Multiloop correlators for two-dimensional quantum gravity

J. Ambjørn and J. Jurkiewicz [1]
Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

and

Yu.M. Makeenko
Institute for Theoretical and Experimental Physics, SU-117 259 Moscow, USSR

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We find explicitly all multi-loop correlators in the complex matrix model to the leading order in $1/N$ and show that they are identical to the even part of the multi-loop correlators in the hermitean matrix model. The scaling limit for the corresponding macroscopic loop correlators is constructed and agrees with the one of the hermitean model to all orders in $1/N^2$. In particular the double scaling limits of the two models will lead to identical "string equations".

1. Introduction

The theory of discretized random surfaces [1–3] has recently attracted much interest since it has been useful in attempts to perform the summation over all genera for non-critical strings [4–6]. In the case where the dimension d of the target space is zero it is also a model of pure two-dimensional gravity and it can be reduced to the problem of solving the $N\times N$ hermitean matrix model in the so called scaling limit [7]. A convenient tool in the study of the matrix model are the loop equations, or alternatively the Dyson–Schwinger equations. In the modern context these equations were considered in a seminal paper by Kazakov for genus zero [8]. In the present paper we generalize the technique of Kazakov and solve completely the $N\times N$ complex matrix model in the large N limit. By a complete solution we mean that all connected multi-loop correlators are calculated explicitly, even away from the scaling limit. The n-loop correlators turn out to be connected in a simple way to the n-point correlators. We construct explicitly the scaling limit of the loop correlators and relate them to the ones of the hermitean matrix model.

[1] Permanent address: Institute of Physics, Jagellonian University, ul. Reymonta 4, PL-30-059 Cracow 16, Poland.

2. The Dyson–Schwinger equations

Let us consider the Dyson–Schwinger equations (DSE's) for the hermitean matrix model and for the complex matrix model. In the first case we take a potential of the form

$$V_{\mathrm{H}}(\varphi)=\sum_{j=1}^{\infty}\frac{g_j^{\mathrm{H}}}{j}\operatorname{tr}\varphi^j\,, \tag{1}$$

while in the second case

$$V_{\mathrm{C}}(\varphi^\dagger\varphi)=\sum_{j=1}^{\infty}\frac{g_j^{\mathrm{C}}}{j}\operatorname{tr}(\varphi^\dagger\varphi)^j\,. \tag{2}$$

In these generalised potentials most g_j's are to be considered as sources for operators $\operatorname{tr}\varphi^j$ etc. and should be put to zero after relations between different correlators have been derived, leaving us with the usual potentials involving only finite sums in (1) and (2).

Expectation values of functions $F_{\mathrm{H}}(\varphi)$ and $F_{\mathrm{C}}(\varphi^\dagger\varphi)$ are defined by

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$$\langle F_H(\varphi)\rangle = \frac{\int d\varphi\, F_H(\varphi)\exp[-NV_H(\varphi)]}{\int d\varphi \exp[-NV_H(\varphi)]}, \tag{3}$$

and

$$\langle F_C(\varphi^\dagger\varphi)\rangle = \frac{\int d\varphi\, d\varphi^\dagger\, F_C(\varphi^\dagger\varphi)\exp[-NV_C(\varphi^\dagger\varphi)]}{\int d\varphi\, d\varphi^\dagger \exp[-NV_C(\varphi^\dagger\varphi)]}. \tag{4}$$

A convenient way to derive the DSE's is to consider in the two cases the field redefinitions [#1]

$$\text{hermitean matrices: } \varphi\to\varphi+\epsilon\varphi^{n+1} \tag{5}$$

$$\text{complex matrices: } \varphi\to\varphi+\epsilon\varphi(\varphi^\dagger\varphi)^n \quad \varphi^\dagger\to\varphi^\dagger. \tag{6}$$

The changes in the measures and the actions are

$$\delta(d\varphi)=\epsilon\, d\varphi \sum_{k=0}^{n} \operatorname{tr}\varphi^k \operatorname{tr}\varphi^{n-k},$$

$$\delta V_H(\varphi)=\epsilon \sum_{j=1}^{\infty} g_j^H \operatorname{tr}\varphi^{j+n}, \tag{7}$$

$$\delta(d\varphi\, d\varphi^\dagger)=\epsilon\, d\varphi\, d\varphi^\dagger \sum_{k=0}^{n} \operatorname{tr}(\varphi^\dagger\varphi)^k \operatorname{tr}(\varphi^\dagger\varphi)^{n-k},$$

$$\delta V_C(\varphi^\dagger\varphi)=\epsilon \sum_{j=1}^{\infty} g_j^C \operatorname{tr}(\varphi^\dagger\varphi)^{j+n}, \tag{8}$$

for hermitean and complex matrices respectively.

