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Developments in determining the gravitational potential using toroidal functions *

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Cohl & Tohline (1999) have shown how the integration/summation expression for the Green's function in cylindrical coordinates can be written as an azimuthal Fourier series expansion, with toroidal functions as expansion coefficients. In this paper, we show how this compact representation can be extended to other rotationally invariant coordinate systems which are known to admit separable solutions for Laplace's equation.

Key words: stars: formation – stars: evolution – galaxies: formation – galaxies: evolution – separation of Laplace's equation – spheroidal coordinates – parabolic coordinates – toroidal coordinates

1. Introduction

The Newtonian gravitational and Coulomb potentials, $\Phi(\mathbf{x}, t)$, due to a mass or charge distribution $\rho(\mathbf{x}, t)$, are important throughout our physical world. In an astrophysical context (cf., §2.1 in Binney & Tremaine 1987), the Newtonian potential is determined by solving the appropriate boundary-value problem that is given by the three-dimensional Poisson equation,

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi \mathrm{G} \rho(\mathbf{x}), \tag{1}$$

where ∇^2 is the Laplacian, and $\mathrm{G} \simeq 6.6742 \times 10^{-8}$ cm^3 g^{-1} sec^{-2} is the gravitational constant (see Gundlach & Merkowitz 2000). Exterior to an isolated mass distribution, physically correct boundary conditions may be imposed on Poisson's equation through the following integral formulation of the potential problem,

$$\Phi(\mathbf{x}) = -\mathrm{G} \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \tag{2}$$

Due to the long-range nature of the Newtonian/Coulomb interaction, a physically correct solution of Poisson's equation and, hence, an accurate determination of the potential "interior" to the boundary can be *highly* boundary value dependent. If inaccurate potential values are given on the boundary, the interior solution will reflect these defects.

In numerical simulations of astrophysical systems, a standard procedure for determining the value of the gravitational potential along the boundary of a computational mesh has been the multipole method. In this method, an approximate solution to eq. (2) is obtained by summing successively higher order multipole moments (monopole, dipole, quadrupole, etc.) of the mass distribution. The multipole moments themselves are normally cast in terms of two quantum numbers — one meridional ℓ, and one azimuthal m — and for practical reasons the series summation is truncated at a finite value of ℓ and m. An alternative to the standard multipole expansion has been given by Cohl & Tohline (1999) in terms of a single sum over the azimuthal quantum number m. This new expression yields the pure azimuthal mode contribution to the Newtonian potential. In terms of the standard multipole description, this new expansion is equivalent to completing the infinite sum over the quantum number ℓ for each given value of the quantum number m (Cohl *et al.* 2000). We now extend this expansion to other rotationally invariant coordinate systems which separate the Laplace equation.

In §2 we describe the rotationally invariant coordinate systems which separate Laplace's equation and briefly describe some of the properties of these systems related to the double summation/integration expressions for

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the reciprocal distance between two points (hereafter, Green's function for Laplace's equation, or just Green's function) in these coordinate systems. In §3 we describe the key expressions for the Green's function in circular cylindrical coordinates, and show how these expressions are consistent with an azimuthal Fourier series expansion for the Green's function whose coefficients are given in terms of toroidal functions. In §4 we describe some of the highly symmetric properties of toroidal functions, such as their behavior for negative degree and order. We also present toroidal function implications of the Whipple formulae for associated Legendre functions. Using these Whipple formulae for toroidal functions, we then obtain a new expansion which is shown to be equivalent to the expansion given in Cohl & Tohline (1999). In §5 we briefly summarize several key mathematical implications of these expansions as they apply to prolate spheroidal, oblate spheroidal, and, parabolic coordinates. In §6 we present these expansions as they apply to bispherical, and toroidal coordinates. All of the coordinate systems presented in §§5 and 6 have known double integration/summation expansions for the Green's function and therefore can be easily related to these new alternative expansions. A detailed treatment in spherical coordinates is being presented elsewhere (Cohl *et al.* 2000). The mathematical relations we derive below are either in the form of an infinite integral or an infinite series expansion over the set of basis functions which separate Laplace's equation.

