On the Linear Control of Underactuated Nonlinear Systems Via Tangent Flatness and Active Disturbance Rejection Control: The Case of the Ball and Beam System

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In this paper, a systematic procedure for controller design is proposed for a class of nonlinear underactuated systems (UAS), which are non-feedback linearizable but exhibit a controllable (flat) tangent linearization around an equilibrium point. Linear extended state observer (LESO)-based active disturbance rejection control (ADRC) is shown to allow for trajectory tracking tasks involving significantly far excursions from the equilibrium point. This is due to local approximate estimation and compensation of the nonlinearities neglected by the linearization process. The approach is typically robust with respect to other endogenous and exogenous uncertainties and disturbances. The flatness of the tangent model provides a unique structural property that results in an advantageous low-order cascade decomposition of the LESO design, vastly improving the attenuation of noisy and peaking components found in the traditional full order, high gain, observer design. The popular ball and beam system (BBS) is taken as an application example. Experimental results show the effectiveness of the proposed approach in stabilization, as well as in perturbed trajectory tracking tasks. [DOI: 10.1115/1.4033313]

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1 Introduction

The analysis and control of UAS has been an active topic of research in recent years. Generally speaking, UAS are not feedback linearizable and in the multivariable case they may have an ill-defined vector relative degree [1]. Approximate feedback linearization allows singularity avoidance at the expense of some weaknesses. The flatness of the linearized system is shown to naturally induce a low-order cascade structure [2], which allows for a simpler (decoupled) and more efficient disturbance and state observer design. In this article, tangent flatness around an equilibrium point and ESO-based ADRC are merged into a systematic procedure for robust feedback controller design in a class of UAS which are non-feedback linearizable. A structural property, revealed by flatness, results in an advantageous low-order cascade decomposition of the LESO design. The BBS is taken as a prototypical application example. Experimental results show the effectiveness of the proposed approach in stabilization, as well as in perturbed trajectory tracking tasks. This article is structured as follows: In Sec. 2, some theoretical preliminaries are introduced. A methodology for controlling a class of nonlinear UAS is proposed in Sec. 3. The same section illustrates the procedure in a direct application to the BBS. Experimental results are provided in Sec. 4. Finally, some conclusions are found in Sec. 5.

2 Preliminary Concepts

2.1 Differentially Flat Systems. It is said that a nonlinear system of the form $\dot{x} = f(x,u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ is *differentially flat*, i.e., linearizable by an endogenous (static) feedback [3], if there exists an endogenous variable, f, having the following properties:

(i) Every system variable may be expressed as a function of f and a finite number of its time derivatives.
(ii) The variable f may be expressed as a function of the system state vector and a finite number of its time derivatives.
(iii) The variable f does not satisfy any differential equation by itself.

2.2 A Class of UAS. Consider the following model of an underactuated nonlinear mechanical system [4]:

$$\begin{aligned} \mathbf{M}_{11}(\mathbf{x})\ddot{x}_1 + \mathbf{M}_{12}(\mathbf{x})\ddot{x}_2 + C_1(x,\dot{x}) &= b(x)u \\ \mathbf{M}_{21}(\mathbf{x})\ddot{x}_1 + \mathbf{M}_{22}(\mathbf{x})\ddot{x}_2 + C_2(x,\dot{x}) &= 0 \end{aligned} \tag{1}$$

where $x^{\mathrm{T}} = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}) \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^{n-p}$, $x_2 \in \mathbb{R}^p$, $n, p \in \mathbb{N}$ are the states, $b(x) \in \mathbb{R}^{n-p}$ is the gain vector, and $u \in \mathbb{R}$ is the control input. The term p denotes the underactuated degrees-of-freedom. The functions $C_1 \in \mathbb{R}^{n-p}$ and $C_2 \in \mathbb{R}^p$ contain the centrifugal and gravity forces. The inertia matrix defined by

$$\mathbf{M}(\mathbf{x}) = \begin{bmatrix} \mathbf{M}_{11}(\mathbf{x}) & \mathbf{M}_{12}(\mathbf{x}) \\ \mathbf{M}_{21}(\mathbf{x}) & \mathbf{M}_{22}(\mathbf{x}) \end{bmatrix} \mathbb{R}^{n\times n}$$

with $\mathbf{M}_{11}(\mathbf{x}) \in \mathbb{R}^{n-p\times n-p}$, $\mathbf{M}_{12}(\mathbf{x}) \in \mathbb{R}^{n-p\times p}$, $\mathbf{M}_{21}(\mathbf{x}) \in \mathbb{R}^{p\times n-p}$, $\mathbf{M}_{22}(\mathbf{x}) \in \mathbb{R}^{p\times p}$ is positive definite for any x.