The invariance of the partition functions

$$Z_H([g])=\int d\varphi \exp[-NV_H(\varphi)],$$

$$Z_C([g])=\int d\varphi\, d\varphi^\dagger \exp[-NV_C(\varphi^\dagger\varphi)] \tag{9}$$

under field redefinitions leads to the following equations for the two models:

$$\left\langle \sum_{k=0}^{n} \operatorname{tr}\varphi^k \operatorname{tr}\varphi^{n-k}\right\rangle = N\left\langle \sum_{j=1}^{\infty} g_j^H \operatorname{tr}\varphi^{n+j}\right\rangle \tag{10}$$

$$\left\langle \sum_{k=0}^{n} \operatorname{tr}(\varphi^\dagger\varphi)^k \operatorname{tr}(\varphi^\dagger\varphi)^{n-k}\right\rangle = N\left\langle \sum_{j=1}^{\infty} g_j^C \operatorname{tr}(\varphi^\dagger\varphi)^{n+j}\right\rangle. \tag{11}$$

[#1] According to our knowledge the first derivation of the DSE for the hermitean matrix model by this method is due to Matsuo [9].

In the hermitean case eq. (10) is valid also for $n=-1$, where the left-hand side becomes zero. Eqs. (10) and (11) are the Dyson–Schwinger equations. We can write them as differential operators in the coupling constants, acting on the partition function

$$L_n Z_{H,C}([g])=0,$$
$$n_H=-1, 0, 1, \ldots, \text{ and } n_C=0, 1, \ldots, \tag{12}$$

where the differential operators are given by

$$L_n=\sum_{k=1}^{\infty}\frac{k(n-k)}{N^2}\frac{\partial^2}{\partial g_k\,\partial g_{n-k}}-\frac{2n}{N}\frac{\partial}{\partial g_n}+\sum_{k=1}^{\infty}(k+n)g_k\frac{\partial}{\partial g_{n+k}}+\delta_{-1,n}Ng_1+\delta_{0,n}N^2. \tag{13}$$

In (13) derivatives with respect to g_j's with $j\leqslant 0$ are understood to be omitted. In view of the nature of the field redefinitions (7) it is not surprising that the L_n's satisfy the commutation relations for a Virasoro algebra as was first observed in ref. [10]. What is maybe more surprising is that the L_n's have exactly the same form for the complex and the hermitean matrix model for $n\geqslant 0$.

We now introduce the following generating functionals for the hermitean and complex matrix models:

$$\chi_H(p)=\frac{1}{N}\sum_{k=0}^{\infty}\frac{\langle \operatorname{tr}\varphi^k\rangle}{p^{k+1}}, \tag{14}$$

$$\chi_C(p)=\frac{1}{N}\sum_{k=0}^{\infty}\frac{\langle \operatorname{tr}(\varphi^\dagger\varphi)^k\rangle}{p^{2k+1}}. \tag{15}$$

These quantities are closely related in the $N\to\infty$ limit to the so called spectral density functions [11]. For a completely general potential, systems described by the one-matrix models can be in many different phases [12]. In the present paper we shall be interested only in the standard perturbative phase, the only one which is connected with the random surface models in $d=0$. The spectral density functions describe the distributions of eigenvalues of φ in the hermitean model ($\sqrt{\varphi^\dagger\varphi}$ in the complex matrix model) in the limit $N\to\infty$. These functions have support on one arc $[x, y]$ in the hermitean model ($[0, \sqrt{z}]$ in the complex model). Consequently we shall assume that non-zero parameters g_i of the potentials admit the existence of the perturbative phase.

To the leading order in $1/N$ we can factorize the product of traces in (10) and (11) and we get the following solutions for $\chi(p)$ corresponding to the perturbative phase:

$$\chi_{\rm H}(p)=\tfrac{1}{2}[V'_{\rm H}(p)-M_{\rm H}(p)\sqrt{(p-x)(P-y)}], \quad (16)$$

$$\chi_{\rm C}(p)=\tfrac{1}{2}[V'_{\rm C}(p)-M_{\rm C}(p)\sqrt{p^2-z}], \quad (17)$$

where

$$V'_{\rm H}(p)=\sum_{j=1}^{\infty} g_j^{\rm H} p^{j-1}, \quad V'_{\rm C}(p)=\sum_{j=1}^{\infty} g_j^{\rm C} p^{2j-1}. \quad (18)$$