2. Rotationally invariant coordinate systems which separate Laplace's equation

It is well-known that Laplace's equation

$$\nabla^2\Phi(\mathbf{x}) = 0, \tag{3}$$

admits a number of different separable solutions for the function $\Phi(\mathbf{x})$ that are given in terms of products of known special functions whose arguments, in turn, are given in terms a triply-orthogonal set of curvilinear coordinates $\{\xi_1, \xi_2, \xi_3\}$ (Bôcher 1894; §10.3 in Morse & Feshbach 1953; §3.6 in Miller 1977). These solutions can be classified as being either simply separable, $\Phi(\mathbf{x}) = \Xi_1(\xi_1)\Xi_2(\xi_2)\Xi_3(\xi_3)$, or $\mathcal{R}$-separable $\Phi(\mathbf{x}) = [\Xi_1(\xi_1)\ \Xi_2(\xi_2)\ \Xi_3(\xi_3)]/\mathcal{R}(\xi_1, \xi_2, \xi_3)$, where the modulation factor $\mathcal{R}(\xi_1, \xi_2, \xi_3)$ (§5.1 in Morse & Feshbach 1953) is a scalar function that has a unique specification for each coordinate system. The simply separable coordinate systems can be geometrically characterized by surfaces that are quadric (second-order), whereas the $\mathcal{R}$-separable coordinate systems are characterized by surfaces which are cyclidic (fourth-order). Using a Lie group theoretic approach, Miller (1977; see also Boyer, Kalnins, & Miller 1976) has demonstrated that there are precisely seventeen conformally independent separable coordinate systems that are either simply separable or $\mathcal{R}$-separable for Laplace's equation. In general, these separable Laplace systems can be put into three classes: a cylindrical class (i.e., invariant under vertical translations), a rotational class (i.e., invariant under rotations about the z-axis), and a more general class.

Here we will focus on the rotational class of Laplace systems, that is, those systems that correspond to the diagonalization of the $z-$component of the angular momentum operator

$$J_z = -i\frac{\partial}{\partial\phi}, \tag{4}$$

where ϕ is the azimuthal coordinate. These coordinate systems share the special property that their eigenfunctions take the form

$$\Phi(\mathbf{x}) = \Psi(R, z)\ \mathrm{e}^{im\phi}, \tag{5}$$

where R represents the distance from the $z-$axis (i.e., the cylindrical radius), z is the vertical height, and

$$J_z\Phi(\mathbf{x}) = m\Phi(\mathbf{x}). \tag{6}$$

If we substitute (5) into Laplace's equation and factor out $\mathrm{e}^{im\phi}$, we obtain a differential equation for Ψ, which is the foundation of generalized axisymmetric potential theory; written in cylindrical coordinates, for example, this key equation takes the form,

$$\frac{\partial^2\Psi}{\partial R^2} + \frac{1}{R}\frac{\partial\Psi}{\partial R} - \frac{m^2}{R^2}\Psi + \frac{\partial^2\Psi}{\partial z^2} = 0. \tag{7}$$

Miller (1977) also has shown that there are precisely nine conformally independent rotational Laplace systems. Five of these coordinate systems (cylindrical, spherical, prolate spheroidal, oblate spheroidal, and, parabolic) are quadric and simply separable for Laplace's equation; the remaining four (one of which is toroidal) are cyclidic and $\mathcal{R}-$separable for Laplace's equation. Moon & Spencer (1961b, §IV) have tabulated ten fourth-order rotational Laplace systems. From Miller's (1977) study, we know that under any conformal symmetry of the Laplace equation, an $\mathcal{R}-$separable coordinate system can be mapped to another $\mathcal{R}-$separable coordinate system. Therefore, one might question whether all of Moon & Spencer's (1961b) rotational Laplace systems have unique double summation/integration Green's function expansions, and how many such coordinate systems exist.

Discussions of the double summation/integration expressions for the Green's function expansions (Jackson 1975, Chapter 3; Morse & Feshbach 1953, Chapter 10) for cylindrical (Cohl & Tohline 1999; see also §3 below) and spherical coordinates (Cohl *et al.* 2000) is abundant in the literature. Double summation/integration expressions for the Green's function for the three remaining quadric rotational Laplace systems (oblate spheroidal, prolate spheroidal, and, parabolic coordinates) also have been previously presented in the literature (Hobson 1931, §§245 and 251; Morse & Feshbach 1953, §10.3) and therefore present themselves easily for comparison. As for the cyclidic coordinate systems, the double summation expressions for the Green's functions in toroidal and bispherical coordinates have been developed (cf., Morse & Feshbach 1953, §10.3; Hobson 1931, §258; see also §6.2 below), but we have been unable to locate the double summation/integration Green's function expansions for the other rotational cyclidic Laplace systems. The three remaining conformally unique cyclidic rotational Laplace systems have $\mathcal{R}-$separable transcendental function solutions to Laplace's equation that are in the form of Lamé functions and Jacobi elliptic functions. We anticipate discussing these less well known cyclidic coordinate systems, as well as the remaining rotational Laplace coordinate systems presented in Moon & Spencer (1961b), in forthcoming investigations.

3. The rotational cylindrical system

The circular cylindrical coordinate system is unique in that it is the only coordinate system that is known to be separable for Laplace's equation as well as being a member of both the rotational and cylindrical classes. In cylindrical coordinates — $\mathbf{x} : \{R\cos\phi, R\sin\phi, z\}$, where the radial coordinate R goes from 0 to ∞, the azimuthal coordinate ϕ goes from 0 to 2π, and the vertical coordinate z goes from $-\infty$ to ∞ — the reciprocal distance between two points $\mathbf{x}$ and $\mathbf{x}'$ can be given in many different forms. For instance, here we write the Green's function for Laplace's equation in terms of the integral of Lipschitz (see problem 3.14 in Jackson 1975 or §13.2 in Watson 1944 or eq. (6.611.1) in Gradshteyn & Ryzhik),

