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SYNTHESIS OF PERFECT SPRING BALANCERS WITH HIGHER-ORDER ZERO-FREE-LENGTH SPRINGS

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ABSTRACT

Static balancing is a well-known technique in mechanism synthesis to achieve equilibrium throughout the range of motion, for instance to eliminate gravity from the equations of motion. Another application of static balancing is in spring-to-spring balancing where the influence of *n* springs on the mechanism behavior (*e.g.* input torque) are balanced by *m* other springs (*n* and *m* both non-zero positive integers). In this category of balanced mechanism, design methodology and examples exist based on zero-free-length springs, *i.e.* linear extension springs in which the force is proportional to the length of the spring, rather than to its elongation. The present paper will present for the first time the design of perfect spring-to-spring balancers with higher-order zero-free-length springs, i.e. springs in which the force is proportional to a (positive integer) power of its length. A general approach will be given together with four new mechanisms incorporating springs ranging from two third-order springs in the simplest example, to four equal thirteenth order springs plus one first order spring in the most complex example.

INTRODUCTION

Static balancing is a technique to provide a mechanism with static equilibrium throughout its range of motion. Fundamentally this is achieved by guaranteeing a constant potential energy. Consequently, no minimum potential energy configuration exists, and neutral stability is obtained. Practically, it means that any configuration of the mechanism can be maintained without any actuator effort [1]. Static balancing has great application in gravity equilibration, but also spring-to-spring balancing examples exist, such as for instance in the Elastor [2] where the buckling of a compression spring is balanced by another spring, or a hand prosthesis in which the elastic counteraction of the cosmetic covering is balanced by a compensation spring [3]. In the latter example, only partial balancing is achieved due to the strong non-linearity of the characteristic of the cosmetic covering. This is the problem that is addressed in this paper.

Many publications on perfect static balancing with linear zero-free-length springs exist [e.g. 4-7]. In [1] a framework was put up for the conceptual design of this class of mechanisms, based on the modification of a particular arrangement of a statically balanced mechanism, named the basic spring force balancer. In the same publication an overview of the current (practical) possibilities of balancing with zero-free-length springs is given, and a vast number of possible mechanisms that can be derived by a number of logical rules to modify, simplify or expand the basic spring force balancer.

This paper will argue that the underlying principle for the basic spring force balancer and its modifications can be explained by one principle, termed the cosine mechanism. However, the cosine mechanism cannot explain one new mechanism that is proposed in the present paper. To resolve this, a second (theoretical) principle is introduced which

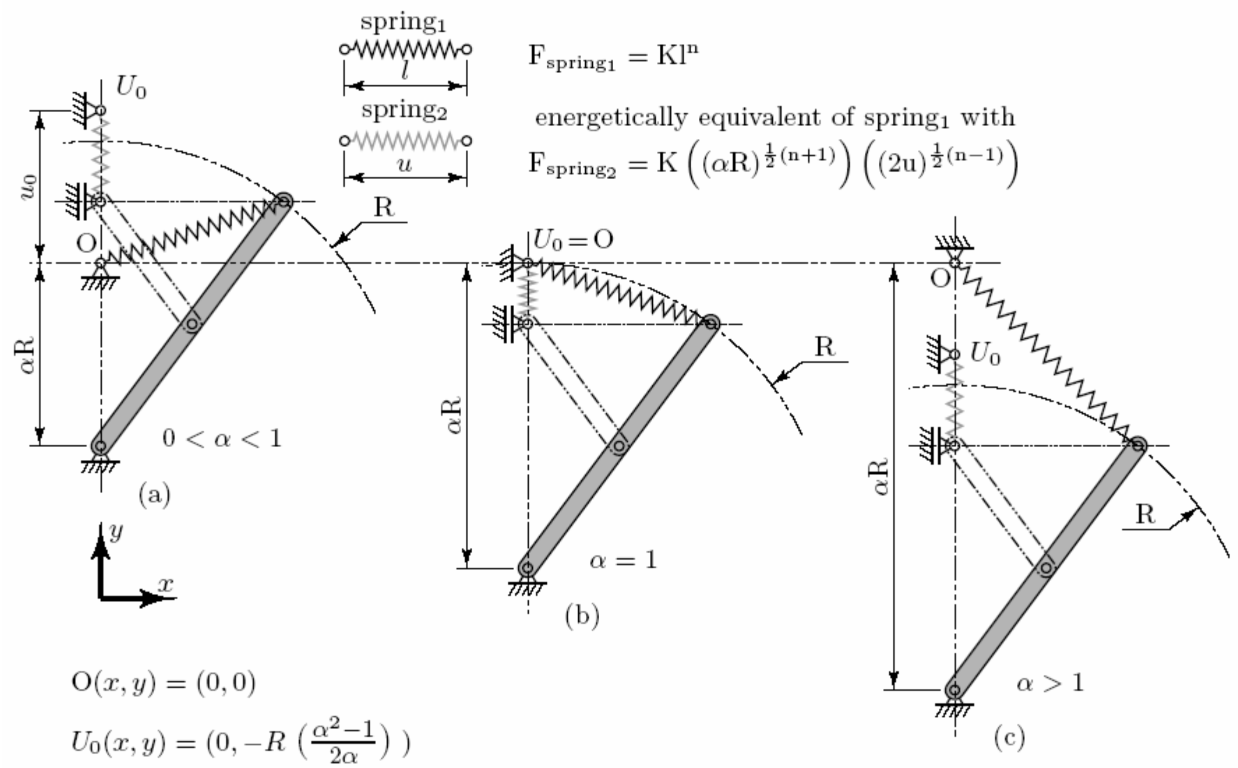

Figure 1. The cosine mechanism. The two springs are energetically equivalent, which means the mechanism acts the same if spring 1 were to be removed and spring 2 added, and vice versa. For different ranges of α namely (a) 0 < α < 1, (b) α = 1, (c) α > 1.

combined with the cosine mechanism can explain all the currently known linkage-type static balancing mechanisms, including the new ones.

Up to now no perfect solutions are known to balance non-linear springs. This paper shows a variety of possible ways to perfectly balance non-linear springs, and aims to investigate their underlying principles. However, this paper by no means claims to show all new possible solutions.

The paper is structured as follows. To extend ideal spring balancing to non-linear springs the term generalized ideal spring is proposed. Subsequently, the cosine mechanisms will be discussed, and a method is proposed to balance generalized ideal springs. This method still has the basic spring force balancer as it main pillar, but three procedures are newly added that tend to linearize the original non-linear force displacement characteristics of generalized ideal springs. These three procedures will be discussed, and new mechanisms will be presented.

GENERALIZED IDEAL SPRINGS

In this paper, the term ideal spring is used in the same way as in [1], namely as a tension spring with zero free length, constant spring rate, limitless strain, and forces acting along its centerline. In other terms, the force-displacement characteristic of an ideal spring is defined as:

$$F = Kx \tag{1}$$

where K is the constant stiffness and where x is the length of the spring (not the elongation) under the influence of force F. Consequently, $F_{x=0} = 0$ with $l_0=0$, where l_0 is the free length, defined as the distance in the force-length diagram from the origin to the intersection of the (extension of the) characteristic and the length-axis. Thus, the free length l_0 of a spring with initial tension F_0 is smaller than the initial length L0, and is calculated as $l_0= L_0- F_0/K$.

