpha)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + Ru - (1 + i\beta)|u|^2 u, \quad x, y \in \left(0, \frac{2\sqrt{2}\pi}{\sqrt{R}}\right), t \in (0, 1]; \tag{4.4a}$$

$$u(x, y, 0) = \frac{\sqrt{2R}}{2} e^{i\left(\frac{\sqrt{R}}{2}x + \frac{\sqrt{R}}{2}y\right)}, \quad x, y \in \left[0, \frac{2\sqrt{2}\pi}{\sqrt{R}}\right]; \tag{4.4b}$$

Table 6
Maximum error and convergence rate in the second example ($R = 2, \alpha = \beta = 1$, and $\mathbf{u}^{\frac{1}{2}}$ was obtained using the analytical solution).

(h_x, h_y, τ)	$e_\infty(h_x, h_y, \tau)$	Rate
$(2\pi/5, 2\pi/5, 1/5)$	0.01239497829524	–
$(2\pi/10, 2\pi/10, 1/20)$	8.097105045953942e-004	3.9362
$(2\pi/20, 2\pi/20, 1/80)$	4.978160430930223e-005	4.0237
$(2\pi/40, 2\pi/40, 1/320)$	3.117868510497469e-006	3.9970

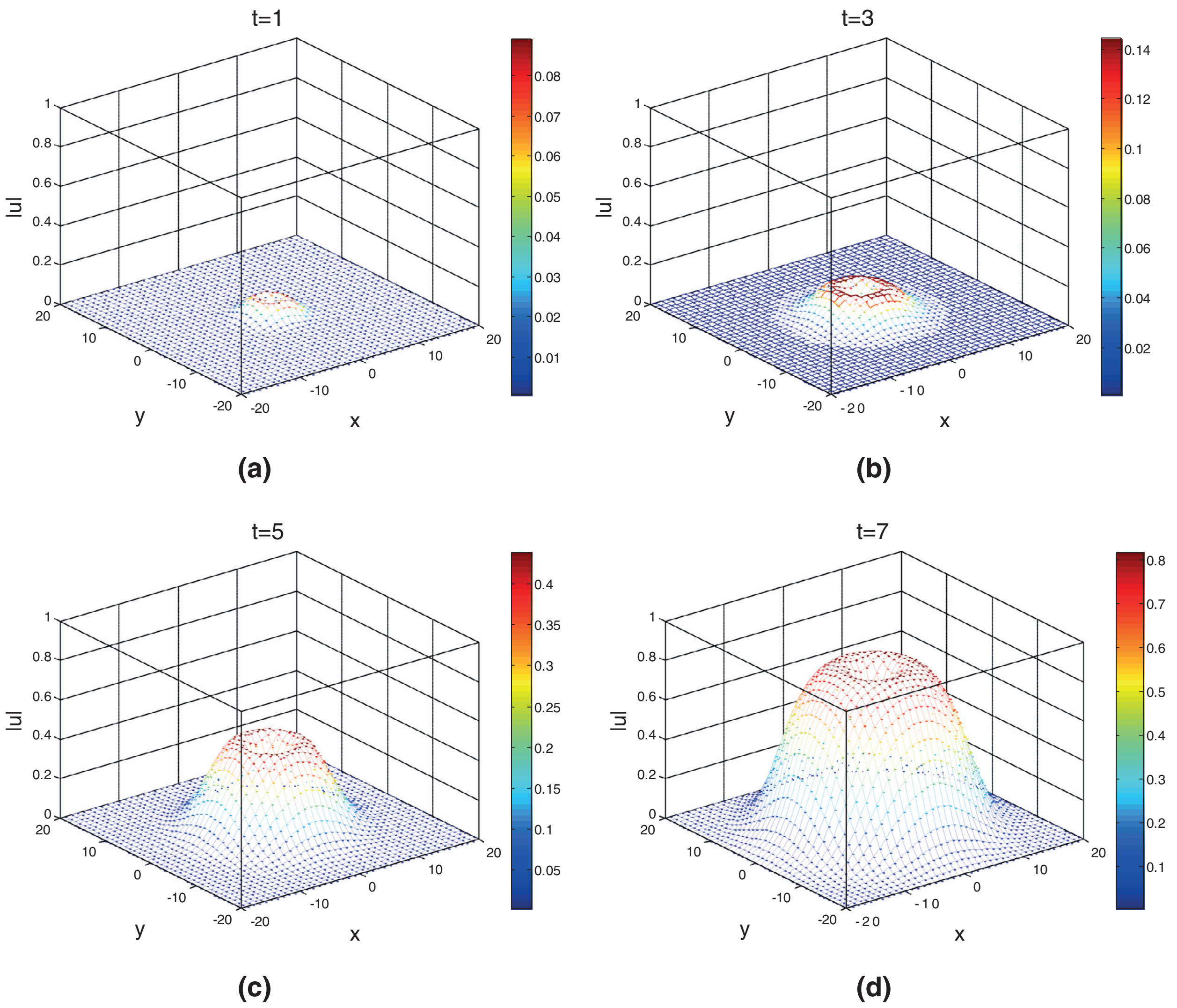

Fig. 2. Solution distributions at various times at (a) $t = 1$, (b) $t = 3$, (c) $t = 5$, and (d) $t = 7$ obtained based on the mesh 40×40 and $\tau = 0.1$ in the third example.

$$u(x, 0, t) = u(x, L_y, t) = \frac{\sqrt{2R}}{2} e^{i(\frac{\sqrt{R}}{2}x - \frac{(\alpha+\beta)R}{2}t)}, \quad t \in [0, 1] \tag{4.4c}$$

$$u(0, y, t) = u(L_x, y, t) = \frac{\sqrt{2R}}{2} e^{i(\frac{\sqrt{R}}{2}y - \frac{(\alpha+\beta)R}{2}t)}, \quad t \in [0, 1], \tag{4.4d}$$

where the analytical solution is $u(x, y, t) = \frac{\sqrt{2R}}{2} e^{i(\frac{\sqrt{R}}{2}x + \frac{\sqrt{R}}{2}y - \frac{(\alpha+\beta)R}{2}t)}$. For simplicity, we used the exact solution for $\mathbf{u}^{\frac{1}{2}}$ in the computation. Results are shown in Tables 5 and 6. Again, as expected, it can be seen from these two tables that the convergence rate of the scheme is approximately fourth-order.