2.2.1 Basic Assumptions. It is assumed that the single input single output (SISO) UAS exhibits a controllable tangent linearization around a natural equilibrium point. The linearized system is assumed to be flat. It includes a lumped disturbance term, which considers internal and external disturbances, considered unknown but absolutely bounded with finite time-bounded derivatives, generated from the linearization process as well as external effects. Also, this term may contain deterministic bounded additive noises of unknown statistics from the measurement process. The ESO-based ADRC will take care of the neglected endogenous disturbances as well as external perturbations by direct estimation and cancellation. The flatness property in the tangent linearized system allows a systematic procedure in the robust flat output feedback controller design for these systems with the help of ESO-based ADRC. In ADRC schemes, the control problem can be approached in terms of estimating the *total disturbance* affecting the system, reducing the control task to the control of a perturbed chain of integrators. An additional advantage of flatness is revealed in a structural property exhibited in the differential parameterization of the system variables via the flat output. This property places in simple terms of measurable (position) variables an intermediate high-order time derivative of the flat output, thus reducing the natural measurement noise effects in the high-order injected flat output phase variables estimation error dynamics. In Sec. 3, a generalized procedure is applied to the popular BBS.

3 A Control Design Procedure Through a Case Study

Consider the well-known BBS [1], shown in Fig. 1, consisting of a ball placed on a beam, which undergoes an angular displacement around a certain pivot, actuated by a direct current (DC) motor directly coupled to the beam by means of a pulley. The dynamical model is represented as follows:

$$\left(m_2+\frac{I_2}{R^2}\right)\ddot{r} - m_2 r\dot{\theta}^2 + m_2 g\sin\theta = 0 \tag{2}$$

$$(m_2 r^2 + m_1 l_1^2 + I_1 + I_p)\ddot{\theta} + 2m_2 r\dot{r}\dot{\theta} + m_2 g r\cos\theta + m_1 g l_1 \sin\theta = N\tau \tag{3}$$

where $r \in \mathbb{R}$ denotes the position of the ball from the center of mass of the beam, $\theta \in \mathbb{R}$ is the angular position of the beam, $m_2 \in \mathbb{R}$ and $R \in \mathbb{R}$ denote, respectively, the ball mass and its radius. $I_2 = \frac{2}{5}m_2R^2 \in \mathbb{R}$ is the ball inertia, $I_1 \in \mathbb{R}$ denotes the beam inertia, $m_1 \in \mathbb{R}$ is the beam mass, and $I_p \in \mathbb{R}$ represents the inertia of the pulley system. $N \in \mathbb{R}$ is a ratio of distances concerning the pulley/motor actuator. The control input (motor torque) $\tau \in \mathbb{R}$ can be expressed as a function of the motor voltage through the approximate relation $\tau(t) = k_\tau V(t)/R_a$, where $k_\tau \in \mathbb{R}$ is the motor torque constant, and $R_a \in \mathbb{R}$ is the motor armature electric resistance.

3.1 Problem Formulation.

Devise a control law in order to manipulate the ball position from an initial value, $r(0)$, to a given final value $r(t_{\text{final}})$ in a finite, prescribed, interval of time $[0, t_{\text{final}}]$, though a smooth rest to rest trajectory.

Step 1: Obtain the Linearized System Around an Equilibrium Point. The system desired equilibrium point is described by: $\bar{r} = \bar{\dot{r}} = \bar{\theta} = \bar{\dot{\theta}} = \bar{V} = 0$. The approximate linearization of the systems (2) and (3) around the equilibrium point is given by

$$\left(m_2+\frac{I_2}{R^2}\right)\ddot{r}_\delta + m_2 g\theta_\delta = 0 \tag{4}$$

