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NEW METHODS TO SOLVE HIGH ORDER POTENTIAL USED IN CALCULATING NON-LINEAR WAVE LOADS

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ABSTRACT
In the present study, non-linear wave loads such as the wave-drift force, wave-drift damping and wave-drift added mass, acting on the body is considered based on the potential theory. To investigate non-linear wave loads, consistent perturbation expansion by means of two small parameters, i.e. the incident wave slope and the low frequency body motion, is performed on a moving frame (body-fixed) coordinate system.

To avoid complicated free surface integrals as much as possible, new approach for the higher order potential in the interaction problem of low frequency motion and waves is suggested in the present work. Instead of integrals, derivative operators are defined to obtain special solutions efficiently

INTRODUCTION
When ocean vehicles work in the wave, we usually need operate to be accurate positioning by means of mooring system or Dynamic Poisoning System (DPS). They may oscillate in horizontal plane with low frequency as a result of slowly varying non-linear wave forces at difference frequency of a pair of ocean wave spectrum component (ω_i - ω_j). The forces are proportion to square of wave height and affected by the slow drift motion. It can be separated two parts. One is proportional to the velocity of the low-frequency motion while the other one is proportional to the acceleration. They are called wave-drift damping and wave drift added mass respectively.

The wave-drift damping had been studied from many kinds of aspect in the past two decades. For example, in the case of the experimental way, T. Kinoshita et. al. developed forced oscillated test system and examined the dependence of wave height, wave frequency and so on. (1988) While in the case of the analytical way, Matsui et. al. (1991), Emmerhoff & Sclavounos (1992), Bao & Kinoshita (1994) evaluated the wave-drift damping in the explicit formula.

The wave-drift added mass have been researched same way as wave-drift damping. For example, in the case of the tank test, Kinoshita, Bao, Yoshida and Ishibashi showed the existence of it and gauged precisely for the different wave length. (2003) Tanizawa simulated by means of full non-linear 2D numerical test. (1997) It based on the potential theory. This numerical test calculated the wave-drift damping as well as wave-drift added mass. For the slow drift motion, the conventional damping coefficient is much smaller than the wave-drift damping and the wave-drift added mass which is as much as 20% of the conventional added mass in some cases. They are not so extreme condition such as the wave length (λ) is 2.7m and the wave height (A) is 0.05m respectively. So the wave-drift added mass cannot be ignored to simulate the resonance frequency even if the wave elevation is not so high.

The formulation of the problem involving the interaction between low-frequency oscillation and incoming waves is suggested by Newman(1993). Two time scales are derived, one associated with the body motion at the frequency σ and the other corresponding to the incident wave frequency ω.

It is difficult to solve for the secularity of free surface condition. Yoshida etc. (2005) suggested new radiation condition similar to Matsui's (1991) one. In this method, to ensure the unique solution for the potential, proper radiation conditions are inserted and special solutions are obtained by the free surface integrals.

The free surface condition is so complicated that the integral over it needs much effort. On the other hand, the Emmerhoff's (1992) approach is simple to calculate. In his method, the special solution is obtained by means of a derivatives operator.

In the present study, we try to follow Emmerhoff's approach to solve the higher order potential in the interaction problem of low frequency motion and waves. A couple of new derivative operators are going to be defined to solve these problems.

2. COORDINATE SYSTEM

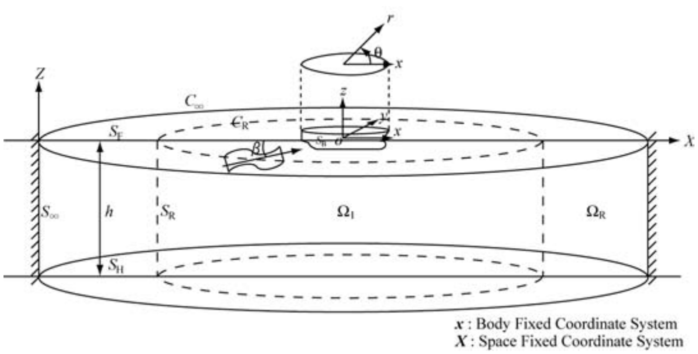

Fig. 1 Coordinate System

The problem to be solved here is that a body with an arbitrary shape is oscillating slowly in a train of regular waves. To simplify description, the body is restrained from responding to the incident waves. In other words, we are going to consider the interaction between the slow oscillations and the ambient diffraction wave field. It is not an essential difficulty to include the effects of linear responses of the body.