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \int_0^\infty dk\ J_0\left(k\sqrt{R^2+R'^2-2RR'\cos(\phi-\phi')}\right)\mathrm{e}^{-k(z_>-z_<)}, \tag{8}$$

where J_0 is the order zero Bessel function of the first kind. Alternatively, this same quantity can be written in a form of the Lipschitz-Hankel integral (§3.11 in Jackson 1975; §13.21 in Watson 1944; eq. (6.671.6) in Gradshteyn & Ryzhik),

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{2}{\pi}\int_0^\infty dk\ \ K_0\left(k\sqrt{R^2+R'^2-2RR'\cos(\phi-\phi')}\right)\cos k(z-z'), \tag{9}$$

where K_0 is the order zero modified Bessel function of the second kind. According to Neumann's *addition theorem* for Bessel functions, the order zero Bessel function of the first kind can be written as a Fourier series expansion over products of Bessel functions of varying order, namely (§11.1 in Watson 1944),

$$\sum_{m=-\infty}^{\infty} J_m(kR)J_m(kR')\ \mathrm{e}^{im(\phi-\phi')} = J_0\left(k\sqrt{R^2+R'^2-2RR'\cos(\phi-\phi')}\right), \tag{10}$$

where J_m is the order m Bessel function of the first kind. In the same vein, the order zero modified Bessel function of the second kind can be expanded as follows using Graf's generalization of Neumann's *addition theorem* (§11.3 in Watson 1944):

$$\sum_{m=-\infty}^{\infty} I_m(kR_<)K_m(kR_>)\ \mathrm{e}^{im(\phi-\phi')} = K_0\left(k\sqrt{R^2+R'^2-2RR'\cos(\phi-\phi')}\right), \tag{11}$$

where I_m and K_m are the order m modified Bessel functions of the first and second kind, respectively. Substituting the two *addition theorems* (10) and (11) into the integral expressions (8) and (9), respectively, yields two double integration/summation expressions for the Green's function (problem 3.14 of Jackson 1975):

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{m=-\infty}^{\infty}\mathrm{e}^{im(\phi-\phi')}\int_0^\infty dk\ J_m(kR)J_m(kR')\ \mathrm{e}^{-k(z_>-z_<)}, \tag{12}$$

and (§3.11 of Jackson 1975),

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty} \mathrm{e}^{im(\phi-\phi')}\int_0^\infty dk\ I_m(kR_<)K_m(kR_>)\,\cos k(z-z'). \tag{13}$$

We can further simplify both of these double summation/integration expressions through the use of known transcendental function solutions to the integrals in both (12) and (13) as given, for example, by eq. (13.22.2) in Watson (1944) and eqs. (6.612.3) and (6.672.4) in Gradshteyn & Ryzhik (1994), namely,

$$\int_0^\infty dk\ J_m(kR)J_m(kR')\ \mathrm{e}^{-k(z_>-z_<)} = \frac{1}{\pi\sqrt{RR'}}\ Q_{m-\frac{1}{2}}\left[\frac{R^2+{R'}^2+(z-z')^2}{2RR'}\right], \tag{14}$$

and,

$$\int_0^\infty dk\ I_m(kR_<)K_m(kR_>)\cos k(z-z') = \frac{1}{2\sqrt{RR'}}\ Q_{m-\frac{1}{2}}\left[\frac{R^2+{R'}^2+(z-z')^2}{2RR'}\right], \tag{15}$$

where $Q_{m-\frac{1}{2}}$ is the odd-half-integer degree Legendre function of the second kind. Hence, both eqs. (12) and (13) lead to the following compact azimuthal Fourier series:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{\pi\sqrt{RR'}}\sum_{m=-\infty}^{\infty} Q_{m-\frac{1}{2}}(\chi)\ \mathrm{e}^{im(\phi-\phi')}, \tag{16}$$

where

$$\chi \equiv \frac{R^2+{R'}^2+(z-z')^2}{2RR'}. \tag{17}$$

Equation (16) is the Green's function expansion that was presented in Cohl & Tohline (1999) as an alternative to the standard multipole expansion technique.

In fact, this result can be easily derived by starting with the algebraic expression for the Green's function in cylindrical coordinates,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{\sqrt{2RR'}}\left[\frac{R^2+{R'}^2+(z-z')^2}{2RR'} - \cos(\phi-\phi')\right]^{-\frac{1}{2}}, \tag{18}$$

and using the Heine identity (§10.2 in Bateman 1959; §74 in Heine 1881; see also §4.5.4 of Magnus, Oberhettinger, and Soni 1966),

$$\sum_{n=-\infty}^{\infty} Q_{n-\frac{1}{2}}(s)\ \mathrm{e}^{in\psi} = \pi\big[2\,(s-\tau)\big]^{-\frac{1}{2}}, \tag{19}$$

where $s=\cosh\sigma \geq 1$, and $-1 \leq \tau = \cos\psi \leq +1$. Using (19) to expand (18) immediately yields (16).