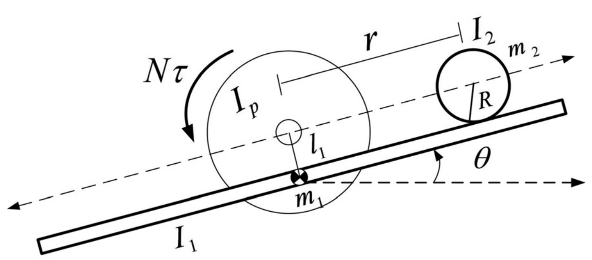

Fig. 1 A schematic of the BBS

$$(m_1 l_1^2 + I_1 + I_p)\ddot{\theta}_\delta + m_2 g r_\delta + m_1 g l_1 \theta_\delta = \frac{k_\tau N}{R_a} V_\delta \tag{5}$$

where $r_\delta = r - \bar{r} = r$, $\dot{r}_\delta = \dot{r} - \bar{\dot{r}} = \dot{r}$, $\theta_\delta = \theta - \bar{\theta} = \theta$, $\dot{\theta}_\delta = \dot{\theta} - \bar{\dot{\theta}} = \dot{\theta}$ and $V_\delta = V - \bar{V} = V$. An alternative writing for the linearized system is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}V_\delta \tag{6}$$

where $\mathbf{x} = \begin{bmatrix} r_\delta & \dot{r}_\delta & \theta_\delta & \dot{\theta}_\delta \end{bmatrix}^{\mathrm{T}}$,

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \dfrac{k_\tau N}{(m_1 l_1^2 + I_1 + I_p)R_a} \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{m_2 g}{m_2 + \dfrac{I_2}{R^2}} & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{m_2 g}{m_1 l_1^2 + I_1 + I_p} & 0 & -\dfrac{m_1 g l_1}{m_1 l_1^2 + I_1 + I_p} & 0 \end{bmatrix}$$

Step 2: Verify the Existence of the Flatness Property, and Obtain a Flat Output as Well as the Input-to-Flat Output Relationship. The linearized model (4) and (5) is controllable and hence, flat with controllability Matrix $\mathbf{K}_c = [B \quad AB \quad A^2B \quad A^3B]$. The incremental flat output can be computed as $f = [0 \quad 0 \quad 0 \quad 1]\mathbf{K}_c^{-1}\mathbf{x}$. We can choose the flat output as the incremental position of the ball, $r_\delta = (f/\varepsilon)$, with $\varepsilon = -(R_a(m_1 l_1^2 + I_1 + I_p)(m_2R^2 + I_2)/\alpha k_\tau N R^2 m_2)$. Define $\alpha = m_2 g/(m_2 + I_2/R^2) \in \mathbb{R}$. All system variables in the linear model BBS and the control input are expressible as functions of the flat output r_δ and a finite number of its time derivatives

$$\theta_\delta = -\frac{1}{\alpha}\ddot{r}_\delta, \quad \dot{\theta}_\delta = -\frac{1}{\alpha}r_\delta^{(3)} \tag{7}$$

$$V_\delta(t) = -\frac{(m_1 l_1^2 + I_1 + I_p)R_a}{\alpha k_\tau N} r_\delta^{(4)} - \frac{R_a m_1 g l_1}{\alpha k_\tau N}\ddot{r}_\delta + \frac{m_2 g r_\delta}{k_\tau N} \tag{8}$$

Step 3: Find the Cascade Form of the UAS. The incremental input-to-flat output relationship for the BBS (8) is represented by means of a fourth-order linear time-invariant system

$$r_\delta^{(4)} = -\frac{\alpha k_\tau N}{(m_1 l_1^2 + I_1 + I_p)R_a} V_\delta(t) - \frac{m_1 g l_1}{(m_1 l_1^2 + I_1 + I_p)}\ddot{r}_\delta + \frac{\alpha m_2 g}{(m_1 l_1^2 + I_1 + I_p)} r_\delta$$

Notice that the linearized system naturally decomposes into a cascade connection of two independent blocks; the first one controlled by the input voltage $V_\delta(t)$, with the corresponding output given by the flat output incremental acceleration $\ddot{r}_\delta = \ddot{r}$. Thus, the linearized acceleration of the ball can be expressed in terms of the angular position of the beam through the relation: $\ddot{r}_\delta = -\alpha\theta_\delta$. The signal $\alpha\theta_\delta$ acts as an auxiliary input to the second block, which consists of an elementary chain of two integrators rendering, simple enough, the estimation of the phase variables $\dot{r}_\delta$ and r_δ (see Fig. 2). This fundamental cascade property simplifies and decouples the ESO design task in the ADRC scheme.