To extend spring force balancing to non-linear springs, the term generalized ideal spring is proposed. A generalized ideal spring is an extended version of an ideal spring. The force displacement characteristic of a generalized ideal spring is defined as:

$$F = Kx^n \tag{2}$$

where again x is the actual length of the spring under the influence of force F, and where K is the stiffness, while still having zero free length, $l_0=0$. Therefore a generalized spring still has $F_{x=0}=0$ for $n>0$. For $n\leq 0$ this means that $F_{x=0}\neq 0$. To limit the possible non-linear springs only generalized springs are considered for $n\geq 0$. The superposition principle for forces

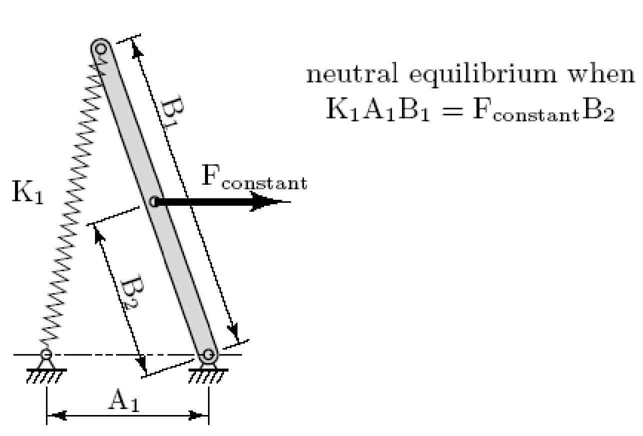

Figure 2. Most general form of the most widely used principle to perfectly balance an ideal spring and a constant force, the basic constant force balancer.

allows the generalized springs to be added up. Integer values for n are therefore of specific interest. A series expansion of a certain force displacement characteristic up to a certain n could then be accurately balanced. The remaining part is assumed negligible, although this may not always be the case. In such a case the solution has become an approximation.

COSINE MECHANISM

The cosine mechanism can be seen in Fig. 1. Projecting the y position of the moving attachment point of the spring onto the line x=0 is the driving principle and therefore the name cosine mechanism is chosen. The cosine mechanism allows for representing the energy stored in a certain (non-linear) spring by another (non-linear) spring. If the energy is the same for all positions the spring can be considered energetically equivalent and this allows for one spring that is physically in a mechanism to be represented by the other (virtual) spring. The two springs do not have the same type of force displacement (i.e. constant, linear or non-linear) characteristics. The advantage is that the virtual spring can have a known solution to balance its type of force displacement characteristic whereas the other spring does not.

Now consider Fig. 1, a cosine mechanism with spring 1, for which:

$$F_{spring1} = Kl^n \tag{3}$$

In order to find the characteristic for spring 2 such that spring 2 is energetically equivalent to spring 1, observe that the arc trajectory of the moving end point of the link is given by

$$x^2 + (y + \alpha R)^2 = R^2 \tag{4}$$

or

$$x^2 = R^2 - (y + \alpha R)^2 \tag{5}$$

The length of spring 1, l, is given by

$$l = \sqrt{x^2 + y^2} = \sqrt{R^2 - (y + \alpha R)^2 + y^2} \tag{6}$$

When defining

$$y = u_0 - u \tag{7}$$

where u_0 is defined as

$$u_0 \stackrel{\text{def}}{=} -R\left(\frac{\alpha^2 - 1}{2\alpha}\right) \tag{8}$$

and u is the length of spring 2. By defining u_0 in this way it is now possible to conveniently write the length of spring 2, u, as a function of the length of spring 1, l. Combining Eqs. (6), (7) and (8) yields:

$$l = \sqrt{2\alpha R u} \tag{9}$$

Using the principle of virtual energy, the following must hold for $F_{spring2}$ to be energetically equivalent for any position to $F_{spring1}$:

$$F_{spring_2} = \left(\frac{dl}{du}\right) F_{spring_1} \tag{10}$$

The force of spring 1, $F_{spring1}$, is given by

$$F_{spring_1} = Kl^n = K(\sqrt{2\alpha R u})^n \tag{11}$$

Since, using equation (9),

$$\frac{dl}{du} = \sqrt{\frac{\alpha R}{2u}} \tag{12}$$

The force of spring2, $F_{spring2}$, therefore has to be

$$F_{spring_2} = \sqrt{\frac{\alpha R}{2u}} K(\sqrt{2\alpha R u})^n \tag{13}$$

Rewriting finally yields

$$F_{spring_2} = K(\alpha R)^{\left(\frac{1}{2}(n+1)\right)} (2u)^{\left(\frac{1}{2}(n-1)\right)} \tag{14}$$

when it is attached at the point U_0 for which the coordinates are

$$U_0(x, y) = \left(0, -R\left(\frac{\alpha^2 - 1}{2\alpha}\right)\right) \tag{15}$$

Basic Constant Force Balancer

The basic constant force balancers (BSFB, Fig. 2, [1]) is a special form of the cosine mechanism. Choosing an ideal spring for spring 1 ($F_{spring1} = Kl^1$) means the force of spring 2 becomes a constant force. Substituting $n=1$ in Eq. (14) results in:

$$F_{spring2} = K(\alpha R) \tag{16}$$

This constant force can be compensated by a constant force in opposite direction, something that can easily be achieved, for instance by gravity or another basic constant force balancer. Consider Fig. 2. With $(\alpha R) = A_1$ and $R = B_1$ it allows for balancing a constant force and an ideal spring when

$$K_1 A_1 B_1 = F_{constant} B_2 \tag{17}$$

When $A_1 = B_1$ allows for using the spring for both positive and negative values of its length.

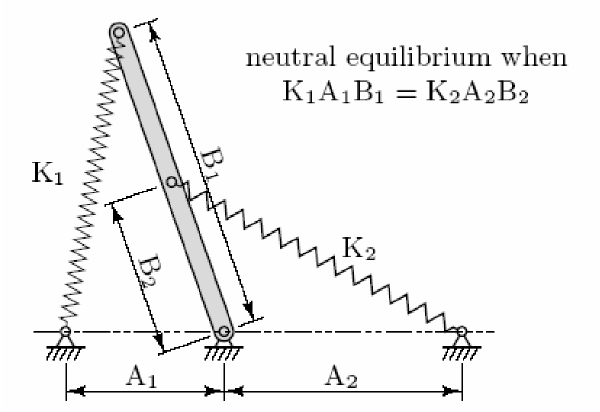

Figure 3. Most general form of the most widely used principle to perfectly balance two linear ideal springs, the basic force spring balancer.

Basic spring force balancer

When combining two basic constant force balancers two ideal springs can be balanced. This results in the basic spring force balancer (BSFB, Fig. 3, [1]). When the two mechanisms generate the same force but in opposite direction they will be in balance throughout their stroke, which gives:

$$\mathrm{K_1A_1B_1 = K_2A_2B_2} \tag{18}$$

When $A_1=B_1$ and/or $A_2=B_2$ means spring 1 and/or spring 2 respectively can be used for negative values of their length.