The frequency of the slow oscillation is designated by σ while the wave frequency is given by ω. It is assumed that $\sigma << \omega$. The low-frequency oscillation is restricted in the horizontal plane, i.e. in the mode of surge, sway or yaw designated by j=1, 2 or 6 respectively. Its displacement and velocity is expressed as follows respectively:

$$\begin{aligned} \xi_j(t) &= \mathrm{Re}\left\{ i\bar{\xi}_j e^{-i\sigma t} \right\} \qquad (j = 1,\ 2 \text{ or } 6) \\ \dot{\xi}_j(t) &= \mathrm{Re}\left\{ \sigma\bar{\xi}_j e^{-i\sigma t} \right\} \end{aligned} \tag{2-1}$$

where an over dot denotes the time differentiation. In (2-1), $\bar{\xi}$ represents the amplitude of the slow oscillation, which is assumed real without losing generality.

A Cartesian coordinate system following the low-frequency oscillations is adopted to describe the problem. The *oxy* plane coincides with the undisturbed free surface while the *z*-axis is pointing upward. The coordinates of moving frame is related to a space-fixed frame, say *OXYZ*, as follows:

$$\begin{aligned} X &= x + \delta_{j1}\xi_1(t) \\ Y &= y + \delta_{j2}\xi_2(t) \end{aligned} \qquad (j = 1 \text{ or } 2) \tag{2-2a}$$

$$\begin{aligned} X &= x\cos\xi_6 - y\sin\xi_6 \\ Y &= x\sin\xi_6 + y\cos\xi_6 \end{aligned} \qquad (j = 6) \tag{2-2b}$$

In (2-2), δ_{ij} represents the Kroenecker delta function, i.e. δ_{ij} =1 as $i = j$ while δ_{ij} = 0 elsewhere.

The time derivative in the space-fixed frame can be transferred to the moving frame by chain-rule differentiation:

$$d/dt = \partial/\partial t - \dot{\xi}_j(t)\partial/\partial x_j \qquad (j = 1 \text{ or } 2) \tag{2-3a}$$

where x_1=x and x_2=y

In the case of low-frequency yaw motion, referring to the moving frame, the incident wave angle β changes with time, i.e. $\beta = \beta_0 - \xi_6(t)$. Therefore, when a time derivative is taken, a term of differentiation with respect to β should be added:

$$d/dt = \partial/\partial t - \dot{\xi}_6(t)\left(\partial/\partial x_6 + \partial/\partial\beta\right) \tag{2-3b}$$

Here, x_6=θ, i.e. the azimuth angle.

3. PERTURBATION EXPANSION

It is assumed the fluid to be inviscous and the flow to be irrotational. Therefore, there exists a velocity potential $\Phi(\boldsymbol{x}, t)$. It is natural to use two time scales to describe these two kinds of motions with low and high frequency respectively. Following the approach of Newman's [4], the velocity potential can be expressed by a perturbation expansion up to the quadratic order in wave amplitude ζ_a as:

$$\begin{aligned} \Phi(\boldsymbol{x},t) = \mathrm{Re}\Big\{ & \phi_1(\boldsymbol{x})e^{-iS_j(t)} + \phi_2^{(0)}(\boldsymbol{x}) + \ldots + \sigma\bar{\xi}\Big[\phi_{0j}(\boldsymbol{x})e^{-i\sigma t} \\ & + \phi_{1j}^{(+)}(\boldsymbol{x})e^{-i(S_j(t)+\sigma t)} + \phi_{1j}^{(-)}(\boldsymbol{x})e^{-i(S_j(t)-\sigma t)} + \phi_{2j}^{(0)}(\boldsymbol{x})e^{-i\sigma t} + \ldots\Big]\Big\} \end{aligned} \tag{3-1}$$

The potentials on the right-hand side of (3-1) depends only on the space position $\boldsymbol{x}$. The number in the subscript indicates the order in wave amplitude while the letter j =1, 2 or 6 denotes that the potential is related to the slow surge, sway or yaw motion respectively. Superscripts are used if needed to denote harmonic time dependence on the wave frequency. Here, potentials with double wave frequency are omitted since they will not contribute to the wave-drift added mass and damping.