Step 4: Implement the Linear ADRC for the Linearized System. To apply the ADRC [5], it is necessary to express the system as an additively disturbed chain of integrators. The linearized system can be expressed as follows:

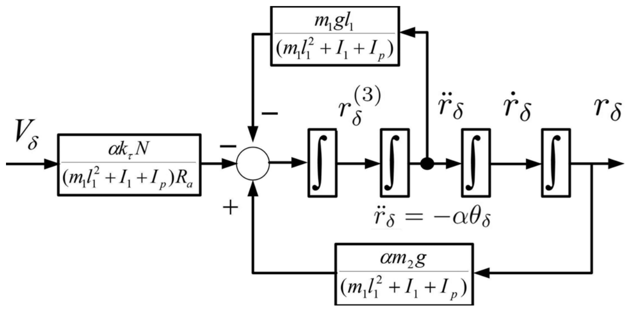

Fig. 2 Cascade structure of BBS

$$r_\delta^{(4)} = -\frac{\alpha k_\tau N}{(m_1 l_1^2 + I_1 + I_p) R_a} V_\delta(t) + \frac{\alpha m_2 g}{(m_1 l_1^2 + I_1 + I_p)} r_\delta - \frac{m_1 g l_1}{(m_1 l_1^2 + I_1 + I_p)} \ddot{r}_\delta + \mathbf{H.O.T.} + \nu(t)$$

where **H.O.T.** stands for high-order terms and $\nu(t)$ includes the nonmodeled dynamics, external disturbances, and/or deterministic additive noises, according to the basic assumptions state before. The flat output trajectory tracking error is defined as $e_{r_\delta} := r_\delta - r^*(t) = r - \bar{r} - (r^*(t) - \bar{r})$; this last term coincides with the tracking error $r - r^*(t)$. The key step in flatness-based ADRC schemes [2] is to treat the tracking error dynamics as the following simplified perturbed model:

$$e_{r_\delta}^{(4)} = -\frac{\alpha k_\tau N}{(m_1 l_1^2 + I_1 + I_p) R_a} V_\delta(t) + \xi(t)$$

where $\xi(t) := \mathbf{H.O.T.} + \nu(t)$ is the lumped disturbance input and $V_\delta^*(t) \in \mathbb{R}$ is the feedforward control input, i.e.

$$\xi(t) = \frac{\alpha k_\tau N}{(m_1 l_1^2 + I_1 + I_p) R_a} V_\delta^*(t) + \frac{\alpha m_2}{(m_1 l_1^2 + I_1 + I_p)} e_{r\delta}(t) - \frac{m_1 g l_1}{(m_1 l_1^2 + I_1 + I_p)} \ddot{e}_{r\delta}(t) + \mathbf{H.O.T.}$$

3.2 LESO Design. Denote, $e_{r_\delta}^{(i)} \in \mathbb{R}$ simply as e_i, $i = 0, 1, \ldots, n-1$. The flat output trajectory tracking error model is given by

$$\begin{aligned} \dot{e}_0 &= e_1 \\ \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= -\frac{\alpha k_\tau N}{(m_1 l_1^2 + I_1 + I_p) R_a} V_\delta(t) + \xi(t) \end{aligned}$$

Notice that the variable e_2 may be expressed as $e_2 = \ddot{r}_\delta - \ddot{r}_\delta^*(t) = -\alpha\theta_\delta - \ddot{r}_\delta^*(t)$, which represents a known input to the second-order pure integration system

$$\begin{aligned} \dot{e}_0 &= e_1 \\ \dot{e}_1 &= e_2 = -\alpha\theta_\delta - \ddot{r}_\delta^*(t) \end{aligned} \tag{9}$$

while the remaining error system is given by

$$\begin{aligned} \dot{e}_2 &= e_3 \\ \dot{e}_3 &= -\frac{\alpha k_\tau N}{(m_1 l_1^2 + I_1 + I_p) R_a} V_\delta(t) + \xi(t) \end{aligned} \tag{10}$$