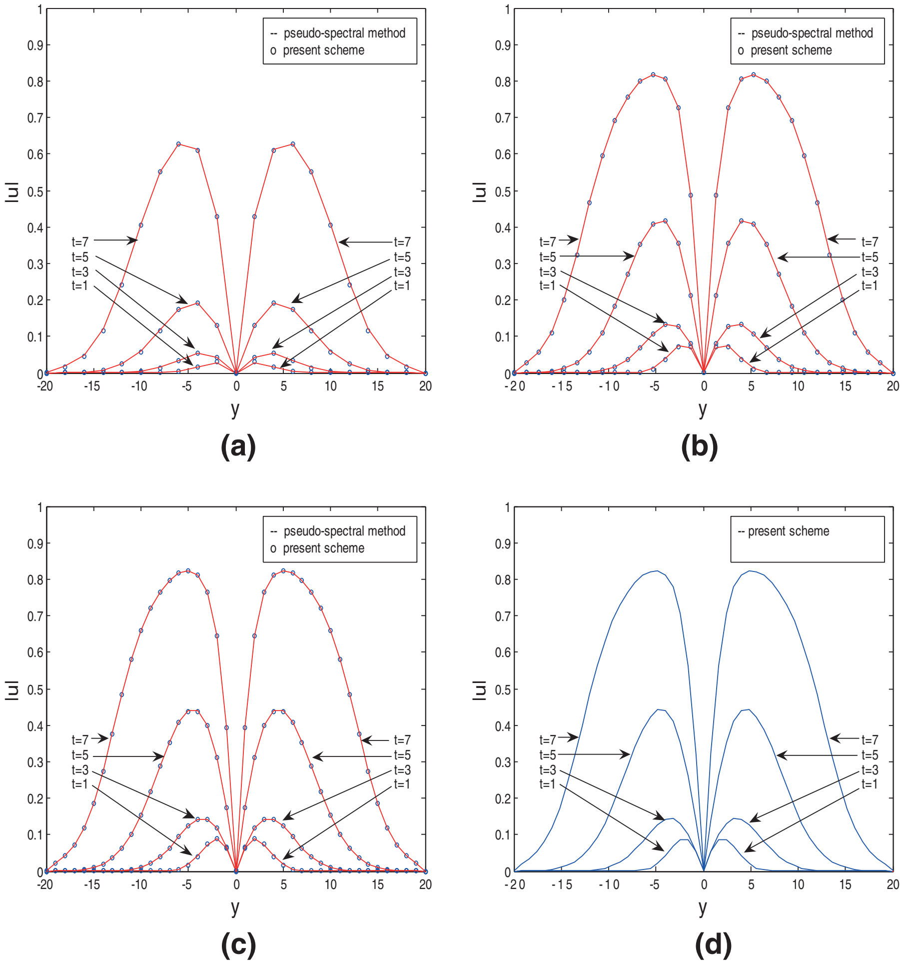

Fig. 3. Solution profiles along the y-axis at various times obtained based on four meshes of (a) 20×20, (b) 30×30, (c) 40×40, (d) 50×50, and $\tau = 0.1$ in the third example.

Finally, we considered a more complex example with no periodic boundary condition, which we could not find the exact solution, as follows:

$$\frac{\partial u}{\partial t} = (1+i)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + u - (1+i)\,|u|^2\,u, \quad x, y \in (-\infty, +\infty)\,, t > 0; \tag{4.5a}$$

$$u(x, y, 0) = \frac{2}{\sqrt{\pi}}(x + iy)e^{-(x^2+y^2)}, \quad x, y \in (-\infty, +\infty). \tag{4.5b}$$

The example is related to the non-equilibrium condensate. For the coefficient of the diffusion term $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is $1 + i$, we expect to have some diffusions in the solution. In our computation, the domain was taken to be $-20 \le x, y \le 20$, where the boundary condition was set to be zero. The number of grid points in both x and y was chosen to be 40×40 with the time step $\tau = 0.1$. For this case, $\mathbf{u}^{\frac{1}{2}}$ was computed using the fourth-order accurate and explicit pseudo-spectral method [39] built in the software MATLAB. Fig. 2 shows the simulation of the solution at various times in $0 \le t \le 7$. As expected, we see from Fig. 2 that the vortex grows and diffuses toward the boundary.

We then chose four different meshes of 20×20, 30×30, 40×40, and 50×50 with the time step $\tau = 0.1$ and compared with the fourth-order accurate and explicit pseudo-spectral method, as shown in Figs. 3 and 4. It can be seen from these two figures that the solutions obtained based on these two methods are not significantly different for the meshes of 20×20, 30×30, and 40×40, as shown in Figs. 3(a)–(c) and 4(a)–(c). However, for the mesh of 50×50, the pseudo-spectral method produces a divergent solution in which the maximum value of $u(x, y) = 4.670916294752263e^{180} + i8.621058722691569e^{180}$ at $t = 0.8$ and $u(x, y) = \text{NaN}$ at $t = 0.9$. On the other hand, our method still produces a stable solution, as shown in Figs. 3(d) and 4(d). This indicates that our method has a better stability condition.