Here, the phase function is defined as

$$S_j(t) = \begin{cases} \omega t - \xi_j(t)\kappa_j & (j = 1 \text{ or } 2) \\ \omega t & (j = 6) \end{cases} \tag{3-2}$$

where $\kappa_1 = k_0 \cos\beta$ and $\kappa_2 = k_0 \sin\beta$ with k_0 to be the wave number of the incident waves. The so-called encountering frequency ω_e is obtained from the time derivative of $S_j(t)$

$$\omega_e = \dot{S}_j(t) = \omega - \dot{\xi}_j(t)\kappa_j \quad (j = 1 \text{ or } 2) \tag{3-3}$$

As mentioned earlier, in the case of low-frequency yaw motion, a term of differentiation with respect to β should be added to a time derivative when referring to the moving frame (see 2-3b). For the convenience of later discussion, we define $\kappa_6 = i\,\partial/\partial\beta$, comparable with the case of j=1 or 2.

Next, the potential $\phi_{1j}^{(\pm)}$ is considered, which is linear in wave amplitude as denoted by the first subscript. It takes account of the interaction between low frequency motions and waves. When the low frequency σ is asymptotically small, this potential is further expanded into a series of σ:

$$\phi_{1j}^{(\pm)} = \tfrac{1}{2}(\hat{\psi}_j \pm \sigma\tilde{\psi}_j) \tag{3-4}$$

4. BOUNDARY VALUE PROBLEMS

The velocity potential is governed by the Laplace equation in the fluid domain and satisfy an impermeable condition on the sea bottom z=-h and on the body surface S_0.

When the free surface condition is considered in the moving frame, it is stated as follows on the exact elevation of the free surface z=$\zeta(x, y, t)$:

$$\begin{aligned}&\partial^2\Phi/\partial t^2 + g\,\partial\Phi/\partial z - 2\dot{\xi}_j(t)\partial^2\Phi/\partial t\partial x_j - \ddot{\xi}_j(t)\partial\Phi/\partial x_j \\ &+ 2\nabla(\partial\Phi/\partial t)\cdot\nabla\Phi - 2\dot{\xi}_j(t)\nabla(\partial\Phi/\partial x_j)\cdot\nabla\Phi \\ &+ \tfrac{1}{2}(\nabla\Phi\cdot\nabla)(\nabla\Phi\cdot\nabla\Phi) = 0 \qquad \text{on } z = \zeta(x,y,t)\end{aligned} \tag{4-1}$$

where the wave elevation ζ of the free surface is evaluated by

$$\begin{aligned}\zeta &= -1/g\left(\partial\Phi/\partial t - \dot{\xi}_j(t)\partial\Phi/\partial x_j + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)_{z=\zeta} \\ &= -1/g\left[\partial\Phi/\partial t - \dot{\xi}_j(t)\partial\Phi/\partial x_j + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi\right. \\ &\quad -1/g\,\partial\Phi/\partial t\,\partial\left(\partial\Phi/\partial t - \dot{\xi}_j(t)\partial\Phi/\partial x_j\right)/\partial z \\ &\quad \left. + \dot{\xi}_j(t)\partial\Phi/\partial x_j\,\partial^2\Phi/\partial t\partial z\right]_{z=0} + O(\Phi^3)\end{aligned} \tag{4-2}$$

In (4-2), the expansion about the mean free surface, i.e. z = 0, has been made. In the same way, the free surface condition in (4-1) is also expanded about the mean free surface and the perturbation expansions are substituted into it to yield the free surface condition for each order of potential:

$$\partial\phi_1/\partial z - \nu\phi_1 = 0 \tag{4-3a}$$

$$\partial\phi_2^{(0)}/\partial z = f_2^{(0)}$$
$$\text{where } f_2^{(0)} = \mathrm{Re}\left\{-i\omega/(2g)\phi_1\,\partial^2\phi_1^*/\partial z^2\right\} \tag{4-3b}$$

$$\partial\phi_{0j}/\partial z = 0 \tag{4-3c}$$

$$\partial\hat{\psi}_j/\partial z - \nu\hat{\psi}_j = \hat{f}_j$$
$$\text{where } \hat{f}_j = -2i\,\partial\phi_1/\partial x_j - 2\kappa_j\phi_1 + 2i\nabla\phi_{0j}\cdot\nabla\phi_1 - i\phi_1\,\partial^2\phi_{0j}/\partial z^2 \tag{4-3d}$$