A set of two decoupled second-order LESO can be proposed for the simultaneous estimation of the flat output tracking error phase variables associated with e_{r_δ} and of the perturbation signal $\xi(t) = z_1 \in \mathbb{R}$. For the phase variables e_0 and e_1 in Eq. (9), one sets

$$\begin{aligned} \dot{\hat{e}}_0 &= \hat{e}_1 + \rho_1(e_0 - \hat{e}_0) \\ \dot{\hat{e}}_1 &= \hat{e}_2 + \rho_0(e_0 - \hat{e}_0) \end{aligned}$$

while for the rest of the dynamics, associated with e_2 and e_3, in Eq. (10), including the disturbance z_1, one synthesizes

$$\begin{aligned} \dot{\hat{e}}_2 &= \hat{e}_3 + \lambda_{\varphi+1}(e_2 - \hat{e}_2) \\ \dot{\hat{e}}_3 &= -\frac{\alpha k_\tau N}{(m_1 l_1^2 + I_1 + I_p) R_a} V_\delta(t) + \hat{z}_1 + \lambda_\varphi(e_2 - \hat{e}_2) \\ \dot{\hat{z}}_j &= \hat{z}_{j+1} + \lambda_{\varphi-j}(e_2 - \hat{e}_2), \quad j = 1, \ldots, \varphi - 1 \\ \dot{\hat{z}}_\varphi &= \lambda_0(e_2 - \hat{e}_2) \end{aligned} \tag{11}$$

where $\varphi \in \mathbb{Z}^+$ is the order of the dynamic state extension in the observer, which can be chosen of rather arbitrary order. In this case, the order $\varphi = 6$ was systematically chosen. The observation error, $\tilde{e}_0 = e_0 - \hat{e}_0$, of the incremental flat output tracking error satisfies the following perturbed dynamics:

$$\ddot{\tilde{e}}_0 + \rho_1 \dot{\tilde{e}}_0 + \rho_0 \tilde{e}_0 = \eta(\dot{\tilde{e}}_0, \tilde{e}_0, e_2)$$

where $\eta(\dot{\tilde{e}}_0, \tilde{e}_0, e_2)$ is a perturbation term depending on the estimation and tracking errors, considering e_2 as a perturbation term, depending on e_0, e_1. An appropriate choice of the design coefficients $\{\rho_1, \rho_0\}$ renders an asymptotically exponentially decreasing estimation error phase variables, $\tilde{e}_0$, $\dot{\tilde{e}}_0$, and $\ddot{\tilde{e}}_0$ toward previously chosen arbitrarily small neighborhoods of the origin. In this case, the parameters ρ_1 and ρ_0 are selected such that a stable second-order dynamics with characteristic polynomial $s^2 + 2\zeta_o\omega_o s + \omega_o^2$, being $\zeta_o, \omega_o \in \mathbb{R}^+$, is matched. The parameters are chosen as follows: $\rho_1 = 2\zeta_o\omega_o$ and $\rho_0 = \omega_o^2$, $\omega_0, \zeta_0 \in \mathbb{R}^+$. The tracking error velocity $\dot{e}_{r_\delta} = e_1$ is accurately estimated for feedback purposes by $\hat{e}_1$. The observation error of the flat output acceleration tracking error $\tilde{e}_2 = e_2 - \hat{e}_2$ generates the following linear reconstruction error dynamics:

$$\tilde{e}_2^{(p+2)} + \lambda_{p+1}\tilde{e}_2^{(p+1)} + \cdots + \lambda_1 \dot{\tilde{e}}_2 + \lambda_0 \tilde{e}_2 = [\xi(t)]^{(\varphi)}$$

A necessary and sufficient condition for having the incremental flat output acceleration estimation error $\tilde{e}_2$, and its associated phase variables, $\tilde{e}_2, \dot{\tilde{e}}_2, \ldots, \tilde{e}_2^{(\varphi+1)}$, ultimately, uniformly, convergent toward a sufficiently small neighborhood of the acceleration estimation error phase space, consists in choosing the observer gains: λ_k, $\{k = 0, \ldots, \varphi + 1\}$, sufficiently large such that the linear injected error dynamics becomes stable. The efficient pole placement procedure, developed in Ref. [6], is hereby adopted, see Ref. [2]. The controller may be readily synthesized as a disturbance canceling ADRC, properly attenuating the total input disturbance, $\xi(t)$, while imposing a desired exponentially dominated tracking error dynamics. Simultaneously, the tracking error phase variable estimates $\hat{e}_1$ and $\hat{e}_3$ are obtained from the proposed observers. The controller is thus given by