Although the use of Heine's identity leads to the main result (16) more quickly, *addition theorems* such as (10) and (11) can be extremely useful when casting various expressions for the Green's function for Laplace's equation in different coordinate systems in terms of single summation/integration expressions like (8), (9), and (16). Such expressions encapsulate the symmetries manifested within these diverse forms. In what follows we offer an interesting twist on the development in the theory of *addition theorems.* We note that the essential ingredient which leads to our new development, eq. (19), was given by Heine (1881) over a century ago in, "Handbuch der Kugelfunktionen,"[1] but it appears not to have been utilized much in practical astrophysics applications over the past century.

[1] Richard Askey, general editor for the section on special functions in the "Encyclopedia of Mathematics and its Applications," in his foreword to Miller (1977) says, in regard to the *addition theorem* for spherical harmonics and the corresponding *addition theorem* for trigonometric functions, "... (these) are among the most important facts known about these functions." Whittaker and Watson (1943) further expound on the *addition theorems* in chapter XV on Legendre functions. Watson (1944) devotes his chapter XI to the discussion of the *addition theorems* known then for Bessel functions. Hobson (1931) devotes his chapter VIII to the discussion of the *addition theorems* known then for associated Legendre functions. Hobson (1931) proclaims of Heine's "Inaugural dissertation...which has hitherto been the only treatise dealing with the functions which could claim to be complete..." and in connection with Eduard Heine,"...he first introduced the Legendre's functions of the second kind, together with the associated functions."

4. The highly symmetric nature of toroidal functions

Associated Legendre functions of the first and second kind, $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ are in general characterized by three complex constants: the degree ν, the order μ, and the argument z. Toroidal (or ring) functions are the associated Legendre functions with odd-half-integer degree and integer order. Here we illustrate some of the striking symmetry characteristics of toroidal functions. We then show how these lead to an alternative azimuthal Fourier series representation of the Green's function for the rotational Laplace systems whose coefficients are given in terms of toroidal functions of the first kind.

4.1. The negative degree and order conditions

Here we present, as examples, the negative degree condition for associated Legendre functions of the first kind (cf., eq. [8.2.1] in Abramowitz & Stegun 1965),

$$P^\mu_{-\nu-\frac{1}{2}}(z) = P^\mu_{\nu-\frac{1}{2}}(z); \tag{20}$$

the negative degree condition for associated Legendre functions of the second kind (cf., eq. (8.2.2) in Abramowitz & Stegun 1965),

$$Q^\mu_{-\nu-\frac{1}{2}}(z) = \frac{1}{\cos\pi(\nu-\mu)}\left[\cos\pi(\nu+\mu)\ Q^\mu_{\nu-\frac{1}{2}}(z) + \pi \mathrm{e}^{i\mu\pi}\sin\nu\pi\ P^\mu_{\nu-\frac{1}{2}}(z)\right]; \tag{21}$$

the negative order condition for associate Legendre functions of the first kind (cf., eq. (8.2.5) in Abramowitz & Stegun 1965),

$$P^{-\mu}_{\nu-\frac{1}{2}}(z) = \frac{\Gamma(\nu-\mu+\frac{1}{2})}{\Gamma(\nu+\mu+\frac{1}{2})}\left[P^\mu_{\nu-\frac{1}{2}}(z) - \frac{2}{\pi}\mathrm{e}^{-i\mu\pi}\sin\mu\pi\ Q^\mu_{\nu-\frac{1}{2}}(z)\right]; \tag{22}$$

and, finally, the negative order condition for associate Legendre functions of the second kind (cf., eq. [8.2.6] in Abramowitz & Stegun 1965),

$$Q^{-\mu}_{\nu-\frac{1}{2}}(z) = \mathrm{e}^{-2i\mu\pi}\frac{\Gamma(\nu-\mu+\frac{1}{2})}{\Gamma(\nu+\mu+\frac{1}{2})}Q^\mu_{\nu-\frac{1}{2}}(z). \tag{23}$$

For toroidal functions with $\nu = n$ and $\mu = m$ being positive integers, with $n > m$, it follows from eqs. (20)—(23),

$$P^m_{-n-\frac{1}{2}}(z) = P^m_{n-\frac{1}{2}}(z), \tag{24}$$

$$Q^m_{-n-\frac{1}{2}}(z) = Q^m_{n-\frac{1}{2}}(z), \tag{25}$$

$$P^{-m}_{n-\frac{1}{2}}(z) = \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})}P^m_{n-\frac{1}{2}}(z), \qquad \text{and,} \tag{26}$$

$$Q^{-m}_{n-\frac{1}{2}}(z) = \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})}Q^m_{n-\frac{1}{2}}(z). \tag{27}$$

It can also be shown with very little algebra that both equations (26) and (27) are invariant under index interchange of the n and m indices.