$$\partial\tilde{\psi}_{js}/\partial z - \nu\tilde{\psi}_{js} = \tilde{f}_j$$
$$\text{where } \tilde{f}_j = \omega^{-1}[2\nu\hat{\psi}_j - i\,\partial\phi_1/\partial x_j - \kappa_j\phi_1 + 2i\nabla\phi_{0j}\cdot\nabla\phi_1 - i\phi_{0j}\left(\partial^2\phi_1/\partial z^2 - \nu^2\phi_1\right)] \tag{4-3e}$$

$$\partial\phi_{2j}^{(0)}/\partial z - \sigma^2/g\,\phi_{2j}^{(0)} = f_{2j}^{(0)} \tag{4-3f}$$

$$\begin{aligned}f_{2j}^{(0)} = \mathrm{Re}\,1/(2g)\Big\{&i\omega\left(\phi_1^*\hat{\psi}_{jzz} - \phi_{1zz}^*\hat{\psi}_j\right) - 3\nu^2(\phi_1^*\phi_{1x_j} \\ &- \phi_1^*\nabla\phi_1\cdot\nabla\phi_{j0}) - 2\nu^2\phi_1\phi_1^*\phi_{j0zz} + 2\nabla\phi_1^*\cdot\nabla\phi_{1x_j} \\ &- \phi_{1x_j}^*\phi_{1zz} + i\kappa_j\phi_1\phi_{1zz}^* + \phi_{1zz}^*\nabla\phi_1\cdot\nabla\phi_{j0} - \nabla\phi_1^* \\ &\cdot\nabla\left(\nabla\phi_1\cdot\nabla\phi_{j0}\right) + \tfrac{1}{2}\left(\phi_{j0zz}\nabla\phi_1\cdot\nabla\phi_1^* - \nabla\phi_{j0}\right. \\ &\left.\cdot\nabla\left(\nabla\phi_1\cdot\nabla\phi_1^*\right)\right) + \delta_{j6}\left(\nabla\phi_1\cdot\nabla\phi_1^* - \tfrac{3}{2}\nu^2\phi_1\phi_1^*\right)_\beta\Big\}\end{aligned}$$

As a summary of this section, we write down the form of the boundary value problem for the potentials $\hat{\psi}_j$ and $\tilde{\psi}_j$ as follows. The corresponding forcing terms in the free surface condition are given in (4-3).

$$\begin{aligned}&\nabla^2\psi_j = 0 && \in\Omega \\ &\partial\psi_j/\partial z - \nu\psi_j = f_j && \in S_F \\ &\partial\psi_j/\partial z = 0 && \in S_H \\ &\partial\psi_j/\partial n = 0 && \in S_B\end{aligned} \tag{4-4}$$

proper far field condition.

5. WAVE LOADS

Once the potential are solved, the hydrodynamic pressure can be evaluated by the Bernoulli equation. In the moving frame following the low-frequency oscillations, the Bernoulli equation can be expressed as:

$$p = -\rho\left(\partial\Phi/\partial t - \dot{\xi}(t)\,\partial\Phi/\partial x + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + gz\right) \tag{5.1}$$

The wave loads are obtained by integrating the hydrodynamic pressure along the instantaneous wetted body surface $\tilde{S}_B$, which consists of a mean wetted body surface S_B under the calm water and a surface wetted by the wave elevation ζ along the water line C_B of the body. The force acting on the body in i-direction (i = 1, 2, 6 representing the direction in surge sway or yaw respectively) may be evaluated by the following integrals.

$$\begin{aligned}F_i(t) &= -\rho\int_{\tilde{S}_B}\left(\frac{\partial\Phi}{\partial t} - \dot{\xi}(t)\frac{\partial\Phi}{\partial x} + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + gz\right)n_i\,ds \\ &= -\rho\int_{S_B}\left(\frac{\partial\Phi}{\partial t} - \dot{\xi}(t)\frac{\partial\Phi}{\partial x} + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)n_i\,ds \\ &\quad + \frac{\rho}{2g}\int_{C_B}\frac{\partial\Phi}{\partial t}\left[\frac{\partial\Phi}{\partial t} - 2\dot{\xi}(t)\frac{\partial\Phi}{\partial x} + \nabla\Phi\cdot\nabla\Phi - \frac{1}{g}\frac{\partial\Phi}{\partial t}\frac{\partial^2\Phi}{\partial t\partial z}\right]n_i\,dl\end{aligned} \tag{5-2}$$

where ρ represents the fluid density. In (5-2), use has been made of the wave elevation ζ shown in (4-2).