$$V_\delta = \frac{(m_1 l_1^2 + I_1 + I_p) R_a}{\alpha k_\tau N} [\hat{z}_1 + k_3 \hat{e}_3 + k_2 e_2 + k_1 \hat{e}_1 + k_0 e_0]$$

where, naturally the tracking errors, e_0 and e_2, themselves are used instead of their estimates. The coefficients of the feedback controller are chosen considering the differential equation of the closed-loop flat output tracking error

$$e_0^{(4)} + k_3 e_0^{(3)} + k_2 \ddot{e}_0 + k_1 \dot{e}_0 + k_0 e_0 = \xi(t) - \hat{z}_1$$

They are chosen so that the associated characteristic polynomial of the dominantly linear closed-loop dynamics $p_c(s) = s^4$

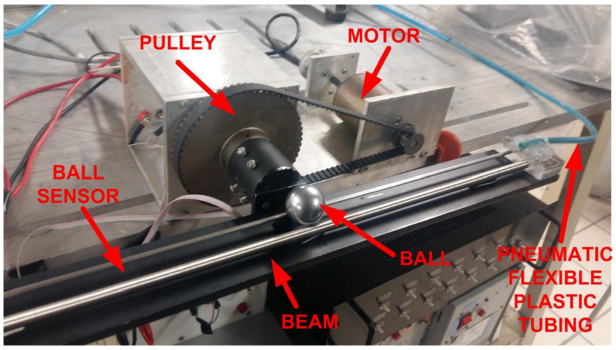

Fig. 3 Block diagram of the ball and beam control system

$+k_3s^3 + k_2s^2 + k_1s + k_0$ becomes stable. In this case, $k_3 = 4\zeta_c\omega_c$, $k_2 = 2\omega_c^2 + 4\zeta_c^2\omega_c^2$, $k_1 = 4\zeta_c\omega_c^3$, $k_0 = \omega_c^4$, with $0 < \zeta_c < 1$, $\omega_c \in \mathbb{R}^+$. One may force the closed-loop tracking error dynamics to exhibit the desired stable fourth-order dynamics with characteristic polynomial coincident with $(s^2 + 2\zeta_c\omega_c s + \omega_c^2)^2$.

4 Experimental Results

The experimental device consists of a 24 V DC motor which drives an aluminum beam via a synchronous belt and a pulley with a ratio $N = 6{:}1$. The angular position of the beam is measured using an incremental optical encoder of 2500 pulses per revolution. A linear sensor, consisting of an etched wire made of a nickel–chromium wire, measured the position of the ball along the beam with a resolution of 25 (mm/V) as shown in Fig. 3. The data acquisition is carried out through a data card, model QPIDe, from Quanser. The control strategy described before was implemented in the MATLAB-SIMULINK Quarc platform. The sampling time was 0.001 (s). The parameters for the beam are $I_1 = 0.0045$ (kg/m^2), $m_1 = 0.065$ (kg), $l_1 = 0.015$ (m), $I_p = 0.001$ (kg/m^2), $m_2 = 0.065$ (kg), and $I_2 = 0.0045$ (kg/m^2). The radius of the ball is, $R = 0.0127$ (m). The parameters of the motor are $R_a = 2.983$ (Ω), $k_\tau = 0.0724$ (N m/A). The initial conditions for the position variables in the system were $[r = 0,\ \theta = 0]$. The observer gain parameters for the observation error $\tilde{e}_0$ were set as $\zeta_o = 3$, $\omega_o = 70$. The observer gain parameters for observation error $\tilde{e}_2$ were set as $T = 5$, $a_0 = 4$, $\alpha_1 = 4.1$. The controller design parameters were $\zeta_c = 0.8$, $\omega_c = 15$. Figure 4 shows the performance of the closed loop trajectory tracking for ball position and acceleration, from the initial position $r_\delta(0) = 0$, toward the equilibrium position $r_\delta(4) = -0.14$ (m), and then moved to the final resting position $r_\delta(12) = 0.14$ (m). The tracking error evolution, the control input, and the lumped disturbance estimation are shown in Fig. 5.