The constants x, y, z and the functions $M_{\rm H,C}(p)$ are uniquely determined by the requirement that for large $|p|$ $\chi(p)$ starts with the term $1/p$ and that $M(p)$ contains no negative powers of p. Using standard methods [11] we can represent $\chi(p)$ by the integrals

$$\chi_{\rm H}(p)=\frac{1}{2\pi {\rm i}}\sqrt{(p-x)(p-y)} \times \int_x^y \frac{V'_{\rm H}(q)\,{\rm d}q}{(q-p)\sqrt{(q-x)(q-y)}}, \quad (19)$$

$$\chi_{\rm C}(p)=\frac{1}{\pi {\rm i}}\sqrt{p^2-z}\int_0^{\sqrt{z}} \frac{qV'_{\rm C}(q)\,{\rm d}q}{(q^2-p^2)\sqrt{q^2-z}}, \quad (20)$$

where x, y are implicitly defined by the integrals

$$Q_{\rm H}(x,y,[g])=-\frac{1}{\pi {\rm i}}\int_x^y \frac{V'_{\rm H}(q)\,{\rm d}q}{\sqrt{(q-x)(q-y)}}=0, \quad (21)$$

$$W_{\rm H}(a,b,[g])=-\frac{1}{\pi {\rm i}}\int_x^y \frac{qV'_{\rm H}(q)\,{\rm d}q}{\sqrt{(q-x)(q-y)}}=2, \quad (22)$$

while z for the complex matrix model is defined by

$$W_{\rm C}(z,[g])=-\frac{2}{\pi {\rm i}}\int_0^{\sqrt{z}} \frac{qV'_{\rm C}(q)\,{\rm d}q}{\sqrt{q^2-z}}=2. \quad (23)$$

Eqs. (19) and (20) can be expanded in power series in x, y or z and $1/p$, an expansion which can also be obtained by the method of Kazakov [8,13]. Here let us give explicit expressions only in the case of the complex matrix model

$$M_{\rm C}(p,z[g])=\sum_{j=1}^{\infty} p^{2j-2}\sum_{k=0}^{\infty} c_k g_{k+j}^{\rm C} z^k, \quad (24)$$

$$W_{\rm C}(z,[g])\equiv\sum_{j=1}^{\infty} c_j g_j^{\rm C} z^j, \quad (25)$$

where the c_j are the coefficients in the power series expansion of $1/\sqrt{1-x}$:

$$c_j=\frac{(2j)!}{4^j(j!)^2}. \quad (26)$$

We note that $W(z)$ plays the role of the cosmological constant. The function $W_{\rm C}(z,[g])$ defined by (23) has a simple scaling property:

$$W_{\rm C}(z,[\lambda g])=\lambda W_{\rm C}(z,[g]). \quad (27)$$

if we scale all coupling constants with a factor λ we get the following equation for $z(\lambda)$:

$$\lambda\frac{{\rm d}W_{\rm C}(z,[g])}{{\rm d}z}\frac{{\rm d}z}{{\rm d}\lambda}+W_{\rm C}(z,[g])=0. \quad (28)$$

For $\lambda=1$ it gives

$$W'_{\rm C}(z)\,{\rm d}z=-2\,{\rm d}\lambda. \quad (29)$$

Since $\chi_{\rm H,C}(p)$ as defined by (14) and (15) are in fact functionals of arbitrary potentials $V_{\rm H,C}(p,[g])$ we can use them to extract information about all connected correlators. In the next section we use this observation to solve completely the part of the complex matrix model model which involves connected correlators of the form

$$\langle {\rm tr}(\varphi^\dagger\varphi)^{n_1}(\varphi^\dagger\varphi)^{n_2}...(\varphi^\dagger\varphi)^{n_k}\rangle_{\rm c}.$$

3. Solution of the complex matrix model

Let us now introduce the generating functional for the connected Green functions in the complex matrix model:

$$\chi(p_1,...,p_n)=\sum_{k_1,...,k_n=1}^{\infty}\frac{\chi_{k_1,...,k_n}}{p_1^{2k_1+1}...p_n^{2k_n+1}}, \quad (30)$$

where the coefficients are the connected n-point functions

$$\chi_{k_1,...,k_n}=N^{n-2}\langle {\rm tr}(\varphi^\dagger\varphi)^{k_1}...{\rm tr}(\varphi^\dagger\varphi)^{k_n}\rangle_{\rm c}. \quad (31)$$

We will denote $\chi(p_1,...,p_n)$ as the n-loop correlator.