4.2. The Whipple formulae

Another aspect of the symmetric nature of toroidal functions under index interchange is further demonstrated by using Whipple's formulae for associated Legendre functions (§3.3.1 in Erdélyi *et al.* 1953). We start with eqs. (8.2.7) and (8.2.8) in Abramowitz and Stegun (1965), namely,

$$P^{-\nu-\frac{1}{2}}_{-\mu-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right) = \frac{(z^2-1)^{1/4}\mathrm{e}^{-i\mu\pi}Q^\mu_\nu(z)}{(\pi/2)^{1/2}\Gamma(\nu+\mu+1)}, \tag{28}$$

and,

$$Q^{-\nu-\frac{1}{2}}_{-\mu-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right) = -i(\pi/2)^{1/2}\Gamma(-\nu-\mu)(z^2-1)^{1/4}\mathrm{e}^{-i\nu\pi}P^{\mu}_{\nu}(z), \tag{29}$$

which are valid only for $\mathrm{Re}\, z > 0$ (since the Legendre function of the second kind becomes discontinuous on the cut) and for all complex ν and μ, except when $\nu = n =$ integer since the Gamma function has poles along the negative real axis. By combining eqs. (28) and (29) with eqs. (20)—(23) along with the negative argument condition for Gamma functions [eq. (6.1.17) of Abramowitz & Stegun (1965)], namely,

$$\Gamma(-z) = \frac{-\pi}{\Gamma(z+1)\sin \pi z}, \tag{30}$$

we can write two general expressions which are valid for all complex ν and μ. Here, for associated Legendre functions of the first kind,

$$P^{\mu}_{\nu-\frac{1}{2}}(z) = \frac{\sqrt{2}\,\Gamma(\mu-\nu+\frac{1}{2})}{\pi^{3/2}(z^2-1)^{1/4}}\left[\pi \sin\mu\pi\; P^{\nu}_{\mu-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right) +\; \cos\pi(\nu+\mu)\;\mathrm{e}^{-i\nu\pi}Q^{\nu}_{\mu-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right)\right], \tag{31}$$

which is equivalent to,

$$Q^{\mu}_{\nu-\frac{1}{2}}(z) = \frac{\mathrm{e}^{i\mu\pi}\Gamma(\mu-\nu+\frac{1}{2})(\pi/2)^{1/2}}{(z^2-1)^{1/4}}\left[P^{\nu}_{\mu-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right) - \frac{2}{\pi}\;\mathrm{e}^{-i\nu\pi}\sin\nu\pi\; Q^{\nu}_{\mu-\frac{1}{2}}\left(\frac{z}{\sqrt{z^2-1}}\right)\right]. \tag{32}$$

Now, we substitute $\nu = n$ and $\mu = m$ (with m and n being positive integers) in eqs. (31) and (32), or, alternatively, combine eqs (28) and (29) with eqs. (24) – (27) in order to derive the Whipple formulae for associated toroidal functions in a convenient form where n and m are interchanged on the two sides of the equations:

$$P^{m}_{n-\frac{1}{2}}(\cosh\eta) = \frac{(-1)^n}{\Gamma(n-m+\frac{1}{2})}\sqrt{\frac{2}{\pi\sinh\eta}}\; Q^{n}_{m-\frac{1}{2}}(\coth\eta), \tag{33}$$

and

$$Q^{m}_{n-\frac{1}{2}}(\cosh\eta) = \frac{(-1)^n\pi}{\Gamma(n-m+\frac{1}{2})}\sqrt{\frac{\pi}{2\sinh\eta}}\; P^{n}_{m-\frac{1}{2}}(\coth\eta). \tag{34}$$

These last two expressions allow us to express toroidal functions of a certain kind (first or second, respectively) with argument hyperbolic cosine, as a direct proportionality in terms of the toroidal function of the other kind (second or first, respectively) with argument hyperbolic cotangent.

4.3. An alternative azimuthal Fourier series expansion

Substituting for $Q_{m-\frac{1}{2}}$ in eq. (16) from eq. (34) with $m = 0$ and $n = m$, we have an alternative form for the Green's function:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sqrt{\frac{\pi}{2RR'(\chi^2-1)^{1/2}}}\sum_{m=-\infty}^{\infty}\frac{(-1)^m}{\Gamma(m+\frac{1}{2})}P^{m}_{-\frac{1}{2}}\left(\frac{\chi}{\sqrt{\chi^2-1}}\right)\;\mathrm{e}^{im(\phi-\phi')}. \tag{35}$$

These two separate ways to write the Heine identity may each be useful in numerical applications. The same thing should be possible in any of the rotational Laplace systems. For instance, in cylindrical coordinates, the infinite integrals over products of Bessel functions given by eqs. (14) and (15) can be expressed in terms of the associated toroidal functions of the first kind as well. New *addition theorems* and definite integrals arise as we demonstrate in the next sections.

5. Three second-order rotational Laplace systems

In this section we present necessary implications of the main results (16) and (35) in the prolate spheroidal, oblate spheroidal, and, parabolic coordinate systems.