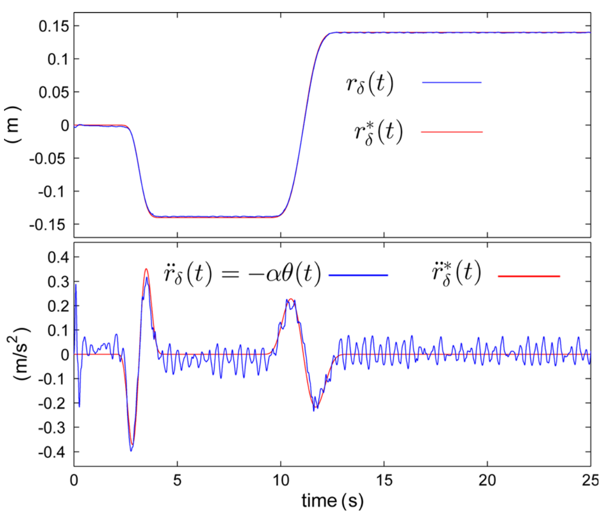

Fig. 4 Performance of closed loop reference trajectory tracking ball position and acceleration

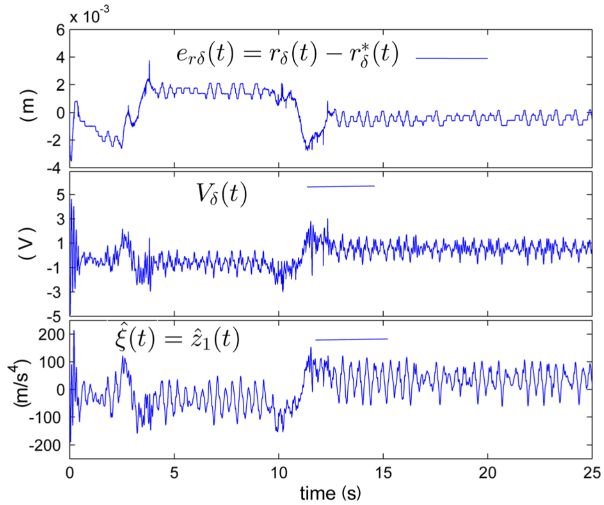

Fig. 5 Tracking error, voltage input control signal, and lumped online disturbance estimation

4.1 Comparison Test. In order to test the performance of the flatness-based ADRC scheme proposed, we carried out a comparative analysis with respect to a state feedback integral (SFI) controller [7] as well as the super twisting (ST) algorithm [8]. The error dynamics is given by $\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}V_{\delta\mathrm{SFI}}$, where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{C} & 0 \end{bmatrix} \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \quad \hat{\mathbf{C}} = [\mathbf{C} \quad 0]$$

$$V_{\delta\mathrm{SFI}} = -\hat{\mathbf{K}}\mathbf{e} \quad \mathbf{e} = \begin{bmatrix} \mathbf{x} - \mathbf{x}^*(t) \\ \hat{\mathbf{C}}\int(\mathbf{x} - \mathbf{x}^*(t)) \end{bmatrix} \quad \mathbf{C} = [1 \quad 0 \quad 0 \quad 0]$$

The SFI controller is specified as

$$V_{\delta\mathrm{SFI}} = -(\kappa_1 e_1 + \kappa_2 e_2 + \kappa_3 e_3 + \kappa_4 e_4 + \kappa_5 e_5) \tag{12}$$

The state stabilization error equation is simplified to $\dot{\mathbf{e}} = (\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}})\mathbf{e}$, the desired closed-loop poles of matrix $\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}$ are specified as $(s - \mu_1)(s - \mu_3)(s - \mu_3)(s - \mu_4)(s - \mu_5) = 0$, being $\mu_1 = -0.9 + 2.6814i$, $\mu_2 = -0.9 - 2.6814i$, $\mu_3 = \mu_4 = \mu_5 = -10$. Then the state-feedback gain matrix $\hat{\mathbf{K}} = [\kappa_1\ \kappa_2\ \kappa_3\ \kappa_4\ \kappa_5]$ can be determined by a traditional pole-placement technique. For the super twisting controller, the following sliding surface and controller were defined:

$$\sigma = -\alpha\dot{\theta}_\delta - r_\delta^{*(3)} + \gamma_2(-\alpha\theta_\delta - \ddot{r}^*) + \gamma_1(\dot{r}_\delta - \dot{r}_\delta^*) + \gamma_0(r_\delta - r_\delta^*)$$