We can get the connected n-point function by differentiating the connected $(n-1)$-point function:

$$\chi_{k_1,...,k_{n-1},k_n}=-k_n\frac{\partial}{\partial g_{k_n}^{\rm C}}\chi_{k_1,...,k_{n-1}}, \quad (32)$$

and this means that in a similar way we can get the n-loop correlator by combining (30) and (32):

$$\chi(p_1,...,p_n)=-\sum_{j=1}^{\infty}\frac{j}{p_n^{2j+1}}\frac{\mathrm{d}}{\mathrm{d}g_j^{\mathrm{C}}}\chi(p_1,...,p_{n-1})\,. \tag{33}$$

At this point it is convenient to introduce the loop insertion operator $-\mathrm{d}/\mathrm{d}g(p)$, where

$$\frac{\mathrm{d}}{\mathrm{d}g(p)}\equiv\sum_{j=1}^{\infty}\frac{j}{p^{2j+1}}\frac{\mathrm{d}}{\mathrm{d}g_j^{\mathrm{C}}}\,. \tag{34}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}g_j^{\mathrm{C}}}=\frac{\partial}{\partial g_j^{\mathrm{C}}}+\frac{\partial z}{\partial g_j^{\mathrm{C}}}\frac{\partial}{\partial z}=\frac{\partial}{\partial g_j^{\mathrm{C}}}-\frac{c_j z^j}{W'(z)}\frac{\partial}{\partial z} \tag{35}$$

we have the following decomposition of $\mathrm{d}/\mathrm{d}g(p)$:

$$\frac{\mathrm{d}}{\mathrm{d}g(p)}=\frac{\partial}{\partial g(p)}-f(p,z)\Omega\left(z,\frac{\partial}{\partial z}\right), \tag{36}$$

where we have introduced the notation

$$\frac{\partial}{\partial g(p)}\equiv\sum_{j=1}^{\infty}\frac{j}{p_n^{2j+1}}\frac{\partial}{\partial g_j^{\mathrm{C}}}\,,\quad \Omega\left(z,\frac{\partial}{\partial z}\right)\equiv\frac{1}{W'(z)}\frac{\partial}{\partial z}\,,$$

$$f(p,z)\equiv\frac{z}{2(p^2-z)^{3/2}}\,. \tag{37}$$

We can now obtain the connected two-loop correlator simply by applying $-\mathrm{d}/\mathrm{d}g(p)$ to (17):

$$\chi(q,p)=-\frac{\mathrm{d}}{\mathrm{d}g(p)}\chi(q)\,. \tag{38}$$

In the process of applying $-\mathrm{d}/\mathrm{d}g(p)$ to $\chi(q)$ it is convenient to note the following relations which follow from the definitions:

$$W'(z)=\tfrac{1}{2}M(p)-(p^2-z)\frac{\partial M(p)}{\partial z}\,, \tag{39}$$

$$\frac{\partial}{\partial g(p)}g(q)=-\frac{pq}{(p^2-q^2)^2}\,, \tag{40}$$

$$\frac{\partial}{\partial g(p)}M(q)=-\frac{1}{2}\frac{\partial}{\partial p}\frac{p}{(p^2-q^2)\sqrt{p^2-z}}\,, \tag{41}$$

$$\frac{\partial}{\partial g(p)}W(z)=f(p,z)\,. \tag{42}$$

From these relations it is a matter of algebra to derive

$$\chi(p,q)=\frac{1}{2(p^2-q^2)^2}\left(\frac{p^2q^2-\frac{1}{2}(p^2+q^2)z}{\sqrt{p^2-z}\sqrt{q^2-z}}-pq\right), \tag{43}$$

and as in ref. [13] we note that the two-loop correlator is universal. It contains no explicit reference to the coupling constants g_j. It appears only implicitly through the variable z and all multicritical models have the same two-loop function.

Further we can write

$$\chi(r,q,p)=-\frac{\mathrm{d}}{\mathrm{d}g(p)}\chi(r,q)\,, \tag{44}$$

and since $\chi(r,q)$ has no explicit dependence of g_j one gets easily

$$\chi(r,q,p)=\frac{1}{W'(z)}\frac{z^2}{16(\sqrt{p^2-z}\sqrt{q^2-z}\sqrt{r^2-z})^3}\,. \tag{45}$$

We will now show that the general formula for the multi-loop function $\chi(p_1,...,p_{n+3})$ is given by

$$\chi(p_1,...,p_{n+3})=\left(\frac{1}{W'(z)}\frac{\partial}{\partial z}\right)^n\frac{1}{2zW'(z)}\prod_{k=1}^{n+3}\frac{z}{2(p_k^2-z)^{3/2}}\,. \tag{46}$$

The proof is a consequence of the following lemma:

Lemma. Let $h(z)$ be a function of z only. We have the following:

$$-\frac{\mathrm{d}}{\mathrm{d}g(p)}\Omega^n\frac{1}{W'(z)}h(z)=\Omega^{n+1}\frac{1}{W'(z)}h(z)f(p,z)\,. \tag{47}$$