Thus, the mechanism can be regarded as a combination of two basic constant force balancers that both generate a constant force, through use of a cosine mechanism, which are equal but opposite to one another at a certain (virtual) point in the mechanism.

NON-LINEAR SPRING BALANCER SYNTHESIS

Three procedures will be discussed namely:

- *Symmetric Design*. By designing the total mechanism symmetrical the even part of the force displacement characteristic is reduced or cancelled.
- *Adding Two Semi-Identical Characteristics*. By adding two semi-identical (non-linear) force displacement characteristics the combined force displacement characteristic can be perfectly or more like a linear ideal spring.
- *Cosine Mechanism for Generalized Ideal Springs*. The cosine mechanism can be used to approximately halve the power (n) of a generalized ideal spring, thus making it more like a (linear) ideal spring.

The power of these methods lies in combining the three, allowing for two (or more) non-linear springs to behave as a linear spring. Once the generalized springs are thus ”linearized” by a certain mechanism it can be balanced by a linear spring with negative stiffness. For this linear spring with negative stiffness, through use of the basic spring force balancer, a copy of itself or a configuration of different generalized ideal springs can be used. This then allows for non-linear to non-linear spring balancing.

Symmetric Design

By keeping the design of the compensating mechanism symmetrical for the system of forces a lot of terms (the even part) of a series expansion are reduced or eliminated. An important division of any series expansion of a random force displacement characteristic is made into an odd and even part. When the mechanisms system of forces is symmetrical by itself with respect to x=0 (*i.e.* $F_x=-F_{-x}$) the even parts are eliminated.

If the mechanisms system of forces is not symmetrical it can be made symmetrical by adding a mirrored copy making the whole mechanism symmetric again. Since exact symmetry for the system of forces is subject to production accuracy of the geometry a small even part will remain.

Odd and Even Part

A general form to write any force displacement characteristic is

$$F^{error} = a_0 + a_1(\frac{x}{D}) + a_2(\frac{x}{D})^2 + a_3(\frac{x}{D})^3 + a_4(\frac{x}{D})^4 + a_5(\frac{x}{D})^5 + \ldots = \sum_{n=0}^{\infty} a_n(\frac{x}{D})^n \tag{19}$$

where x is the displacement and D is the maximum stroke. This can be divided into two parts, the odd part:

$$F_{odd} \stackrel{\text{def}}{=} a_1(\frac{x}{D}) + a_3(\frac{x}{D})^3 + a_5(\frac{x}{D})^5 + \ldots \tag{20}$$

and the even part:

$$F_{even} \stackrel{\text{def}}{=} a_0 + a_2(\frac{x}{D})^2 + a_4(\frac{x}{D})^4 + \ldots \tag{21}$$

The big difference is that at $x = 0 \ldots -\mathrm{D}$ and $x = 0 \ldots +\mathrm{D}$ the odd forces have opposite signs whereas the even forces have the same signs. Therefore an exact copy of the total mechanism but mirrored around $x = 0$ will have

$$(F_{\text{odd}})_{\text{mirrored}} = F_{\text{odd}} \tag{22}$$

$$(F_{\text{even}})_{\text{mirrored}} = -F_{\text{even}} \tag{23}$$

Combining with a Mirrored Copy

By combining two compensating mechanisms, one an exact copy of the other but mirrored with respect to x=0, the resulting mechanism has no even part and double the odd part of a single version. Even with a (large) misalignment the even part vanishes (when x^*=0 is redefined with respect to the line of symmetry). The even terms are responsible for a small extra odd part. This extra odd part will purely (for x^2) or mainly (x^4

Table 1 : Combined force displacement characteristic of two misaligned even or odd generalized springs of opposite sign.

$F_{1-2} = A\left((\frac{x}{D}) + \alpha_{ecc}\right)^n - A\left((\frac{x}{D}) - \alpha_{ecc}\right)^n$			
n=even		n=odd	
2	$4A\alpha_{ecc}(\frac{x}{D})$	1	$2A\alpha_{ecc}$
4	$8A\alpha_{ecc}^3\left((\frac{x}{D}) + \frac{1}{\alpha_{ecc}^2}(\frac{x}{D})^3\right)$	3	$6A\alpha_{ecc}\left((\frac{x}{D})^2 + \frac{1}{3}\alpha_{ecc}^2\right)$
6	$12A\alpha_{ecc}^5\left((\frac{x}{D}) + \frac{10}{3\alpha_{ecc}^2}(\frac{x}{D})^3 + \frac{1}{\alpha_{ecc}^4}(\frac{x}{D})^5\right)$	5	$20A\alpha_{ecc}^3\left((\frac{x}{D})^2 + \frac{1}{10}\alpha_{ecc}^2 + \frac{1}{2\alpha_{ecc}^2}(\frac{x}{D})^4\right)$

Table 2 : Combined force displacement characteristic of two misaligned odd or even generalized springs of the same sign.

$F_{1+2} = A\left((\frac{x}{D}) + \alpha_{ecc}\right)^n + A\left((\frac{x}{D}) - \alpha_{ecc}\right)^n$			
n=odd		n=even	
1	$2A(\frac{x}{D})$	0	$2A$
3	$6A\alpha_{ecc}^2\left((\frac{x}{D}) + \frac{1}{3\alpha_{ecc}^2}(\frac{x}{D})^3\right)$	2	$2A\left((\frac{x}{D})^2 + \alpha_{ecc}^2\right)$
5	$10A\alpha_{ecc}^4\left((\frac{x}{D}) + \frac{2}{\alpha_{ecc}^2}(\frac{x}{D})^3 + \frac{1}{5\alpha_{ecc}^4}(\frac{x}{D})^5\right)$	4	$12A\alpha_{ecc}^2\left((\frac{x}{D})^2 + \frac{1}{6}\alpha_{ecc}^2 + \frac{1}{6\alpha_{ecc}^2}(\frac{x}{D})^4\right)$

or higher) consist of a linear term. The odd terms with a misalignment produce a small odd error again. This is discussed in more detail and with a broader application in the next main section.

Purely Odd Force Displacement Characteristics

When the system of forces of the compensating mechanism is symmetric around x=0 for every position, the force displacement characteristic will be purely odd (i.e. have no even part). This can be easily understood when such a mechanism is regarded as two halves added together. Since the second half is an exact mirrored copy of the first there is no possible remaining even part in the force displacement characteristic. This has interesting implications. For instance this also includes cases were the springs used have purely even force displacement characteristics (for instance $F_{\text{spring}}=a_2(x/D)^2$) but are used in a mechanism that has a symmetric system of forces with respect to x=0. This raises the following unanswered question: does a purely odd force mean the mechanism is therefore symmetric? A search was done for an example of a mechanism that would disprove this; a purely odd force that is produced by a non-symmetrical mechanism. No such example was found but no further attempt was made to prove this. What remains is a mere advice that in almost any case this seems to be true.