The wave forces are expanded into a perturbation series in the same way as the velocity potential i.e.

$$\begin{aligned}F_i(t) = \mathrm{Re}\Big\{&F_{1i}e^{-iS_j(t)} + F_{2i}^{(0)} + \ldots + \sigma\bar{\xi}\Big[F_{0ij}e^{-i\sigma t} \\ &+ F_{1ij}^{(+)}e^{-i(S_j(t)+\sigma t)} + F_{1ij}^{(-)}e^{-i(S_j(t)-\sigma t)} + F_{2ij}^{(0)}e^{-i\sigma t} + \ldots\Big]\Big\}\end{aligned} \tag{5-3}$$

We are interested in the force with a time factor of low frequency, i.e. F_{0ij} and F_{2ij}. Here, F_{0ij} is the hydrodynamic force when the body is slowly oscillating in calm water while F_{2ij} is in quadratic order of wave amplitude. Each of them can be separated into real and imaginary parts as followings;

$$\begin{aligned} F_{0ij} &= -(-i\sigma A_{0ij} + B_{0ij}) \\ F_{2ij}^{(0)} &= -(-i\sigma A_{2ij} + B_{2ij}) \end{aligned} \tag{5-4}$$

The Real part of them is in phase with the velocity of the low frequency oscillations, which gives conventional (B_{0ij}) or wave-drift damping (B_{2ij}). The imaginary parts, in phase with the acceleration, gives conventional (A_{0ij}) or wave drift added mass (A_{2ij}).

The conventional damping and added mass for the slow drift motion are well known (Kinoshita etc 2002). We are interested in the terms of the wave-drift damping and wave-drift added mass, which can be written in the following form;

$$\begin{aligned} \frac{B_{2ij}}{\rho\zeta_0^2\omega} = \mathrm{Re}\Bigg[& \int_{S_B} \left\{ -\frac{\partial \phi_2^{(0)}}{\partial x_j} + \nabla\phi_{0j}\cdot\nabla\phi_2^{(0)} + \frac{1}{2\nu}\nabla\hat{\psi}_j\cdot\nabla\phi_1^* \right\} n_i ds \\ & - \frac{1}{2\nu}\int_{C_B} \left(\nu\hat{\psi}_j - i\frac{\partial\phi_1}{\partial x} - \kappa_j\phi_1 + i\nabla\phi_{0j}\cdot\nabla\phi_1 \right)\phi_1^* n_i dl \Bigg] \end{aligned} \tag{5-5a}$$

$$\begin{aligned} \frac{A_{2ij}}{\rho\zeta_0^2} = \mathrm{Im}\Bigg[& \int_{S_0} \left\{ i\nu\phi_{2j} - \frac{1}{2\nu}\nabla\tilde{\psi}_j\cdot\nabla\phi_1^* \right\} n_i ds + \frac{1}{2}\int_{C_0} \Big\{ (\hat{\psi}_j + \tilde{\psi}_j)\phi_1^* \\ & + i\nu\phi_{0j}\phi_1\phi_1^* - \frac{i}{2\nu}\phi_{0j}\nabla\phi_1\cdot\nabla\phi_1^* \Big\} n_i dl \Bigg] \end{aligned} \tag{5-5b}$$

It can be observed from (5-5a, b) that only the potential $\hat{\psi}_j$ will contribute to the wave-drift damping while both potentials of $\hat{\psi}_j$ and $\tilde{\psi}_j$ are required to evaluate the wave-drift added mass. Therefore, an efficient way to solve the higher order potential $\tilde{\psi}_j$ is desired.

6. SOLUTIONS OF THE BVP

6.1 Approach for the Lower Order Potentials

In the present study, the boundary value problems presented in section 4 are solved by the hybrid method. The fluid domain, designated by Ω, is divided into an inner region Ω_I and an outer region Ω_R by a virtual cylindrical surface S_R with a radius equal to r_0 (see Fig. 1). Different expressions are used to represent the solutions for each order of potentials in these two regions. They will be matched on the common surface S_R.