Proof. First we note that

$$-\frac{\partial}{\partial g(p)}\frac{1}{W'(z,[g])}=-\frac{1}{W'(z)^2}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial g(p)}W(z,[g])\right)=\frac{1}{W'(z)^2}\frac{\partial}{\partial z}f(p,z)\,. \tag{48}$$

It follows that

$$\left[-\frac{\partial}{\partial g(p)}, \Omega\right]=\Omega f(p,z)\Omega-f(p,z)\Omega^2, \tag{49}$$

and by induction that

$$\left[-\frac{\partial}{\partial g(p)}, \Omega^n\right]=\Omega^n f(p,z)\Omega-f(p,z)\Omega^{n+1}, \tag{50}$$

and therefore

$$\left[-\frac{\partial}{\partial g(p)}, \Omega^n\frac{1}{W'(z)}\right]$$
$$=\Omega^{n+1}\frac{1}{W'(z)}f(p,z)-f(p,z)\Omega^{n+1}\frac{1}{W'(z)}, \tag{51}$$

and finally, if $h(z)$ denotes a function with *no* explicit dependence on $[g]$, we have from the decomposition (36)

$$-\frac{\mathrm{d}}{\mathrm{d}g(p)}\left(\Omega^n\frac{1}{W'(z)}\right)h(z)$$
$$=\left(\Omega^{n+1}\frac{1}{W'(z)}\right)f(p,z)h(z), \tag{52}$$

which proves (46).

As already noted W plays the role of the cosmological constant. With this interpretation the differentiation

$$\left(\frac{\mathrm{d}}{W'(z)\,\mathrm{d}z}\right)^n=\frac{\mathrm{d}^n}{\mathrm{d}W^n} \tag{53}$$

is just the differentiation with respect to the cosmological constant. It is remarkable that one gets the complete n-loop correlator by such a simple differentiation.

We can directly take the scaling limit of (46) for any multicritical model. The mth multicritical point is defined [8] by fine tuning the coupling constants g_j such that

$$W^{(k)}(z_c)=0, \quad k=1,\dots,m-1,$$
$$W^{(m)}(z_c)\neq 0. \tag{54}$$

Let us discuss in some detail the scaling limit in the simplest (and probably most interesting) case: $m=2$. In this case for z close to z_c

$$W'(z,[g])=-2g(z-z_c)+\dots, \tag{55}$$

where g is a non-universal positive constant. The scaling limit and the scaling parameter a are now introduced in the standard way [13,14] as

$$p_i^2=z_c+a\pi_i \tag{56}$$

$$z=z_c-a\sqrt{\Lambda}. \tag{57}$$

It follows that

$$\frac{1}{W'(z)}\frac{\mathrm{d}}{\mathrm{d}z}=\frac{-1}{ga^2}\frac{\mathrm{d}}{\mathrm{d}\Lambda}, \tag{58}$$

and to leading order in the scaling parameter a we get (for $n\geqslant 3$, for $n=1$, 2 the formulas can be found in ref. [13])

$$\chi(\pi_1,\dots,\pi_n;\Lambda)$$
$$=\frac{C_n}{a^{7n/2-5}}\frac{\mathrm{d}^{n-3}}{\mathrm{d}\Lambda^{n-3}}\left(\frac{1}{\sqrt{\Lambda}}\prod_{i=1}^{n}\frac{1}{(\pi_i+\sqrt{\Lambda})^{3/2}}\right), \tag{59}$$

where C_n is a non-universal constant which with the choice (55) is given by

$$C_n=(-1)^{n-1}\left(\frac{g^2}{4z_c}\right)\left(\frac{z_c}{2g}\right)^n. \tag{60}$$

Λ is the renormalized cosmological constant and the limit $\pi_i\to\infty$ corresponds to n times differentiation of the free energy, which goes as $\Lambda^{2-\gamma}$, where the critical exponent $\gamma=-\frac{1}{2}$ for $m=2$. This agrees with (59).

Formula (59) can readily be generalized to multicritical models where $m>2$. Let us only mention here that $\chi(p_1,\dots,p_n)$ is "renormalized" as follows:

$$\chi(\pi_1,\dots,\pi_n;\Lambda)_{(\mathrm{R})}=a^{(m+3/2)n-2m-1}\chi(p_1,\dots,p_n), \tag{61}$$

in accordance with (59).