Symmetry of the System of Forces vs. Geometric Symmetry

The emphasis in the previous sections is on symmetry of the system of forces and not geometry. The reason is that an asymmetric geometry can still lead to a symmetric system of forces. Another unanswered question arises: does a symmetric geometry always have a symmetric system of forces? Again a search is done to an example of a mechanism that would disprove this. No such example was found and no further attempt was made to prove this. What again remains is a mere advice that in almost any case this seems to be true.

Adding Two Semi-Identical Characteristics

Pistecky [8] mentions that adding two non-linear force-displacement characteristic can give a surprisingly near-linear force displacement characteristic. The term semi-identical is used for force displacement characteristics for non linear springs of same power (n) and absolute amplitude (or stiffness), but of possible different sign of amplitude. If two (semi-)identical force displacement characteristics that have an opposite eccentricity with respect to x=0 are added, interesting properties are the result. This is further investigated which results in Tables 1 and 2.

The form in which the semi-identical force displacement characteristics are written results from the following.

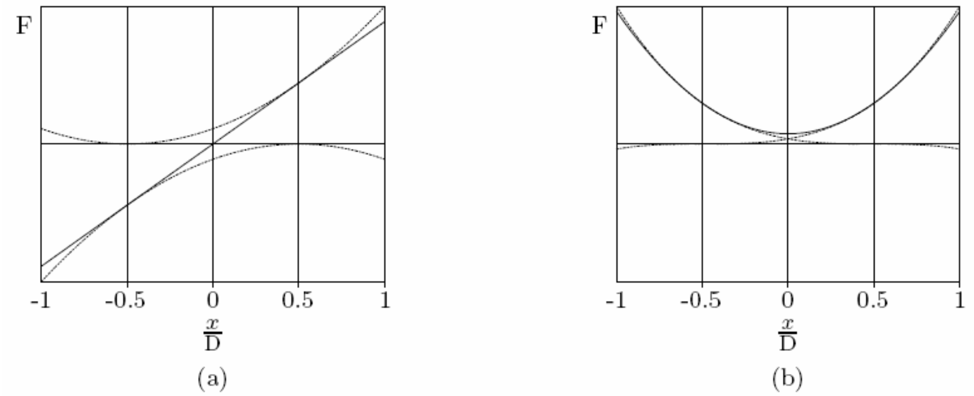

Figure 4: Qualitative graphs of the combined force displacement characteristic of two mis-aligned (a) even or (b) odd generalized springs of opposite sign.

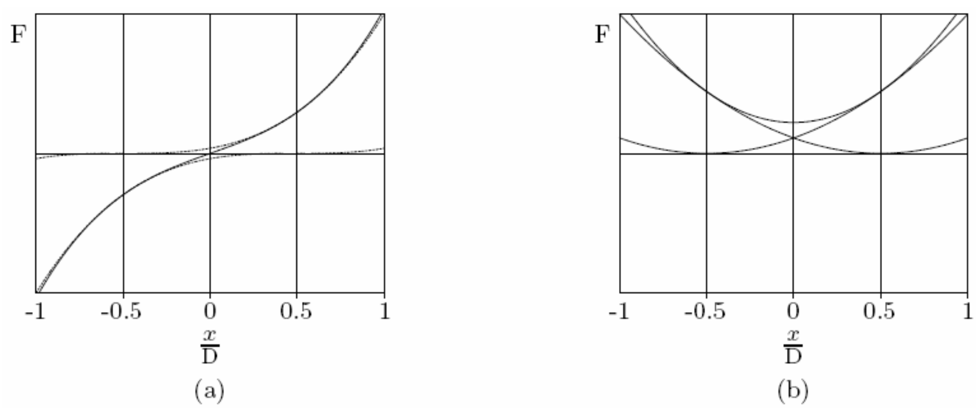

Figure 5: Qualitative graphs of the combined force displacement characteristic of two mis-aligned (a) odd or (b) even generalized springs of the same sign.

Displacements are considered over a stroke $x = -D \ldots 0 \ldots D$. A generalized ideal spring is used with

$$F_{\mathrm{spring}} = A\left(\frac{x^*}{D}\right)^n \tag{24}$$

and

$$F_{\mathrm{spring}} = 0 \quad \text{at} \quad \frac{x}{D} = -\alpha_{\mathrm{ecc}} \tag{25}$$

the resulting force displacement characteristic will be

$$F_1 = A\left(\left(\frac{x}{D}\right) + \alpha_{\mathrm{ecc}}\right)^n \tag{26}$$

Similarly a second (semi-identical) force displacement characteristic is created which is

$$F_2 = A\left(\left(\frac{x}{D}\right) - \alpha_{\mathrm{ecc}}\right)^n \tag{27}$$

To F_1, a negative F_2 and a positive F_2, respectively, are added (since forces cannot not physically be subtracted), giving

$$F_{1-2} = F_1 - F_2 = A\left(\left(\frac{x}{D}\right) + \alpha_{\mathrm{ecc}}\right)^n - A\left(\left(\frac{x}{D}\right) - \alpha_{\mathrm{ecc}}\right)^n \tag{28}$$

$$F_{1+2} = F_1 + F_2 = A\left(\left(\frac{x}{D}\right) + \alpha_{\mathrm{ecc}}\right)^n + A\left(\left(\frac{x}{D}\right) - \alpha_{\mathrm{ecc}}\right)^n \tag{29}$$

F_{1-2} and F_{1+2} are evaluated for positive integer values and rewritten to a simpler form, which is then put in Tables 1 and 2. Qualitative graphs of the two semi-identical force displacement characteristics and their combined resulting force displacement characteristic are given in Figs. Fig. 4 and Fig. 5. The figures and tables are ordered in such a way to show the similarities in the combined force displacement characteristic between different values for n and adding or ”subtracting” of the semi-identical force-displacement characteristics.

The direct applications of these tables and figures are:

- Adding the force displacement characteristics of two even generalized springs with stiffness of opposite sign, see Fig. 4a, or adding the force displacement characteristics of two odd generalized springs with stiffness of the same sign, see Fig. 5a, will result in a combined force-displacement characteristic that is purely odd and consist of a purely or largely linear term.
- Adding two force displacement characteristics resulting in a combined force displacement characteristic of the form

Table 3. Force displacement characteristics of equivalent spring 2, see Fig. 1, for various functions of spring1, using a cosine mechanism.

n	F_{spring_1}	F_{spring_2}
	Kl^n	$K\left((\alpha R)^{\frac{1}{2}(n+1)}\right)\left((2u)^{\frac{1}{2}(n-1)}\right)$
7	Kl^7	$8K\alpha^4R^4u^3$
5	Kl^5	$4K\alpha^3R^3u^2$
3	Kl^3	$2K\alpha^2R^2u$
2	Kl^2	$K\sqrt{2\alpha^3R^3u}$
1	Kl	$K\alpha R$
-1	$K\frac{1}{l}$	$K\frac{1}{u}$
-2	$K\frac{1}{l^2}$	$\frac{K}{2\sqrt{2}\sqrt{\alpha R}}\frac{1}{u\sqrt{u}}$
-3	$K\frac{1}{l^3}$	$\frac{K}{4\alpha R}\frac{1}{u^2}$
-5	$K\frac{1}{l^5}$	$\frac{K}{8\alpha^2R^2}\frac{1}{u^3}$

$A((x/D)+\alpha_{ecc})^2$- $A((x/D)-\alpha_{ecc})^2$, see Table 1, or of the form $A((x/D)+\alpha_{ecc})^1$+$A((x/D)-\alpha_{ecc})^1$, see Table 2, can be rewritten to a force-displacement characteristic of a perfectly linear ideal spring. For the first this is somewhat surprising and is mentioned in [8], for the second (the already linear spring) it is obvious.