5.1. Prolate spheroidal coordinates

In prolate spheroidal coordinates — $\mathbf{x}:\{a\sinh\sigma\sin\theta\cos\phi, a\sinh\sigma\sin\theta\sin\phi, a\cosh\sigma\cos\theta\}$, where $\cosh\sigma$ goes from 1 to ∞, $\cos\theta$ goes from -1 to $+1$, and ϕ goes from 0 to 2π — the surface $\sigma=$ constant is a prolate spheroid and the surface $\theta=$ constant is a hyperboloid of revolution of two sheets. The Green's function can be written as follows (§245 of Hobson 1931):

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{a}\sum_{\ell=0}^{\infty}(2\ell+1)\sum_{m=-\ell}^{\ell}(-1)^m\left[\frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)}\right]^2 P_\ell^m(\cos\theta)P_\ell^m(\cos\theta')P_\ell^m(\cosh\sigma_<)Q_\ell^m(\cosh\sigma_>)\ \mathrm{e}^{im(\phi-\phi')}. \tag{36}$$

Consequently, the following two expressions must be valid *addition theorems*:

$$\sum_{\ell=|m|}^{\infty}(2\ell+1)\left[\frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)}\right]^2 P_\ell^m(\cos\theta)P_\ell^m(\cos\theta')P_\ell^m(\cosh\sigma_<)Q_\ell^m(\cosh\sigma_>)=\frac{Q_{m-1/2}(\chi)}{A_m(\sigma,\sigma',\theta,\theta')}, \tag{37}$$

and

$$\sum_{\ell=|m|}^{\infty}(2\ell+1)\left[\frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)}\right]^2 P_\ell^m(\cos\theta)P_\ell^m(\cos\theta')P_\ell^m(\cosh\sigma_<)Q_\ell^m(\cosh\sigma_>)=\frac{P^m_{-1/2}(\chi/\sqrt{\chi^2-1})}{B_m(\sigma,\sigma',\theta,\theta')}, \tag{38}$$

where

$$\chi=\frac{\cosh^2\sigma+\cosh^2\sigma'-\sin^2\theta-\sin^2\theta'-2\cosh\sigma\cosh\sigma'\cos\theta\cos\theta'}{2\sinh\sigma\sinh\sigma'\sin\theta\sin\theta'}, \tag{39}$$

$A_m\equiv\pi(-1)^m\sqrt{\sinh\sigma\sinh\sigma'\sin\theta\sin\theta'}$, and, $B_m\equiv\pi^{-1/2}\Gamma(m+1/2)\sqrt{2(\chi^2-1)^{1/2}\sinh\sigma\sinh\sigma'\sin\theta\sin\theta'}$.

5.2. Oblate spheroidal coordinates

In oblate spheroidal coordinates — $\mathbf{x}:\{a\cosh\sigma\sin\theta\cos\phi, a\cosh\sigma\sin\theta\sin\phi, a\sinh\sigma\cos\theta\}$, where $\sinh\sigma$ goes from 0 to ∞, $\cos\theta$ goes from -1 to $+1$, and ϕ goes from 0 to 2π — the surface $\sigma=$ constant is an oblate spheroid and the surface $\theta=$ constant is a hyperboloid of revolution of one sheet. With a Green's function (§251 of Hobson 1931) similar to eq. (36) except for the replacement of cosh by $i\sinh$, addition theorems analogous to eqs. (37) and (38) follow, with the same replacement of cosh by $i\sinh$ and an interchange of sinh and cosh in eq. (39).

5.3. Parabolic coordinates

In parabolic coordinates — $\mathbf{x}:\{\lambda\mu\cos\phi, \lambda\mu\sin\phi, (\lambda^2-\mu^2)/2\}$, where λ goes from 0 to ∞, μ goes from 0 to ∞, and ϕ goes from 0 to 2π — the surfaces $\lambda=$ constant and $\mu=$ constant are both paraboloids of revolution. The Green's function can be written as follows (eq. (10.3.68) in Morse & Feshbach 1953):

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{2}{a}\int_0^\infty dkk\sum_{m=-\infty}^{\infty}J_m(k\lambda)J_m(k\lambda')I_m(k\mu_<)K_m(k\mu_>)\ \mathrm{e}^{im(\phi-\phi')}. \tag{40}$$

Consequently, the following two expressions must be valid *definite integrals*:

$$\int_0^\infty dkk\ J_m(k\lambda)J_m(k\lambda')I_m(k\mu_<)K_m(k\mu_>)=\frac{Q_{m-1/2}(\chi)}{2\pi\sqrt{\lambda\lambda'\mu\mu'}}, \tag{41}$$

and

$$\int_0^\infty dkk\ J_m(k\lambda)J_m(k\lambda')I_m(k\mu_<)K_m(k\mu_>)=\frac{P^m_{-1/2}(\chi/\sqrt{\chi^2-1})}{E_m(\lambda,\lambda',\mu,\mu')}, \tag{42}$$

where

$$\chi=\frac{4\lambda^2\mu^2+4\lambda'^2\mu'^2+(\lambda^2-\lambda'^2+\mu'^2-\mu^2)^2}{8\lambda\lambda'\mu\mu'}, \tag{43}$$

and, $\quad E_m\equiv 2^{3/2}\pi^{-1/2}(-1)^m\Gamma(m+\tfrac{1}{2})\sqrt{(\chi^2-1)^{1/2}\lambda\lambda'\mu\mu'}$.