4. The hermitean matrix model

We can introduce the generating functional for observables like

$$\chi_{k_1,\dots,k_n}\equiv N^{n-2}\langle\mathrm{tr}\,\varphi^{k_1}\dots\mathrm{tr}\,\varphi^{k_n}\rangle_c \tag{62}$$

in the same way as for the complex matrix model and the multiloop correlators

$$\chi(p_1,\dots,p_n)=\sum_{k_1,\dots,k_n=1}^{\infty}\frac{\chi_{k_1,\dots,k_n}}{p_1^{k_1+1}\dots p_n^{k_n+1}}, \tag{63}$$

the only difference being that we have also odd powers of p_i appearing in the expression for $p_1 ... p_n \chi(p_1, ..., p_n)$. It is convenient to split $p_1 ... p_n \chi(p_1, ..., p_n)$ into an even part, having only even powers of any of the p_i's, and the rest, which we somewhat misleadingly denote as the odd part. The odd part contains at least one odd power of one of the p_i's:

$$\chi(p_1, ..., p_n) = \chi^{\mathrm{E}}(p_1, ..., p_n) + \chi^{\mathrm{O}}(p_1, ..., p_n) , \tag{64}$$

where

$$p_1 ... p_n \chi^{\mathrm{E}}(p_1, ..., p_n) = \sum_{k_1, ..., k_n} \frac{\chi_{2k_1, ..., 2k_n}}{p_1^{2k_1} ... p_n^{2k_n}} . \tag{65}$$

We can apply the method used in the former section to the hermitean model. We get the multiloop correlators by applying the loop insertion operator $-\mathrm{d}/\mathrm{d}g(p)$ an appropriate number of times to $\chi(p)$. If we consider the case where we perturb around an even potential like

$$V_0^{\mathrm{E}}(\varphi) = \sum_{k=1}^{\infty} \frac{g_{2k}^{\mathrm{H}}}{2k} \operatorname{tr} \varphi^{2k} , \tag{66}$$

we have that $y = -x = \sqrt{z}$ and

$$\chi_{\mathrm{H}}^{\mathrm{E}}(p) = \chi_{\mathrm{C}}(p) . \tag{67}$$

As long as we only want to bring down even powers $\operatorname{tr} \varphi^{2n}$ we can act with the even part of the loop insertion operator

$$\frac{\mathrm{d}}{\mathrm{d}g_{\mathrm{E}}(p)} \equiv \sum_{k=1}^{\infty} \frac{2k}{p^{2k+1}} \frac{\mathrm{d}}{\mathrm{d}g_{2k}^{\mathrm{H}}} , \tag{68}$$

and everything becomes identical to the derivation for the complex matrix model, except for a factor two caused by the replacement $g_k^{\mathrm{C}} \operatorname{tr}(\varphi^\dagger \varphi)^k / k$ with $g_{2k}^{\mathrm{H}} \operatorname{tr}(\varphi)^{2k}/2k$. We keep the non-zero values of $g_{2k}^{\mathrm{H}} = g_k^{\mathrm{C}}$ the same in the two cases. This means that also the non-universal constants z_c and g remain the same and

$$\chi^{\mathrm{E}}(p_1, ..., p_n) = 2^{n-1} \chi_{\mathrm{C}}(p_1, ..., p_n) . \tag{69}$$

In the general case the formulas become somewhat complicated away from the critical point and we will only give here explicitly the first few loop correlators. Let us introduce the notation

$$p_x = p - x , \quad p_y = p - y , \quad R(p) = p_x p_y . \tag{70}$$

The loop insertion operator can then be written as

$$-\frac{\mathrm{d}}{\mathrm{d}g(p)} = -\frac{\partial}{\partial g(p)} + \frac{p_y}{M(x)[R(p)]^{3/2}} \frac{\partial}{\partial x} + \frac{p_x}{M(y)[R(p)]^{3/2}} \frac{\partial}{\partial y} . \tag{71}$$

The n-loop correlator is now given by

$$\chi(p_1, ..., p_n) = (-1)^{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}g(p_2) ... \mathrm{d}g(p_n)} \chi(p_1) . \tag{72}$$

Let us here give the first few multiloop correlators explicitly:

$$\chi(p) = \tfrac{1}{2}[V'(p) - M(p)\sqrt{R(p)}] , \tag{73}$$

$$\chi(p, q) = \frac{1}{2(p-q)^2} \left(\frac{pq - \frac{1}{2}(p+q)(x+y) + xy}{\sqrt{R(p)R(q)}} - 1 \right) , \tag{74}$$

$$\chi(p, q, r) = \frac{y-x}{8[R(p)R(q)R(r)]^{3/2}} \left(\frac{p_x q_x r_x}{M(y)} - \frac{p_y q_y r_y}{M(x)} \right) . \tag{75}$$