In the inner region, according to the Green's identity, the potentials can be expressed by the following integral over its boundary surface:

$$C_P\phi_P = \int_S \left(G_{PQ}\,\partial\phi_Q/\partial n - \phi_Q\,\partial G_{PQ}/\partial n \right) ds_Q \tag{6-1}$$

where P is the field point while Q is the source point distributed on the boundary surface S surrounding the inner region Ω_1. It consists of the body surface S_B, part of the free surface S_F in the inner region, the sea bottom S_H and the virtue surface S_R. C_P is the shape coefficient at the field point P. The value of C_P can be determined by assuming the potential to be a unit constant, i.e. $\phi_P = \phi_Q = 1$ and consequently,

$$C_P = -\int_S \partial G_{PQ}/\partial n\, ds \tag{6-2}$$

G_{PQ} designates the Green's function. In the present study, the fundamental solution for a pulsating source and its mirror image referring to the sea bottom S_H is used as the Green's function, i.e.

$$G_{PQ} = \frac{1}{r_{PQ}} + \frac{1}{r_{PQ'}} \tag{6-3a}$$

where

$$\begin{aligned} r_{PQ} &= \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2} \\ r_{PQ'} &= \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P + z_Q + 2h)^2} \end{aligned} \tag{6-3b}$$

This part of discussion is well known in the literature of boundary element method. Hence, our discussion will be concentrated on the outer region.

In the outer region, the solutions are expressed by the eigen-function expansion. A cylindrical coordinate system, O-$r\theta z$, is also adopted in order to be used in the outer region for convenience.

Different eigen-functions are used for different boundary value problems. For example, for the low-frequency potential ϕ_{0j}, a rigid wall condition is imposed on the free surface by neglecting terms of order $O(\sigma^2)$ or higher. Its solution in the outer region can be written as:

$$\phi_{0j} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} R_{mn}^{0j}(r) Z_m^{0j}(z) (A_{mn}^{0j}\cos n\theta + B_{mn}^{0j}\sin n\theta) \tag{6-4a}$$

where

$$R_{mn}^{0j}(r) = \begin{cases} r_0^n / r^n & m = 0 \\ K_n(\lambda_m r)/K_n(\lambda_m r_0) & m > 0 \end{cases} \tag{6-4b}$$

$$Z_m^{0j}(z) = \cos\lambda_m(z+h); \qquad \lambda_m = m\pi/h \tag{6-4c}$$

and K_n is the modified Bessel function of the second kind of order n.

For the linear potential ϕ_1, which satisfies a homogeneous free surface condition, the expansion is readily written as:

$$\phi_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} R_{mn}^{1}(r) Z_m^{1}(z) (A_{mn}^{1}\cos n\theta + B_{mn}^{1}\sin n\theta) \tag{6-5a}$$

where

$$R_{mn}^{1j}(r) = \begin{cases} H_n(k_0 r)/H_n(k_0 r_0) & m = 0 \\ K_n(k_0 r)/K_n(k_0 r_0) & m > 0 \end{cases} \tag{6-5b}$$

$$Z_m^{10}(z) = \begin{cases} \cosh k_0(z+h)/\cosh k_0 h & m = 0 \\ \cos k_m(z+h)/\cos k_m h & m > 0 \end{cases} \tag{6-5c}$$

and H_n is the Hankel function of the first kind of order n, while k_m (m>0) is the positive solution of the dispersion relation:

$$\nu = -k_m \tan k_m h \tag{6-6}$$

For the potential $\hat{\psi}_j$, in addition to the general terms same to (6-5) except the coefficients A_{mn} and B_{mn}, some special solutions should be added to satisfy the inhomogeneous free surface condition. The special solutions take the following form:

$$\hat{\psi}_{1j}^{U} = -2\left(i\,\partial/\partial x + k_0 \cos\beta\right)\partial\phi_1/\partial\nu \tag{6-7}$$

$$\hat{\psi}_{1j11}^{S} = i\int_{S_F''}\left(2\nabla\phi_{0j}\cdot\nabla\phi_1 - \phi_1\,\partial^2\phi_{0j}/\partial z^2\right)G^P ds \tag{6-8}$$