- Table 1 and figure Fig. 4a show that, even with a large misalignment, adding two compensating mechanisms, one a mirrored copy of the other, the even part vanishes. The combined even terms produce a small extra odd part. This extra odd part will purely (for x^2) or largely consist of a linear term, see Table 1 for even values of *n*. The odd terms of a single version combined also produce a small odd error, see Table 2 for odd values of *n*. It also shows the qualitative similar combined result between certain odd en even terms (*i.e.* the pair *n*=1 and *n*=2 or the pair *n*=3 and *n*=4 give similar combined results).
- It shows that misalignment of a positive and a negative (linear) ideal spring results in a constant force, see Table 1 for *n*=1.

A few other points of interest are mentioned that do not directly result in a practical design but are worth mentioning.

- The combined force displacement characteristic can only be equal to zero for any value of *x* for F_{1-2}. Apart from the trivial case when α_{ecc}=0 (subtracting the exact same function) there also the less trivial case when *n*=0 (two constant forces of same value but opposite sign). The basic spring force balancer is based on this. The fact that this combination is the only less trivial case where force displacement characteristic is zero for all *x* might hint towards the basic spring force balancer being the only possible solution for compensating two identical ideal springs but is left for future research.
- Adding two semi-identical force displacement characteristics produces a combined force displacement characteristic of the same kind (both are odd or both are even). "Subtracting" two semi-identical force displacement characteristics produces a combined force displacement characteristic of the other kind (odd turns to even and even turns to odd).
- When expanding the single force displacement characteristic $F_1=A((x/D)+\alpha_{ecc})^n$ it will always keep containing a term $A(x/D)^n$ for any α_{ecc}.

Cosine Mechanism For Generalized Ideal Springs

The cosine mechanism, see Fig. 1, can be used with generalized ideal springs and not just with (linear) ideal springs. The cosine mechanism approximately halves the power (*n*) of a generalized ideal spring thus making it "much more linear". The choice of U_0 at the specific location $U_0(x,y) = (0,-R((\alpha^2-1)/2\alpha))$ results in both the energetically equivalent springs, spring 1 and spring 2, being generalized ideal springs.

The transformation of the force displacement characteristic from $F_{spring1}=Kl^n$ to $F_{spring2}$ was derived previously and resulted in Eq. (14), shown again below:

$$F_{spring_2} = K(\alpha R)^{\left(\frac{1}{2}(n+1)\right)}(2u)^{\left(\frac{1}{2}(n-1)\right)} \tag{30}$$

Substituting different values of *n* gives Table 3. The table shows that for odd integer values of *n* (i.e −5,−3,−1, 1, 3, 5 etc.) the generalized energetically equivalent spring 2 has integer values for its power (respectively −3,−2,−1, 0, 1, 2, 3). Some other points of interest are

- The specific location $U_0(x,y) = (0,-R((\alpha^2-1)/2\alpha))$ is independent of *n*. Thus if spring 1 consists of multiple generalized springs (*i.e.* a series expansion), spring 2 can be seen as of the same number of generalized springs, each transformed with Eq. (14).
- The sign of the stiffness does not change from spring 1 to spring 2 over the possible range that the cosine mechanism operates in. When using a spring 1 with positive stiffness the equivalent spring 2 will never have a negative stiffness at any point throughout its possible stroke (it can, however, have zero stiffness).

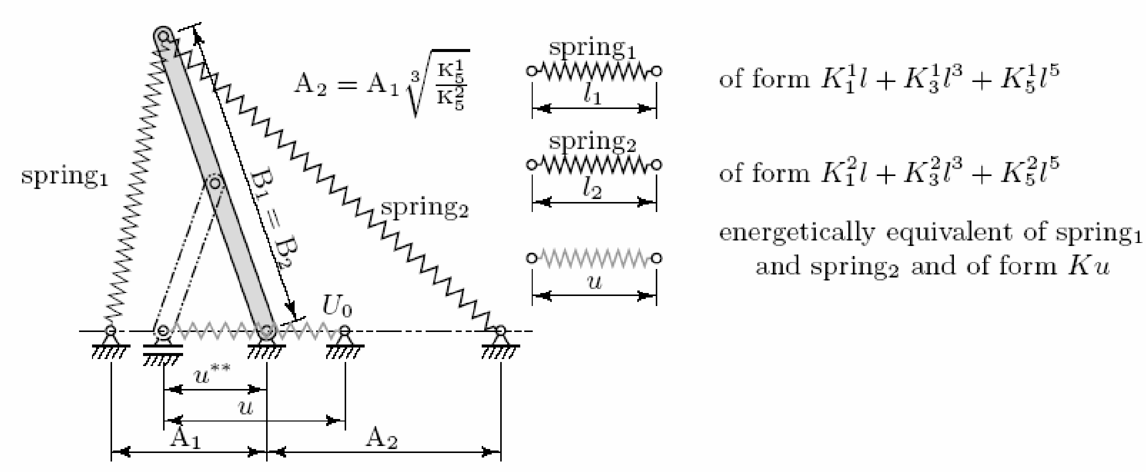

Figure 6: The proposed basic spring force linearizer which allows for two non-linear springs of general form $K_1 l + K_3 l^3 + K_5 l^5$ to behave as one linear spring of form Ku, under the condition shown.

- When $n = -1$, spring 1 and spring 2 are equal. It appears to be a special case, although no direct practical use can be thought of at this moment.
- When $\alpha = 1$, spring 1 can also be used for negative values of its length.

NEW SOLUTIONS FOR PERFECTLY BALANCING GENERALIZED IDEAL SPRINGS

No known mechanism existed up to now that perfectly balances equal non-linear springs. This section shows some new possible mechanisms to perfectly balance non-linear (generalized ideal) springs and is not intended to show all new possible solutions. Since the new mechanisms are first in their kind they are worth mentioning. The new mechanisms are explained by the way there were derived, in the conviction that using a similar way allows for many other new mechanisms for non-linear spring balancing.

The new mechanisms for balancing generalized ideal springs still have the basic spring force balancer as their main pillar. What is different is that, instead of using ideal linear springs in the basic spring force balancer, mechanisms are used that act as a perfect ideal linear spring while using (non-linear) generalized ideal springs.

First mechanisms are shown that use non-linear generalized ideal springs that act as a linear ideal spring.

- The cosine mechanism when used for a generalized ideal springs of form $K_3 l^3$ can act as a linear ideal spring. See Fig. 1 for n=3.
- The basic spring force linearizer is introduced in the next section. The basic spring force linearizer which allows for two non-linear springs of general form $K_1 l + K_3 l^3 + K_5 l^5$ to behave as a linear spring.