For the general asymmetric case it seems difficult to organize the expression in the same nice way as for the complex matrix model. However, in the scaling limit everything simplifies drastically. In order to make an easier contact with the complex matrix model we will assume that we are expanding around an even potential of the form (66), and that we have the $m=2$ multicritical case. First we note that for an even potential we have

$$y = -x = \sqrt{z} , \tag{76}$$

$$M(x) = M(y) = 2W'(z) , \tag{77}$$

$$\frac{\partial}{\partial g(p)} M(x) = -\frac{\partial}{\partial p} \frac{p_y}{[R(p)]^{3/2}} , \tag{78}$$

$$\frac{\partial}{\partial g(p)} M(y) = -\frac{\partial}{\partial p} \frac{p_x}{[R(p)]^{3/2}} , \tag{79}$$

and therefore we can write the three-point function as

$$\chi(p, q, r) = \frac{1}{4W'(z)} \frac{z^2 + z(pq + pr + qr)}{[(p^2 - z)(q^2 - z)(r^2 - z)]^{3/2}} . \tag{80}$$

the scaling limit is obtained when

$$y=\sqrt{z_c}-\frac{a}{2}\frac{\sqrt{\Lambda_1}}{\sqrt{z_c}}, \quad x=-\sqrt{z_c}+\frac{a}{2}\frac{\sqrt{\Lambda_2}}{\sqrt{z_c}},$$

$$p_i=\sqrt{z_c}+\frac{a}{2}\frac{\pi_i}{\sqrt{z_c}}. \tag{81}$$

In (81) conventions are chosen to match (56) and (57). We note that in this limit

$$p_{iy}=\frac{a}{2\sqrt{z_c}}(\pi_i+\sqrt{\Lambda_1}) \tag{82}$$

while

$$p_{ix}=2\sqrt{z_c}+... \tag{83}$$

remains finite. It is easy to convince oneself that in the scaling limit we can drop the dependence of correlators on $M(x)$ and Λ_2 since this dependence will give only the subleading terms. In effect we get for $\chi(\pi_1, ..., \pi_n)$ the same result as for $\chi^E(\pi_1, ..., \pi_n)$ except for a factor 2^{n-1} and Λ_1 replacing Λ. We therefore get that in the scaling limit for $n>2$

$$\chi(\pi_1, ..., \pi_n; \Lambda)=\tfrac{1}{2}\cdot 2^n\chi^E(\pi_1, ..., \pi_n; \Lambda)$$
$$=\tfrac{1}{4}\cdot 4^n\chi_C(\pi_1, ..., \pi_n; \Lambda). \tag{84}$$

We notice that we can construct a sensible scaling limit also for negative values of p_i:

$$p_i=-\sqrt{z_c}+\frac{a}{2}\frac{\pi_i}{\sqrt{z_c}}. \tag{85}$$

In this case the role of the endpoints x, y is exchanged or Λ_1 is replaced by Λ_2. Otherwise the resulting expression is unchanged.

We can easily extend the above arguments to include the case of the general potential. In this case relation (77) is no longer valid. However at the critical point either $M(x)$ or $M(y)$ vanish. Let us assume that it is $M(y)$ and that for y close to y_c

$$M(y)=-8g\sqrt{z_c}\,(y-y_c)+..., \tag{86}$$

where

$$z_c=y_c-x_c. \tag{87}$$

The scaling limit in this case is

$$y=y_c-\frac{a}{2}\frac{\sqrt{\Lambda_1}}{\sqrt{z_c}}, \quad p_i=y_c+\frac{a}{2}\frac{\pi_i}{\sqrt{z_c}},$$

$$x=x_c+.... \tag{88}$$

With these conventions we again reproduce exactly the result for the symmetric potential.

5. The double scaling limit

The relation (84) can easily be extended to higher orders in $1/N^2$ by means of the DSE's. The general DSE's can be written both away from the scaling limit and when the scaling limit is taken for each term in the $1/N$ expansion [13,14,10]. It is this last limit which has our interest. Rather than performing the general analysis, which is straightforward, but cumbersome (mainly because of notation), let us illustrate the iteration of the DSE's for $\chi(p)$. First we introduce the $1/N^2$ expansion:

$$\chi(p_1, ..., p_n)$$
$$=\chi^{(0)}(p_1, ..., p_n)+\frac{1}{N^2}\chi^{(1)}(p_1, ..., p_n)+.... \tag{89}$$