where S_F'' denotes the part of the free surface in the outer region, i.e. $r_0 < r < \infty$ at z = 0, and G^P represents a Green's function suitable for the problem of pressure distribution over the free surface. Its definition and eigen-function expansion are described as followings (see Wehausen & Laitone 1960):

$$G^P(r,\theta,z;\rho,\psi,\zeta) = \sum_{n=-\infty}^{\infty} g_n(r,z;\rho,\zeta)e^{in(\theta-\psi)} \tag{6-9a}$$

where

$$g_n(r,z;\rho,\zeta) = i2\pi C_0\begin{pmatrix} H_n(k_0 r)J_n(k_0\rho) \\ J_n(k_0 r)H_n(k_0\rho)\end{pmatrix} Z_0^{10}(z)Z_0^{10}(\zeta) + 4\sum_{m=1}^{\infty} C_m \begin{pmatrix} K_n(k_m r)I_n(k_m\rho) \\ I_n(k_m r)K_n(k_m\rho)\end{pmatrix} Z_m^{10}(z)Z_m^{10}(\zeta)\begin{pmatrix} r>\rho \\ r<\rho\end{pmatrix} \tag{6-9b}$$

$$C_0 = k_0/[h(k_0^2-\nu^2)+\nu];\quad C_m = k_m/[h(k_m^2+\nu^2)-\nu] \tag{6-9c}$$

These special solutions given in (6-7) and (6-8) can also be expressed in eigen-function expansion after some tedious deduction.

6.2 New Approach for the Higher Order Potential

When the potential $\tilde{\psi}_j$ is considered (defined Equ. (3-4)), difficulty arises from the first forcing term, i.e. $2\nu\hat{\psi}_j$, since $\hat{\psi}_j$ itself satisfies an inhomogeneous free surface condition

Ordinary, the special solution for the inhomogeneous boundary condition can be obtained by an integral over the free surface. In the case of potential $\tilde{\psi}_j$, linear in wave slope but an order of low frequency squared, one of the difficult parts to solve is this part because potential $\hat{\psi}_j$ itself is secular at far field. So we are going to consider this problem in another way.

As mentioned before the potential $\hat{\psi}_j$ consists of three parts, i.e.

$$\hat{\psi}_j = \hat{\psi}_j^U + \hat{\psi}_j^s + \hat{\psi}_j^g \tag{6-10}$$

Similarly, the potential $\tilde{\psi}_j$ contains three parts corresponding to different parts of potential $\hat{\psi}_j$ as followings;

$$\tilde{\psi}_j = \tilde{\psi}_j^U + \tilde{\psi}_j^s + \tilde{\psi}_j^g \tag{6-11}$$

We can obtain solutions correspondingly. The first one is obtained by applying a derivative operator to the first order potential ϕ_1. the operator involves a double derivative with respect to the parameter ν:

$$\tilde{\psi}_j^U = \nu(i\,\partial/\partial x_j + \kappa_j)\,\partial^2\phi_1/\partial\nu^2 \tag{6-12}$$

The second one consists of two parts. The first part is the ν-derivative applied to the second special solution of $\hat{\psi}_j^S$ while the second part is an integral over the free surface.

$$\tilde{\psi}_j^s = 2\nu[\partial\hat{\psi}_j^s/\partial\nu - \int_{S_F} G\,\partial\hat{f}_j^s/\partial\nu\, ds] \tag{6-13}$$

The third one is the ν-derivative applied to the general solution part of the potential $\hat{\psi}_j^g$.

$$\tilde{\psi}_j^g = 2\nu\,\partial\hat{\psi}_j^g/\partial\nu \tag{6-14}$$

The remaining forcing terms can easily be treated in the same way as in solving the potential $\hat{\psi}_j$

7. DISCUSSION

It can be observed from (6-13), a free surface integral, i.e. $\int_{S_F} G\,\partial\hat{f}_j^s/\partial\nu\, ds$, still exists in the new solutions. It is important to examine whether this integral is convergent or not. Since the derivative of the forcing term $\hat{f}_j^s$ referring to the parameter ν is complicated to calculate, we are going to change this form to the differential of the Green function by means of the differential formula of the product as followings;

$$\int_{S_F} G\,\partial\hat{f}_j^s/\partial\nu\, ds = \partial\hat{\phi}_j^s/\partial\nu - \int_{S_F}\partial G/\partial\nu\, f_j^s ds \tag{7-1}$$

Then the behavior of integrand at large distance (r >> 1) is shown in Table 1.