A mechanism acting as a linear spring can then be balanced by

- a negative ideal linear spring of correct stiffness
- a positive ideal linear spring when using a basic spring force balancer
- another mechanism acting as a linear spring when using a basic spring force balancer

The last two options (placing the mechanism in a basic spring force balancer) usually result in complicated mechanisms. However it is found that these complicated mechanisms can often be rewritten to (very) simple mechanisms. As long as the springs have the same length (thus the same energy) for every position it can be moved to another location in the mechanism. Finding (or creating) other locations so a spring has the same length in every position in the complicated mechanism, is often not difficult. Once the springs have been repositioned the mechanism can then be rewritten to another (simpler) form since the energy is the same in every position. The difficulty is finding sensible new locations that will result in a simpler mechanism.

Basic Spring Force Linearizer

In the above, the following powerful properties were derived:

- By designing the total mechanism symmetrical the force displacement characteristic will be purely odd.
- By adding two semi-identical (non-linear) force displacement characteristics the combined force displacement characteristic can be perfectly or "more" linear.

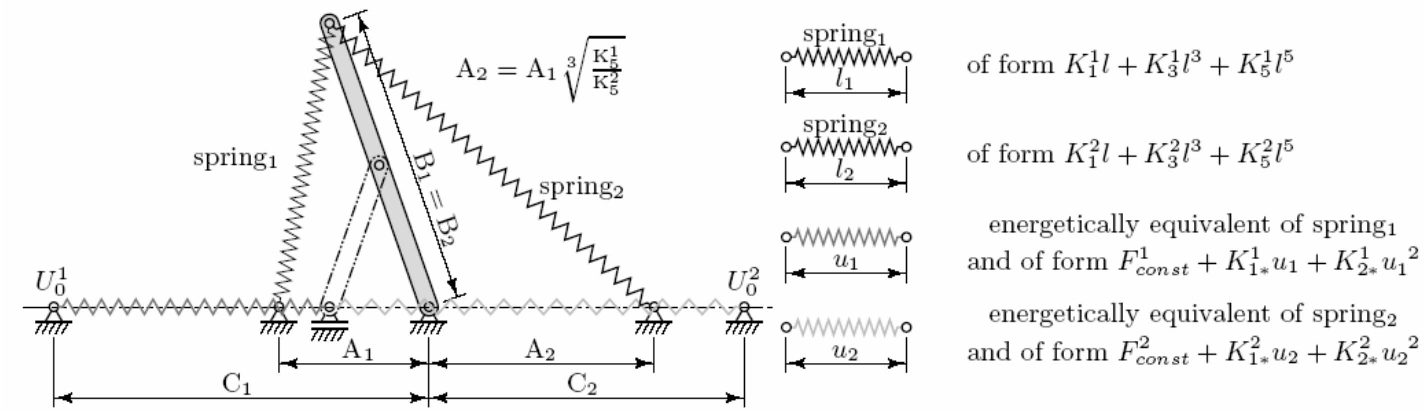

Figure 7: The proposed basic spring force lineariser. Each of the two non-linear springs act as a equivalent spring of form $F_{const} + K_{1*}u + K_{2*}u^2$. Under the condition for A2 and A1, the quadratic terms combined result in a linear term. Thus the two equivalent springs combined result in a FDC of only linear and constant terms which can be rewritten, when choosing the correct u_0, to an ideal spring.

- The cosine mechanism can be used to approximately halve the power (*n*) of a generalized ideal spring thus making it more like a linear ideal spring.

The power of these properties lies in combining the three, allowing for two non-linear springs to behave as a linear spring. The element that follows is proposed as the basic spring force linearizer, see Fig. 6. The springs used are purely odd and of form $K_1 l + K_3 l^3 + K_5 l^5$ (even terms in a single spring are eliminated by designing the single spring symmetrical). All the terms are transformed by using the cosine mechanism and the resulting constant terms (caused by $K_1 l$), the resulting linear terms (caused by $K_3 l^3$) and the resulting quadratic terms (caused by $K_5 l^5$) are combined. The quadratic terms can be seen as semi identical force displacement characteristics when

$$A_2 = A_1\sqrt[3]{\frac{K_5^1}{K_5^2}} \tag{31}$$

Thus the two equivalent springs combined result in a force displacement characteristic of only linear and constant terms. The linear and constant terms can be rewritten, when choosing the correct u0, to an ideal spring. Therefore odd generalized springs up to $K_5 l^5$ are used, higher odd terms will act approximately linear but will consist of some (small) non-linear parts. The following explains this in more detail. Consider Fig. 7 and Fig. 1. Since

$$\alpha R = \alpha_1 B_1 = A_1 \tag{32}$$

this then yields

$$\alpha_1 = \frac{A_1}{B_1} \tag{33}$$

Using Eq. 8 to determine u_{10} this then yields

$$C_1 = \alpha_1 R + u_o^1 = R\left(\alpha_1 - \left(\frac{\alpha_1^2 - 1}{2\alpha_1}\right)\right) = R\left(\frac{\alpha_1^2 - 1}{2\alpha_1}\right) = \frac{1}{2}\left(A_1 + \frac{{B_1}^2}{A_1}\right) \tag{34}$$

The equivalent spring force (force needed for equilibrium) generated by spring 1 can be
written as

$$F^1_{\text{equiv}} = K_1^1 A_1 + 2K_3^1 {A_1}^2 (u^{**} + C_1) + 4K_5^1 {A_1}^3 (u^{**} + C_1)^2 \tag{35}$$

See Fig. 6 for the definition of u**. It is chosen to require $B_1 = B_2$ which at the moment it is not very limiting. The presented theory on the basic spring force linearizer can be extended to allow for $B_1 \neq B_2$ but this is left to be investigated in the future. When $B_1 = B_2$ the equivalent spring force (force needed for equilibrium) generated by spring 2 in the same point can be written as

$$F^2_{\text{equiv}} = -K_1^2 A_2 + 2K_3^2 {A_2}^2 (u^{**} - C_2) - 4K_5^2 {A_2}^3 (u^{**} - C_2)^2 \tag{36}$$