6. Two fourth-order rotational Laplace systems

In this section we present necessary implications of the main results (16) and (36) in the bispherical and toroidal coordinate systems.

6.1. Bispherical coordinates

In bispherical coordinates — $\mathbf{x} : \{a \sin\theta \cos\phi/(s-\tau), a\sin\theta\sin\phi/(s-\tau), a\sinh\sigma/(s-\tau)\}$, where $s \equiv \cosh\sigma$ goes from 1 to ∞, $\tau \equiv \cos\theta$ goes from -1 to $+1$, and ϕ goes from 0 to 2π — the surfaces $\sigma =$ constant are spheres and the surface $\theta =$ constant is a spindle-shaped cyclide. According to eq. (10.3.74) of Morse & Feshbach (1953),

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{a}\big[(s-\tau)(s'-\tau')\big]^{1/2} \sum_{\ell=0}^{\infty} \mathrm{e}^{-(\ell+\frac{1}{2})(\sigma_> - \sigma_<)} \sum_{m=-\ell}^{\ell} \frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)} P_\ell^m(\cos\theta) P_\ell^m(\cos\theta')\, \mathrm{e}^{im(\phi-\phi')}, \tag{44}$$

Consequently, the following two expressions must be valid *addition theorems*:

$$\sum_{\ell=|m|}^{\infty} \frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)}\, \mathrm{e}^{-(\ell+\frac{1}{2})(\sigma_> - \sigma_<)} P_\ell^m(\cos\theta) P_\ell^m(\cos\theta') = \frac{Q_{m-1/2}(\chi)}{\sqrt{\sin\theta\sin\theta'(s-\tau)(s'-\tau')}}, \tag{45}$$

and

$$\sum_{\ell=|m|}^{\infty} \frac{\Gamma(\ell-m+1)}{\Gamma(\ell+m+1)}\, \mathrm{e}^{-(\ell+\frac{1}{2})(\sigma_> - \sigma_<)} P_\ell^m(\cos\theta) P_\ell^m(\cos\theta') = \frac{P^m_{-1/2}(\chi/\sqrt{\chi^2-1})}{F_m(\sigma,\sigma',\theta,\theta')}, \tag{46}$$

where

$$\chi = \frac{\sin^2\theta(s'-\tau')^2 + \sin^2\theta'(s-\tau)^2 + \big[(s'-\tau')\sinh\sigma - (s-\tau)\sinh\sigma'\big]^2}{2\sin\theta\sin\theta'(s-\tau)(s'-\tau')}, \tag{47}$$

and $\quad F_m \equiv 2^{1/2}\pi^{-3/2}(-1)^m\Gamma(m+\frac{1}{2})\sqrt{(\chi^2-1)^{1/2}\sin\theta\sin\theta'(s-\tau)(s'-\tau')}.$

6.2. Toroidal coordinates

In toroidal coordinates — $\mathbf{x} : \{a\sinh\sigma\cos\phi/(s-\tau), a\sinh\sigma\sin\phi/(s-\tau), a\sin\psi/(s-\tau)\}$, where $s \equiv \cosh\sigma$ goes from 1 to ∞ and both ϕ and ψ go from 0 to 2π — the surface $\sigma =$ constant is a torus (which is a cyclide) and the surface $\psi =$ constant is a spherical bowl. By inserting eq. (8.795.2) from Gradshteyn & Ryzhik (1994) into the single summation expression for the reciprocal distance between two points in toroidal coordinates as given by Bateman (1959, §10.3, eq. 26) we derive,[2]

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{a\pi}\big[(s-\tau)(s'-\tau')\big]^{1/2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{in(\psi-\psi')} \sum_{m=-\infty}^{\infty} (-1)^m \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} \times P^m_{n-\frac{1}{2}}(\cosh\sigma_<) Q^m_{n-\frac{1}{2}}(\cosh\sigma_>) \mathrm{e}^{im(\phi-\phi')}. \tag{48}$$

Consequently, the following two expressions must be valid *addition theorems*:

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} P^m_{n-\frac{1}{2}}(\cosh\sigma_<)\; Q^m_{n-\frac{1}{2}}(\cosh\sigma_>)\, \mathrm{e}^{in(\psi-\psi')} = \frac{(-1)^m Q_{m-1/2}(\chi)}{\sqrt{\sinh\sigma\sinh\sigma'(s-\tau)(s'-\tau')}}, \tag{49}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} P^m_{n-\frac{1}{2}}(\cosh\sigma_<)\; Q^m_{n-\frac{1}{2}}(\cosh\sigma_>)\, \mathrm{e}^{in(\psi-\psi')} = \frac{P^m_{-1/2}(\chi/\sqrt{\chi^2-1})}{G_m(\sigma,\sigma',\psi,\psi')}, \tag{50}$$

where

$$\chi = \coth\,\sigma\coth\sigma' - \operatorname{csch}\sigma\operatorname{csch}\sigma'\cos(\psi-\psi'), \tag{51}$$

and $\quad G_m \equiv \pi^{-3/2}\Gamma(m+\frac{1}{2})\sqrt{2(\chi^2-1)^{1/2}\sinh\sigma\sinh\sigma'(s-\tau)(s'-\tau')}.$