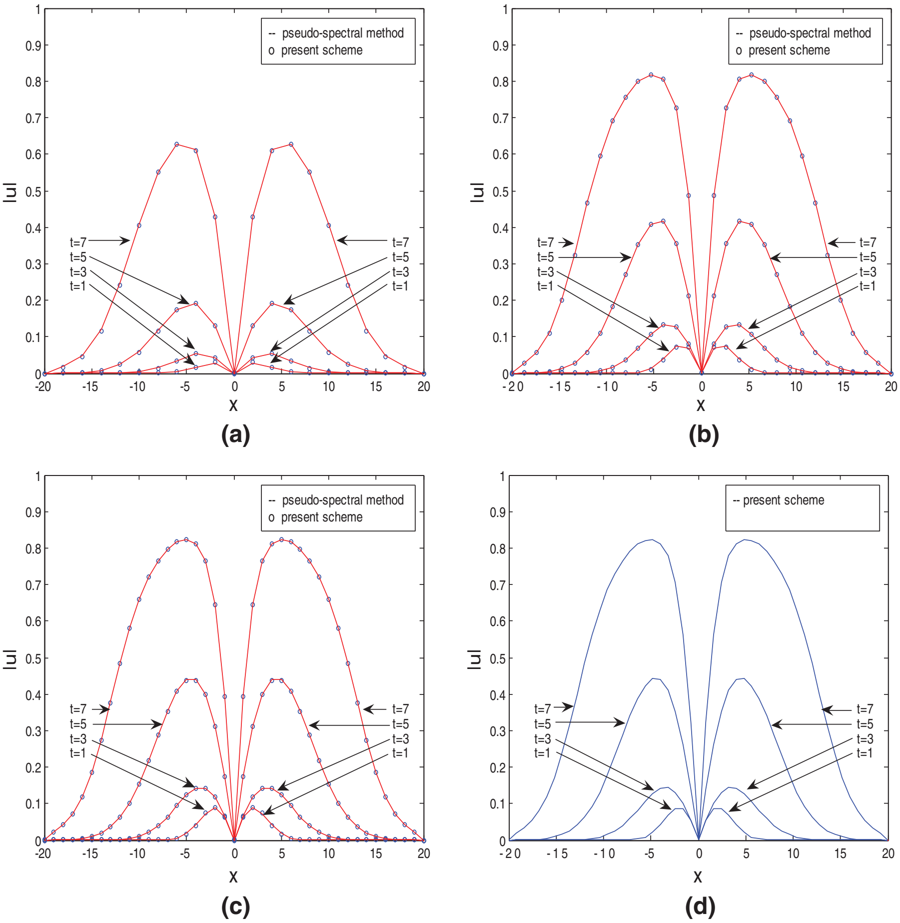

Fig. 4. Solution profiles along the *x*-axis at various times obtained based on four meshes of (a) 20×20, (b) 30×30, (c) 40×40, (d) 50×50, and $\tau = 0.1$ in the third example.

5. Conclusion

In this study, we have developed a new, simple, and accurate finite difference iterative scheme for solving the 1D and 2D complex Ginzburg–Landau equations with initial and periodic boundary conditions, respectively. Coupled with the Crank–Nicolson finite difference technique and the fourth-order compact finite difference method for spatial variables, the new scheme is proved to be unconditionally stable and provides fourth-order accurate numerical solutions with respect to the spatial variables. Numerical errors and convergence rates of the solutions have been tested by several examples. Results show that the maximal l_∞-norm errors are small as expected, and the convergence rates of the numerical solutions are fourth-order with respect to the spatial variables. Further research will be focused on the applications of the new method to practical physics and engineering problems, such as the phenomenological Ginzburg–Landau complex superconductivity model [22]:

$$\eta \frac{\partial \psi}{\partial t} + i\eta k\phi\psi + \left(\frac{i}{k}\nabla + \mathbf{A}\right)^2 \psi - \psi + |\psi|^2 \psi = 0, \quad \text{in } \Omega \times (0, T); \tag{5.1a}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi + \text{curl}\,\text{curl}\mathbf{A} + R\left[\left(\frac{i}{k}\nabla\psi + \mathbf{A}\boldsymbol{\psi}\right)\bar{\psi}\right] = 0, \quad \text{in } \Omega \times (0, T); \tag{5.1b}$$

with the initial and boundary conditions

$$\left(\frac{i}{k}\nabla\psi + \mathbf{A}\boldsymbol{\psi}\right) \cdot \mathbf{n} = 0, \text{curl}\mathbf{A} = H, \text{ on } \partial\Omega \times (0, T); \tag{5.1c}$$

$$\psi(\mathbf{x}, 0) = \psi_0(x), \qquad \mathbf{A}(\mathbf{x}, 0) = \mathbf{A}_0(x), \quad \text{in } \Omega; \tag{5.1d}$$

where Eq. (5.1) may be solved using the present method. Here, ψ is a complex valued function and is referred to as the order parameter so that $|\psi|^2$ gives the relative density of the superconducting electron pairs, and the normal and the pure superconducting states are characterized accordingly by $|\psi|^2 = 0$ and $|\psi|^2 = 1$. $\bar{\psi}$ stands for the complex conjugate of ψ. $\mathbf{A}$ is a real vector potential for the total magnetic field and ϕ is a real scalar function called electric potential. H is the applied magnetic field that points out of the (x_1, x_2)-plane. η, k are positive constants which are related to the known physical quantities.