In order for the quadratic terms to result in a linear term the amplitudes of the quadratic terms have to be the same. Thus

$$4K_5^1 {A_1}^3 = 4K_5^2 {A_2}^3 \tag{37}$$

making

$$A_2 = A_1\sqrt[3]{\frac{K_5^1}{K_5^2}} \tag{38}$$

When defining

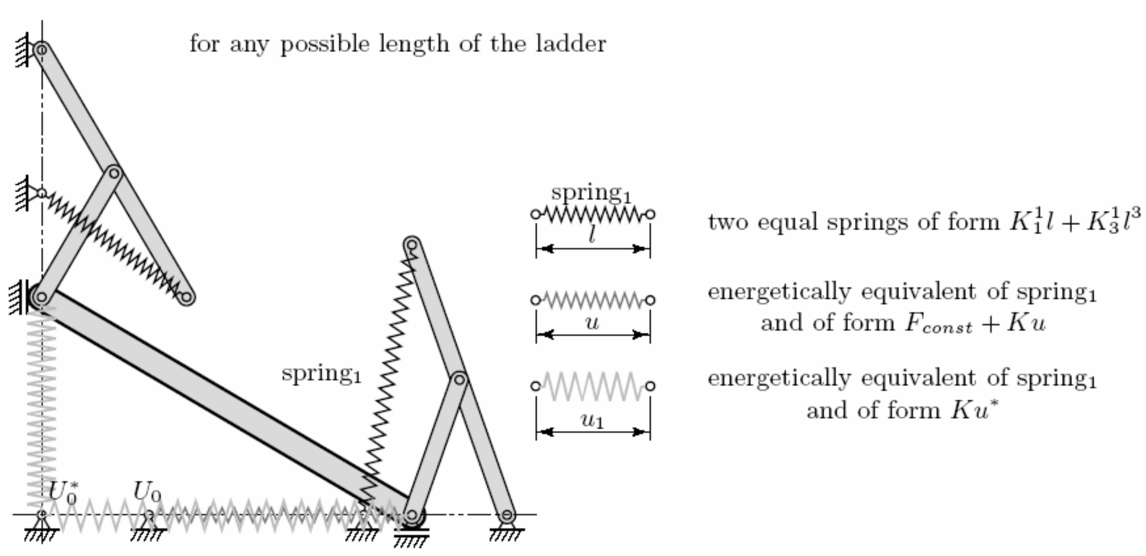

Figure 8: New mechanism that balances two springs of form $K_1l + K_3l^3$.

$$\Delta u^* = \frac{(C_2 - C_1)}{2} \tag{39}$$

the displacement is redefined to

$$u^* = u^{**} - \Delta u^* \tag{40}$$

$$C_3 = C_2 - \Delta u^* = C_1 + \Delta u^* \tag{41}$$

Using the property derived previously

$$A\left((\frac{x}{D}) + \alpha_{ecc}\right)^2 - A\left((\frac{x}{D}) - \alpha_{ecc}\right)^2 = 4A\alpha_{ecc}(\frac{x}{D}) \tag{42}$$

which can be rewritten to

$$AD(x + \alpha_{ecc}D)^2 - AD(x - \alpha_{ecc}D)^2 = 4A\alpha_{ecc}Dx \tag{43}$$

The combined quadratic terms of the equivalent spring forces (force needed for equilibrium) generated by spring 1 and spring 2 can now be written as

$$\begin{gathered} 4K_5^1 {A_1}^3 (u^* + C_3)^2 + 4K_5^1 {A_1}^3 (u^* - C_3)^2 = \\ 4(4K_5^1 {A_1}^3) C_3 u^* = 16 K_5^1 {A_1}^3 C_3 u^* \end{gathered} \tag{44}$$

So when $A_2 = A_1(K^1{}_5/K^2{}_5)^{1/3}$ and $B_1 = B_2$ the combined total equivalent spring forces (force needed for equilibrium) generated by spring1 and spring 2 can be rewritten to an ideal linear spring. Since it now consists only of linear terms and constant terms, redefining the displacement to yet another *u*, see Fig. 6, will result in an ideal linear spring. The expression for the point of U_0 is very elaborate and not given for the general case.

When $K^2{}_5 = K^1{}_5$ the expressions become much simpler. When also $K^2{}_3 = K^1{}_3$ and $K^2{}_1 = K^1{}_1$, so using two identical springs, the combined total equivalent spring forces can then be written as

$$F_{equiv}^{1+2} = \left(4K_3^1 {A_1}^2 + 8K_5^1 {A_1}^2 ({A_1}^2 + {B_1}^2)\right) u \tag{45}$$

which is an ideal linear spring with a zero displacement at $u^{**}=0$. When K^13 and $K^1{}_5$ have different signs there can be perfect balance when

$$K_3^1 + 2K_5^1({A_1}^2 + {B_1}^2) = 0 \tag{46}$$

However when adjusting A_1 and/or B_1 for this perfect balance the force caused by the term K_3l^3 and the term K_5l^5 are of the same order of magnitude. At the position where $l_1 = l_2$ the length l_1 can be written as

$$2({A_1}^2 + {B_1}^2) = 2l_1 \tag{47}$$

The mechanism adjusted for this perfect balance requires

$$K_5 = -\frac{K_3^1}{2l_1^2} \tag{48}$$

Thus the force caused by the term K5l5 can be written as

$$K_5 l_1^5 = -\frac{1}{2} K_3^1 l_1^3 \tag{49}$$

Any other position than $l_1 = l_2$ will result in either l_1 or l_2 becoming larger. Consequently the force due to K_5l^5 relative to K_3l^3 in one of the two springs will become larger. A spring in practice will also likely consist of terms K_7l^7 and higher. So when the terms K_3l^3 and K_5l^5 are of the same order of magnitude the terms K_7l^7 and higher will likely not be negligible. So balancing in this way in practice seems difficult to realize. The basic spring force linearizer will be used as a building block in the next section.

New Mechanisms

This section shows some possible ways to perfectly balance non-linear springs and is not intended to show all new possible solutions. The following new mechanisms for balancing non-linear ideal springs will be explained.

- two springs of form $K_1l + K_3l^3$

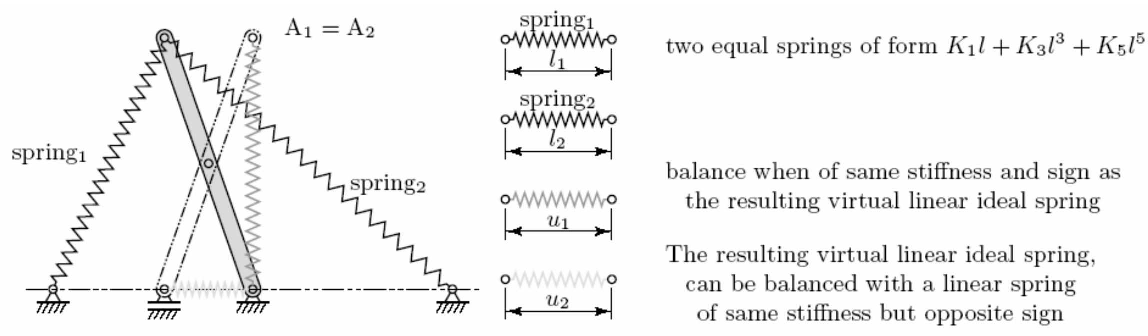

Figure 9: New mechanism that balances two springs of form $K_1l + K_3l^3 + K_5l^5$ and a linear ideal spring.

- two springs of form $K_1l + K_3l^3 + K_5l^5$ and one ideal spring
- four equal springs of form $K_1l + K_3l^3 + K_5l^5$.
- four equal springs of form $K_1l+K_3l^3+K_5l^5+K_7l^7 +K_9l^9+K_{11}l^{11} +K_{13}l^{13}$ and one ideal spring.

No values for stiffness are derived. What is shown is that when the springs are of the mentioned form then there is a solution.