[2] There is an obvious typographical error in the argument of a gamma function in eq. (10.3.81) of Morse & Feshbach (1953). Care must also be taken in using their eqs. (10.3.53), (10.3.63), and (10.3.81) because of a phase convention that Morse & Feshbach (p. 1286) adopt regarding associated Legendre functions (see §10.3 in Morse & Feshbach; Moon & Spencer 1961a). We have adopted Hobson's (1931) notation for the associated Legendre functions $P^\mu_\nu(z)$ and $Q^\mu_\nu(z)$ so as to keep imaginary elements out of our results. There is also an error in Morse & Feshbach's rendering of the Heine identity, eq. (19), in two separate locations: eq. (10.3.79) and the last equation that appears in their chap. 10. In both locations, the Neumann factor $\epsilon_m = 2 - \delta_{m0}$, is missing. The Neumann factor is a natural consequence of going from a complex exponential notation (Fourier series) to a cosine representation (purely real) of a series.

7. Conclusion

There are many problems in physics and astrophysics whose solution requires an accurate determination of Newtonian and/or Coulomb potentials. Whether one attempts to derive the solution to such problems using analytical or numerical techniques, more often than not the potential function is expressed in terms of a Green's function expansion in spherical harmonics. Here we have demonstrated that, in a number of different rotationally invariant coordinate systems which separate Laplace's equation, the Green's function can be written as an azimuthal Fourier series expansion with toroidal functions as expansion coefficients. Along the way, we also have derived a number of *addition theorems* and *definite integrals* that are likely also to have practical uses outside of the context of potential theory. Our unification in terms of zero-order toroidal functions of the second kind of several rotationally invariant coordinate systems suggests that toroidal functions may represent a basis set that is better suited for general studies of nonaxisymmetric mass/charge distributions than, for example, spherical harmonics. In the future, we intend to investigate the Lie group theoretical implications of these results (Srivastava & Manocha 1984; Miller 1968, 1972, 1977).

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References

Abramowitz, M., Stegun, I.A.: 1965, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York
Bateman, H.: 1959, *Partial Differential Equations of Mathematical Physics*, Cambridge University Press, New York
Binney, J., Tremaine, S.: 1987, *Galactic Dynamics*, Princeton University Press
Bôcher, M.: 1894, *Die Reihenentwickelungen der Potentialtheorie*, Leipzig
Boyer, C.P., Kalnins E.G., Miller jr., W.: 1976, Nagoya Math. J. 60, 35
Cohl, H.S., Tohline, J.E.: 1999, Astrophys. J. 527, 86
Cohl, H.S., Rau, A.R.P., Tohline, J.E., Browne, D.A., Cazes, J.E., Barnes, E.: 2000, *Alternative Fourier Expansions for Inverse Square Law Forces*, in preparation
Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: 1953, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York
Gradshteyn, I.S., Ryzhik, I.M.: 1994, *Table of Integrals, Series, and Products*, Academic Press, New York
Gundlach, J.H., Merkowitz, S.M.: 2000, Phys. Rev. Lett. 85, 14, 2869
Heine, E.: 1881, *Handbuch der Kugelfunktionen, Theorie und Anwendungen*, Physica-Verlag, Wuerzburg, p. 1961
Hobson, E. W.: 1931, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press
Jackson, J.D.: 1975, *Classical Electrodynamics*, John Wiley & Sons, New York
Magnus, W., Oberhettinger, F., Soni, R.P.: 1966, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer-Verlag, New York
Miller jr., W.: 1968, *Lie Theory and Special Functions*, Academic Press, New York
Miller jr., W.: 1972, *Symmetry Groups and Their Applications*, Academic Press, New York
Miller jr., W.: 1977, *Symmetry and Separation of Variables*, Addison-Wesley Publishing Company, London
Moon, P., Spencer, D.E.: 1961a, *Field Theory for Engineers*, D. Van Nostrand Company, Princeton
Moon, P., Spencer, D.E.: 1961b, *Field Theory Handbook: Including Coordinate Systems, Differential Equations and Their Solutions*, Springer-Verlag, Berlin
Morse, P., Feshbach, H.: 1953, *Methods of Theoretical Physics*, McGraw-Hill, New York
Srivastava, H.M., Manocha, H.L.: 1984, *A Treatise on Generating Functions*, John Wiley & Sons, New York
Watson, G.N.: 1944, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, New York
Whittaker, E.T., Watson, G.N.: 1943, *A Course of Modern Analysis*, Cambridge University Press, New York

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