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References

[1] M. Lin, W. Duan, Wave packet propagating in an electrical transmission line, Chaos Solitons Fractals 24 (2005) 191–196.
[2] F. Ndzana, A. Mohamadou, T. Kofane, Modulated waves and chaotic-like behaviours in the discrete electrical transmission line, J. Phys. D: Appl. Phys. 40 (2007) 3254–3262.
[3] E. Kengne, R. Vailancourt, Propagation of solitary waves on lossy nonlinear transmission lines, Int. J. Modern Phys. B 1 (2009) 1–18.
[4] E. Kengne, C. Tadmon, T. Nguyen-Ba, R. Vaillancourt, On the dissipative complex Ginzburg-Landau equation governing the propagation of solitary pulses in dissipative nonlinear transmission lines, Chin. J. Phys. 47 (2009) 80–90.
[5] E. Kengne, R. Vaillancourt, 2D Ginzburg-Landau system of complex modulation for coupled nonlinear transmission lines, J. Infrared Millimeter Waves 30 (2009) 679–699.
[6] J. Duan, P. Holmes, Fronts, domain walls and pulses in a generalized Ginzburg-Landau equations, Proc. Edinb. Math. Soc. 38 (1995) 77–97.
[7] J.M. Soto-Crespo, N. Akhmediev, Exploding solition and front solutions of the complex cubic-quintic Ginzburg-Landau equation, Math. Comput. Simul. 69 (2005) 526–536.
[8] E.N. Tsoy, A. Ankiewicz, N. Akhmediev, Dynamical models for dissipative localized waves of the complex Ginzburg-Landau equation, Phys. Rev. E 73 (2006) 036621.
[9] N.K. Efremidis, D.N. Christodoulides, K. Hizanidis, Two-dimensional discrete Ginzburg-Landau solitions, Phys. Rev. A 76 (2007) 043839.
[10] W.H. Renninger, A. Chong, F.W. Wise, Dissipative solitons in normal-dispersion fiber lasers, Phys. Rev. A 77 (2008) 023814.
[11] Q. Du, M.D. Gunzburger, J.S. Peterson, Analysis and approximation of the Ginzburg-Landau model of superconductivity, SIAM Rev. 34 (1992) 54–81.
[12] Q. Du, M.D. Gunzburger, J. Deang, Finite element approximation of a periodic Ginzburg-Landau model for type-II superconductors, Numer. Math. 64 (1993) 85–114.
[13] Q. Du, Discrete gauge invariant approximations of time dependent Ginzburg-Landau model of superconductivity, Math. Comput. 67 (1998) 965–986.
[14] Q. Du, R.A. Nicolaides, X. Wu, Analysis and convergence of covolume approximation of the Ginzburg-Landau model of superconductivity, SIAM J. Numer. Anal. 35 (1998) 1049–1072.
[15] J. Deang, Q. Du, M.D. Gunzburger, Modeling and computation of random thermal fluctuations and material defects in the Ginzburg-Landau model for superconductivity, J. Comput. Phys. 181 (2002) 45–67.
[16] Q. Du, L. Ju, Numerical simulations of the quantized vortices on a thin superconducting hollow sphere, J. Comput. Phys. 201 (2004) 511–530.
[17] Q. Du, L. Ju, Approximations of a Ginzburg-Landau model for superconducting hollow spheres based on spherical centroidal voronoi tessellations, Math. Comput. 74 (2005) 1257–1280.
[18] A. Doelman, Finite-dimensional models of the Ginzburg-Landau equation, Nonlinearity 4 (1991) 231–250.
[19] Q. Wang, Z.D. Wang, Simulating the time-dependent $d_{x^2-y^2}$ Ginzburg-Landau equations using the finite-element method, Phys. Rev. B 54 (1996) R15645.
[20] G.J. Lord, Attractors and inertial manifolds for finite-difference approximations of the complex Ginzburg-Landau equation, SIAM J. Numer. Anal. 34 (1997) 1483–1512.
[21] P. Takac, A. Jungel, A nonstiff Euler discretization of the complex Ginzburg-Landau equation in one space dimension, SIAM J. Numer. Anal. 34 (1997) 292–328.
[22] Z. Chen, S. Dai, Adaptive Galerkin methods with error control for a dynamical Ginzburg-Landau model in superconductivity, SIAM J. Numer. Anal. 38 (2001) 1961–1985.
[23] J. Willers, E.H. Twizell, A finite-difference solution of the Ginzburg-Landau equation, J. Differ. Eq. Appl. 9 (2003) 1059–1068.
[24] Y. Zhang, W. Bao, Q. Du, Numerical simulation of vortex dynamics in Ginzburg-Landau-Schrödinger equation, Eur. J. Appl. Math. 18 (2007) 607–630.
[25] P. Degond, S. Jin, M. Tang, On the time splitting spectral method of the complex Ginzburg-Landau equation in large time and space scale, SIAM J. Sci. Comput. 30 (2008) 2466–2487.
[26] L. Zhang, Long-time behavior of finite difference approximations for the two-dimensional complex Ginzburg-Landau equation, Numer. Funct. Anal. Optimization 31 (2010) 1190–1211.
[27] G.Z. Tsertsvadze, On the convergence of difference schemes for the Kuramoto-Tsuzuki equation and for systems of reaction diffusion type, J. Comput. Math. Math. Phys. 31 (1991) 698–707.
[28] Z.Z. Sun, Q. Zhu, On Tsertsvadze's difference scheme for the Kuramoto-Tsuzuki equation, J. Comput. Appl. Math. 98 (1998) 289–304.
[29] X. Hu, Numerical methods on compact difference schemes for 1D nonlinear Kuramoto-Tsuzuki equation, Numer. Methods Partial Differ. Eq., in press.
[30] S.K. Lele, Compact finite difference schemes with spectral-like resolution, J. Computat. Phys. 103 (1992) 16–42.
[31] T. Wang, B. Guo, Unconditional convergence of two conservative compact difference schemes for non-linear Schrödinger in one dimension, Sci. Sin. Math. 41 (2011) 207–233.