$$V_{\delta\mathrm{SM}} = \frac{(m_1 l_1^2 + I_1 + I_p)R_a}{\alpha k_\tau N}\left[-a\sqrt{|\sigma|}\,\mathrm{sign}(\sigma) + y\right]$$

$$\dot{y} = -M\,\mathrm{sign}(\sigma)$$

For the sliding surface $\sigma = 0$, the set of gains were chosen such that the following desired characteristic polynomial was matched: $s^3 + \gamma_2 s^2 + \gamma_1 s + \gamma_0 = (s + p)(s^2 + 2\zeta_m\omega_m + \omega_m^2)$ with $p = 8$, $\omega_m = 8$, $\zeta_m = 1$, the controller gains was chosen as $a = 70\ M = 0.64$. For both controllers, the measured states $[r_\delta, \dot{r}_\delta, \theta_\delta, \dot{\theta}_\delta]$ were processed by means of low-pass filters $30\pi/(s + 30\pi)$, in order to reduce the noise in the derivative estimation. A disturbance consisting of an air load with a value of approximately 0.1 (N) was applied as a kind of step with duration of 1 (s) at the times $t = 15$ and $t = 20$ (s). It was produced using a

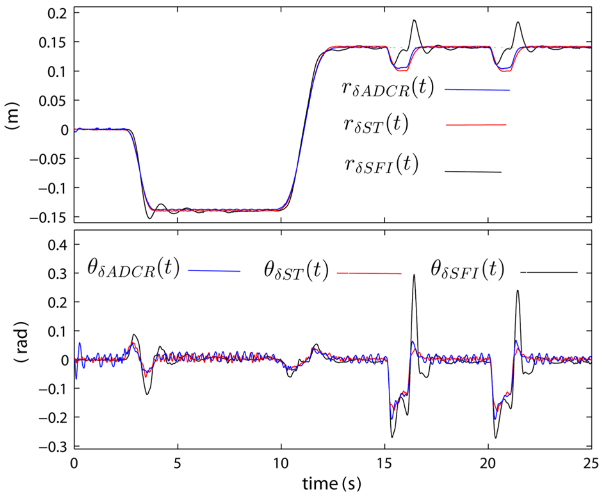

Fig. 6 Tracking trajectory performance

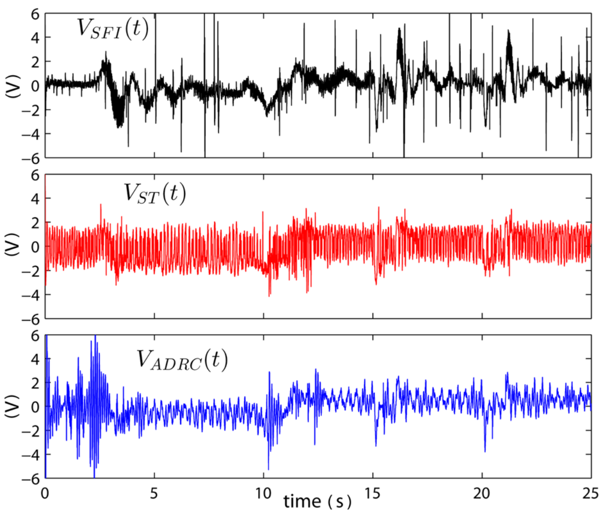

Fig. 7 Control inputs in the comparison test

pneumatic flexible plastic tubing with interior diameter 4 (mm) (see Fig. 3), and activated by means of solenoid air control valve, the pressure of the air was set to 3.44738 (Bar). Figure 6 shows the tracking trajectory results and Fig. 7 shows the control signals. Notice that the sliding mode controller and the proposed scheme obtained better results with respect to the ones of the SFI. Notice that the proposal is quite competitive using purely linear methods in relation to one of the most important robust controllers.

5 Concluding Remarks

The problem of controlling a class of UAS, in a trajectory tracking task, was solved by means of exploiting the flatness associated with the linearized model and via an extended state observer based linear ADRC. The scheme uses the tangent linearization system model of the BBS around an arbitrary equilibrium point. The traditional drawbacks present in standard linear control schemes were overcome by using a particular structure of the LESO, of the Luenberger type, induced by the flatness of the system. The scheme achieves highly competitive experimental results when compared with other linear and nonlinear controllers. Extensions of this result for the multivariable case are considered as future investigation.

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