Table 1

Integrand	r >>1
$\partial G/\partial\nu$ ($rH_n'^{(1)}$ or $rJ_n'^{(1)}$)	$\sim\ r^{\frac{1}{2}}e^{\pm ik_0 r}$
$\hat{f}_j^s$	$\sim\ r^{-2}$

As a summary of Table 1, the integrand behaves as follows for r>>1:

$$G\,\partial\hat{f}_j^s/\partial\nu \sim r^{-\frac{3}{2}}e^{\pm ik_0 r} \tag{7-2}$$

Hence the integral will be convergent. However, the convergent rate is not good enough to integrate it numerically. Special treatments are needed when dealing with the propagating modes in the eigen-function expansion. These propagating modes are separated from the other terms and can be integrated analytically. The remaining terms, i.e. the evanescent modes , which are convergent at least in proportion to the radius, can be calculated numerically.

CONCLUSION

In the present study, higher order potential used in the calculation of the wave-drift added mass is solved by means of proper derivative operator. It is easy to confirm that the free surface integral appeared in the new solutions is convergent. The new solutions have an advantage of saving much effort from tedious numerical integral over the free surface.

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APENDIX A

VALIDATION OF THE SPECIAL SOLUTION

Obviously, all the solutions given in (6-12, 13, 14) satisfy the governing equation since the order of applying the derivative operator and the Laplace operator can be changed. We are going to concentrate our discussion on the proof that these solutions satisfy the corresponding inhomogemeous free surface conditions.

As shown in (6-12) the solution $\tilde{\psi}_j^U$ is expressed as follows;

$$\tilde{\psi}_j^p = \nu\wp\{\partial^2\phi_1/\partial\nu^2\} \tag{A-1}$$

where $\wp\{\}$ is defined as;

$$\wp\{\} = \left(i\,\partial/\partial x_j + \kappa_j\right)\{\ \} \tag{A-2}$$

When (A- 1) is substituted into the free surface condition, the following calculation can be derived;

$$\begin{aligned}
\text{L.H.S} &= \partial\left[\nu\wp\left\{\partial^2\phi_1/\partial\nu^2\right\}\right]/\partial z - \nu\left[\nu\wp\left\{\partial^2\phi_1/\partial\nu^2\right\}\right] \\
&= \nu\wp\left[\partial^3\phi_1/\partial\nu^2\partial z - \nu\,\partial^2\phi_1/\partial\nu^2\right] \\
&= \nu\wp\left\{\partial^2/\partial\nu^2\left(\partial\phi_1/\partial z - \nu\phi_1\right) + 2\,\partial\phi_1/\partial\nu\right\} \\
&= 2\nu\wp\left\{\partial\phi_1/\partial\nu\right\} \\
&= 2\nu\hat{\psi}_j^U = \text{R.H.S}
\end{aligned} \tag{A-3}$$

So the solution for $\tilde{\psi}_j^U$ satisfies the free surface condition.

When the second solution $\tilde{\psi}_j^s$ in (6-13) is considered, it should be noticed that the second part of it, i.e. the free surface integral, represents a potential caused by the pressure distribution of $2\nu\partial\hat{f}_j^s/\partial\nu$ over the free surface. Inserting the second solution into the free surface condition, we have:

$$\begin{aligned}
\text{L.H.S.} &= \partial/\partial z\,(2\nu\,\partial\hat{\psi}_j^s/\partial\nu) - \nu(2\nu\,\partial\hat{\psi}_j^s/\partial\nu) \\
&\quad - 2\nu\,\partial\hat{f}_j^s/\partial\nu \\
&= 2\nu\left[\partial\left(\partial\hat{\psi}_j^s/\partial z - \nu\hat{\psi}_j^s\right)/\partial\nu + \hat{\psi}_j^s - \partial\hat{f}_j^s/\partial\nu\right] \\
&= 2\nu\hat{\psi}_j^s = \text{R.H.S}
\end{aligned}$$

This shows that the corresponding free surface is satisfied.

Finally the third one $\tilde{\psi}_j^g$ is considered. Since the general solution of $\hat{\psi}_j^g$ satisfies a homogeneous free surface condition, the validation is quite simple and is referred to Emmerhoff and Sclavounos (1992).