[32] T. Wang, B. Guo, Q. Xu, Fourth-order compact and energy conservative difference schemes for the nonlinear Schrödinger equation in two dimensions, J. Comput. Phys. 243 (2013) 382–399.
[33] T. Wang, Optimal point-wise error estimate of a compact difference scheme for the coupled Gross-Pitaevskii equations in one dimension, J. Sci. Comput. 59 (2014) 158–186.
[34] T. Wang, Optimal point-wise error estimate of a compact difference scheme for the coupled nonlinear Schrödinger equations, J. Comput. Math. 32 (2014) 58–74.
[35] T. Wang, Optimal point-wise error estimate of a compact difference scheme for the Klein-Gordon-Schrödinger equation, J. Math. Anal. Appl. 412 (2014) 155–167.
[36] M. Lees, Alternating direction and semi-explicit difference methods for parabolic partial differential equations, Numer. Math. 3 (1961) 398–412.
[37] B. Guo, On discrete energy method (i), Chin. Univ. J. Numer. Math. 4 (1984) 327–344.
[38] W. Dai, An unconditionally stable three-level explicit difference scheme for the Schrödinger equation with a variable coefficient, SIAM J. Numer. Anal. 29 (1992) 174–181.
[39] J. Yang, Nonlinear Waves in Integrable and Non-integrable Systems, SIAM, 2010.
[40] R.K. Mohanty, V. Gopal, High accuracy arithmetic average type discretization for the solution of two-space dimensional nonlinear wave equations, Int. J. Model. Simul. Sci. Comput. 3 (2012) 1150005.
[41] D.W. Peaceman, H.H. Rachford Jr., The numerical solution of parabolic and elliptic differential equations, J. Soc. Ind. Appl. Math. 3 (1955) 28–41.
[42] Z.Z. Sun, Numerical Methods of Partial Differential Equations, 2nd edition, Science Press, Beijing, 2012 (in Chinese).
[43] Y. Zhou, Application of Discrete Functional Analysis to the Finite Difference Methods, International Academic Publishers, New York, 1990.
[44] K.E. Atkinson, An Introduction to Numerical Analysis, 2nd edition, John Wiley & Sons, New York, 1989.
[45] K.W. Morton, D.F. Mayers, Numerical Solution of Partial Differential Equations, Cambridge University Press, London, 1994.