Two Springs of Form $K_1l +K_3l^3$.

The cosine mechanism when used for a generalized ideal springs of form K_3l^3 can act as a linear ideal spring with zero length at point U^*_0. See Fig. 1 for n=3. The linear component K_1l creates a constant force. Combining the two forces gives an equivalent ideal linear spring with zero length at a certain point U_0. Combining two of these cosine mechanism into a BSFB allows for balancing two springs of form $K_1l + K_3l^3$, see Fig. 8. The elliptic trammel, a special form of the BSFB, is used for clarity and simplicity but it can also be used with the general BSFB.

Two Springs of Form $K_1l +K_3l^3 +K_5l^5$ and one Ideal Spring.

The resulting linear ideal spring from a basic spring force linearizer can be balanced by a linear ideal spring with opposite stiffness or with the same stiffness when it is combined again into a BSFB. Figure 9 shows this implemented when the two springs are equal, again for the BSFB an elliptic trammel is used for simplicity.

Four Equal Springs of Form $K_1l +K_3l^3 +K_5l^5$.

The resulting virtual linear ideal spring from a basic spring force linearizer can be balanced by another basic spring force linearizer. When put into the mechanism of Fig. 10b the two resulting linear ideal springs are connected by a virtual ladder in a virtual elliptic trammel. Thus the four equal springs of form $K_1l + K_3l^3 + K_5l^5$ in Fig. 10a can be balanced by this very simple and elegant mechanism. Perfect balance is achieved for any values of R_1 and R_2. When $A_1 = B_1$ all the springs can be used for either positive or negative values of its length. Similar to an elliptic trammel that has four possible working area this mechanism will then have, because of its four springs and two possible signs for the displacement, $2^4 = 16$ different working areas.

Four Equal Springs of Form $K_1l +K_3l^3 +K_5l^5 +K_7l^7 +K_9l^9 +K_{11}l^{11} +K_{13}l^{13}$ and a linear ideal spring.

When two springs of form $K_1l + K_3l^3 + K_5l^5+ K_7l^7 +K_9l^9+ K_{11}l^{11}+ K_{13}l^{13}$ are used in a basic spring force linearizer, the resulting virtual spring will be of form $K^*_1u +K^*_3u^3 +K^*_5u^5$. When put into the mechanism of Fig. 11 the two resulting virtual springs are connected by a virtual basic spring force linearizer, which in its turn acts as a linear spring of stiffness K^{**}_1. Thus the four equal springs in Fig. 11 can be balanced by an linear ideal spring with opposite stiffness $-K^{**}_1$ or with the same stiffness K^{**}_1 when it is combined again into a BSFB.

CONCLUSION

This paper presented for the first time mechanisms in which non-linear zero-free-length springs are perfectly statically balanced against one another. The concept of higher order zero-free-length springs was introduced, and defined as springs in which the spring force is proportional to a (positive) power of its length (rather than its elongation). Several techniques were proposed to balance these springs. Finally, four examples of increasing complexity were given. Ultimately, this technique should make the static balancing possible of complex force-deflection characteristics such as the non-linear elastic forces in the cosmetic covering in hand prostheses.

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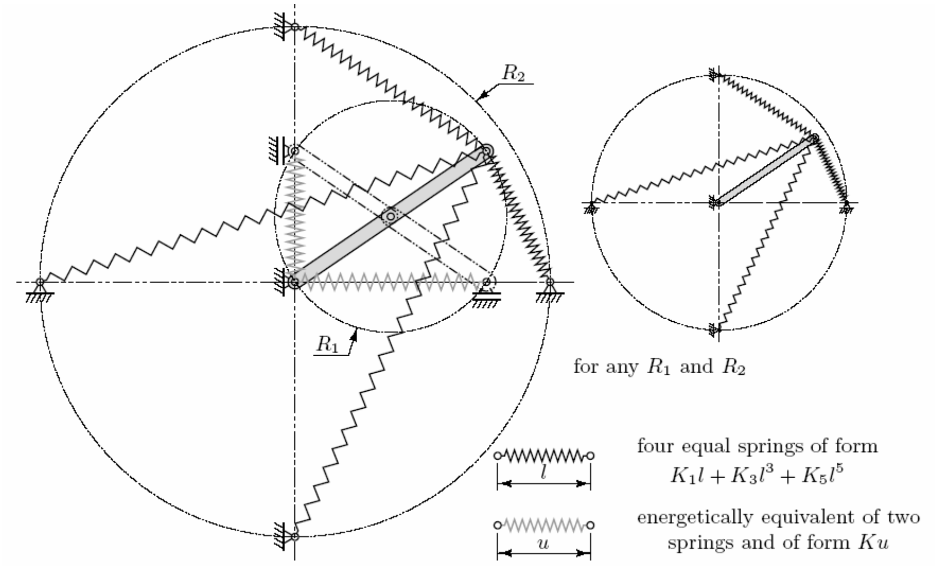

Figure 10: New mechanism that perfectly balances four equal springs of form $K_1l+K_3l^3+K_5l^5$ for any R_1 and R_2 (a) The two resulting linear ideal spring are connected by a virtual ladder in an virtual elliptic trammel resulting in perfect balance between the four equal springs (b) the resulting very simple and elegant mechanism.

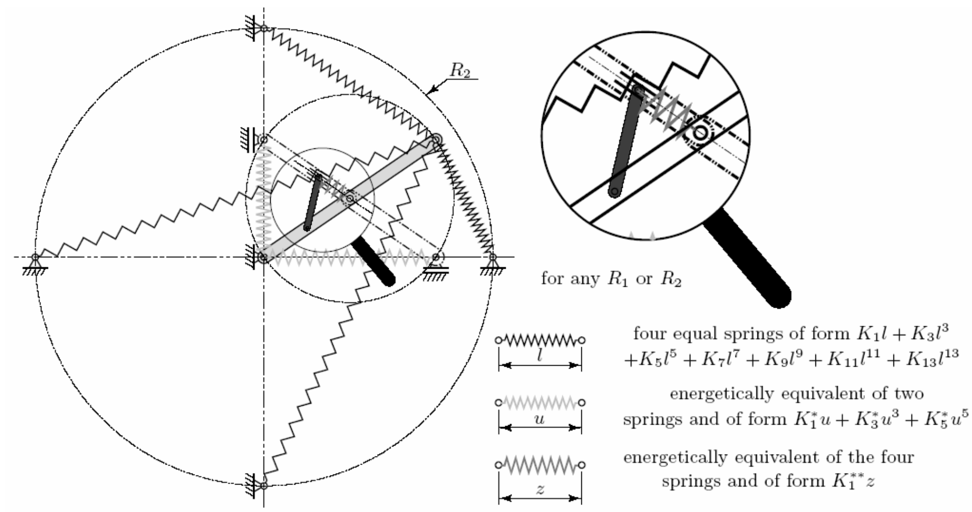

Figure 11: New mechanism that balances four equal springs of form $K_1l + K_3l^3 + K_5l^5 + K_7l^7 + K_9l^9 + K_{11}l^{11} + K_{13}l^{13}$ and a linear ideal spring.

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