The first equations for $\chi(p)$ can in the case of an $m=2$ multicritical model be written as (see refs. [14,13])

$$\chi^{(0)}(p)=\tfrac{1}{2}[V'(p)-M(p)\sqrt{p^2-z}], \tag{90}$$

$$\chi^{(1)}(p)=\frac{\chi^{(0)}(p,p)-\chi^{(0)}(p_0,p_0)}{M(p)\sqrt{p^2-z}}, \tag{91}$$

where p_0 is the zero of $M(p)$. $p_0^2>z$ and analyticity requires that this pole has to be cancelled. In (91) this is seen explicitly to take place. As p_0 also scales we get from (84) that

$$\chi_H^{(1)}(\pi; \Lambda)=4\chi_C^{(1)}(\pi; \Lambda). \tag{92}$$

By iteration of the DSE's we get

$$\chi_H^{(k)}(\pi_1, ..., \pi_n, \Lambda)=4^{k+n-1}\chi_C^{(k)}(\pi_1, ..., \pi_n, \Lambda), \tag{93}$$

and the factor 4^{k+n-1} is just a reflection of the fact that the two models in the scaling limit differ by a factor four for each power of $1/N^2$ extracted from

$$\left\langle\frac{1}{N}\operatorname{tr}\phi^{n_1}...\frac{1}{N}\operatorname{tr}\phi^{n_k}\right\rangle.$$

We can now write down the renormalized macroscopic multiloop correlators introducing the scaling relations (61). If for the mth multicritical model we denote the double scaling expansion parameter by

$$\kappa^2 = \frac{1}{N^2 a^{2m+1}}, \tag{94}$$

we have the following expansion [9] in this limit:

$$\chi(\pi_1, ..., \pi_n)_{(R)} = \chi^{(0)}(\pi_1, ..., \pi_n)_{(R)} + ... + \kappa^{2k}\chi^{(k)}(\pi_1, ..., \pi_n)_{(R)} + ... \tag{95}$$

The relations between the full renormalized multiloop correlators of the two models can finally be written as

$$\chi_H(\pi_1, ..., \pi_n; \Lambda; \kappa^2)_{(R)} = 4^{n-1}\chi_C(\pi_1, ..., \pi_n; \Lambda; 4\kappa^2)_{(R)} . \tag{96}$$

We end with a few remarks. Since the explicit formulas for the multiloop correlators (59) are in the scaling limit very simple to the leading order in the $1/N^2$ and the DSE's in this limit are not that complicated, it seems possible to find the explicit formulas to any order in $1/N^2$. In the case of the complex matrix model it might even by possible to do it away from the scaling limit since also here we have simple algebraic formulas for the multiloop correlators given by (46). It might have some interest to investigate this further, since it is possible to go from the complex matrix model to the unitary matrix model by a proper adjustment of the coupling constants. It is known that the unitary models lead to different "string" equations in the scaling limit [15]. From ref. [16] it is known that the unitary limit corresponds to a different critical point and to a choice of spectral density different from the usual "perturbative" one. The possible choices of spectral densities, and the corresponding different phases have been analysed in detail in the case of the hermitean model [14,17,18] and the methods used there might give a simple relation between the Painlevé I "string" equations of the complex matrix model and the Painlevé II "string" equations of the unitary matrix models.

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References

[1] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B 257[FS14] (1985) 433.
[2] F. David, Nucl. Phys. B 257[FS14] (1985) 543.
[3] V.A. Kazakov, I.K. Kostov and A.A. Migdal, Phys. Lett. B 157 (1985) 295.
[4] M.R. Douglas and S.H. Shenker, preprint RU-89-34 (October 1989).
[5] E. Brézin and V.A. Kazakov, Phys. Lett. B 236 (1990) 144.
[6] D.J. Gross and A.A. Migdal, Phys. Rev. Lett. 643 (1990) 127.
[7] F. David, Nucl. Phys. B 257[FS14] (1985) 45.
[8] V.A. Kazakov, Mod. Phys. Lett. A 4 (1989) 2125.
[9] Y. Matsuo, unpublished.
[10] M. Fukuma, H. Kawai and R. Nakayama, Continuum Schwinger–Dyson equations and universal structures in two-dimensional quantum gravity, preprint KEK-90-27, UT-562.
[11] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, Commun. Math. Phys. 59 (1978) 35.
[12] J. Jurkiewicz, Phys. Lett. B 245 (1990) 178.
[13] J. Ambjørn and Yu.M. Makeenko, Properties of the loop equations for the hermitean matrix model and for two-dimensional quantum gravity, Niels Bohr Institute preprint NBI-HE-90-22, Mod. Phys. Lett., to appear.
[14] F. David, Loop equations and non-perturbative effects in two dimensional quantum gravity, Saclay preprint SPhT/90-43.
[15] V. Periwal and D. Shevitz, Phys. Rev. Lett. 64 (1990) 1326.
[16] E.M. Ilgenfritz, Yu.M. Makeenko and T.V. Shahbazyan, Phys. Lett. B 172 (1986) 81.
[17] F. David, Phases of the large N matrix model and non-perturbative effects in 2d gravity, Saclay preprint SPhT-90-090.
[18] G. Bhanot, G. Mantal and O. Narayan, Phys. Lett. B 251 (1990) 388.