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Marginalist and efficient values for TU games

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Abstract

We derive an explicit formula for a marginalist and efficient value for TU game which possesses the null-player property and is either continuous or monotonic. We show that every such value has to be additive and covariant as well. It follows that the set of all marginalist, efficient, and monotonic values possessing the null-player property coincides with the set of random-order values, and, thereby, the last statement provides an axiomatization without the linearity axiom for the latter which is similar to that of Young for the Shapley value. Another axiomatization without linearity for random-order values is provided by marginalism, efficiency, monotonicity and covariance.

Keywords: Transferable utility game; Value; Axiomatic characterization; Efficiency; Marginalism

1. Introduction

Cooperative game theory is intensively developed now. Many new solutions appear and most of them have axiomatizations. But the standard approach is first to construct a solution or a set of them and then to characterize their properties in terms of axioms. Meanwhile the social choice theory related very closely to that of cooperative games seems to be much more conceptually developed. It starts with a desirable set of properties the solutions should have, and after that the entire set of mappings representing these solutions is described. We adopt the social choice theoretical approach to cooperative game solutions and start with their axiomatic characterizations thereupon deriving a functional description for the associated set of them.

We consider single-valued solutions usually called values. The most famous value for

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cooperative games is the Shapley value (Shapley, 1953) which is linear in its definition. However, as was first shown by Young (1985) and then strengthened by Chun (1989), (1991) the Shapley value may be characterized without the linearity axiom. It turns out that the Shapley value is not the only case, and a class of values defined without the linearity assumption and which appear to be linear is rather wide. We exploit the property of marginalism introduced by Young in his characterization of the Shapley value, what means a dependence of any player's payoff upon only his/her marginal utilities. We derive an explicit formula for a marginalist and efficient value which possesses the null-player property and is either continuous or monotonic. We show that every such value must be additive and covariant as well.

It follows that the set of all marginalist, efficient and monotonic values possessing the null-player property coincides with the set of random-order values introduced and studied by Weber (1988), and thereby the last statement provides an axiomatization without the linearity axiom for the latter which is similar to that of Young for the Shapley value. Another axiomatization without linearity for random-order values is provided by marginalism, efficiency, monotonicity and covariance. So, removing the symmetry assumption in Young and replacing it by monotonicity and either null-player or covariance it is possible to characterize the whole class of random-order values. Thus, it is shown that Young's result is robust without symmetry provided we assume monotonicity and something else. Therefore, this paper closes an open question raised by Hart in the survey (Hart, 1990).

Note that an axiomatic characterization of weighted Shapley values, which are a subset of random-order ones (Monderer et al., 1992), without imposing the additivity axiom but still requiring homogeneity, was obtained by Chun (1991). It is also worth noting a recent work of Nowak (1997) presenting an axiomatization of the Banzhaf value without the additivity assumption.

The paper is organized as follows. In Section 2 we introduce basic definitions and notation. Section 3 presents our main results. Section 4 provides some concluding remarks.

2. The framework

An *n-person game in characteristic function form* or a *transferable utility game* (*TU game*) is a pair $\langle N,v\rangle$ where $N=\{1,\ldots,n\}$ and v is a mapping $v{:}2^N\to R^1$ such that $v(\emptyset)=0$. N is the set of *players* and 2^N denotes the family of all *coalitions* $S\subset N$. In this context a class of games with a fixed set N is naturally identified with the Euclidean space R^{2^n-1} of vectors v, $v=\{v_S\}_{S\neq\emptyset}^{S\subseteq N}$ (we shall occasionally refer to $v_\emptyset$ setting $v_\emptyset=0$). A *value* is a mapping $\xi{:}R^{2^n-1}\to R^n$ which associates with every game $v\in R^{2^n-1}$ a vector $\xi(v)\in R^n$. The real number $\xi_i(v)$ represents the *payoff* to player i in the game v.

A value ξ is *efficient* if for all $v\in R^{2^n-1}$

$$\sum_{i\in N}\xi_i(v)=v_N.$$

A value ξ is *marginalist* if for all $v \in R^{2^n-1}$ and every $i \in N$ $\xi_i(v)$ depends only upon the ith marginal utility vector $\{v_{S\cup\{i\}} - v_S\}_{S\subset N\setminus\{i\}}$, i.e.

$$\xi_i(v) = \phi_i(\{v_{S\cup\{i\}} - v_S\}_{S\subset N\setminus\{i\}})$$

where $\phi_i{:}R^{2^{n-1}} \to R^1$.

Let Π be the set of n! permutations of N and let for all $v \in R^{2^n-1}$ and for any $\pi \in \Pi$, $\pi N = \{i_1, i_2, \ldots i_n\}$, $x^\pi(v) \in R^n$ denote the n-dimensional marginal contribution vector

$$x^\pi_{i_k}(v) = v_{\{i_1,\ldots,i_{k-1},i_k\}} - v_{\{i_1,\ldots,i_{k-1}\}}. \tag{1}$$

It is not difficult to check that for all $v \in R^{2^n-1}$ and for all $\pi \in \Pi$ the vector $x^\pi(v)$ is efficient. One can easily see that for every marginalist value the payoff to any player $i \in N$ may be represented in the form

$$\xi_i(v) = \phi_i(\{x^\pi_i(v)\}_{\pi\in\Pi}) \tag{2}$$

where $\phi_i{:}R^{n!} \to R^1$.

A value ξ is *monotonic* if for any $i \in N$ for all $v, v' \in R^{2n-1}$

$$\left.\begin{array}{ll} v'_S \geq v_S, & \forall S \ni i, \\ v'_S = v_S, & \forall S \not\ni i, \end{array}\right\} \Rightarrow \xi_i(v') \geq \xi_i(v).$$

Remark. Under marginalism monotonicity is equivalent to all functions ϕ_i, $i \in N$, in the definition of marginalism and in particular in Eq. (2), being monotonic.

Notice that this notion of monotonicity coincides with the coalitional monotonicity in Young (1985).

A player i is a *null-player* in the game $v \in R^{2^n-1}$ if $v_{S\cup\{i\}} = v_S$ for every $S \subset N\backslash\{i\}$. A value ξ possesses *the null-player property* if for all $v \in R^{2^n-1}$ for every null-player i in v $\xi_i(v) = 0$.

A value ξ is *additive* if for any $v_1, v_2 \in R^{2^n-1}$

$$\xi(v_1 + v_2) = \xi(v_1) + \xi(v_2).$$

A value ξ is *covariant* if for all $v \in R^{2^n-1}$ for any $\alpha > 0$ and $\beta \in R^n$

$$\alpha\xi(v) + \beta = \xi\left(\left\{\alpha v_S + \sum_{j\in S} \beta_j\right\}_{\substack{S\subset N\\ S\neq\emptyset}}\right).$$

A value ξ is a *random-order value* if for all $v \in R^{2^n-1}$ and every $i \in N$

$$\xi_i(v) = \sum_{\pi\in\Pi} r_\pi x^\pi_i(v)$$

for some probability distribution $\{r_\pi{:}\pi \in \Pi\}$ over the set Π.

In the sequel the Shapley value in a game $v \in R^{2^n-1}$ we denote by Sh(v). For the mean value of any vector $x = \{x_i\}_{i=1}^n$ we use notation $\bar{x}$, i.e. $\bar{x} = \Sigma_{i=1}^n x_i$. By **0** and **e** we denote the vectors with all components equal to 0 or to 1, respectively.

3. Main results

We start with the next lemma. The similar statement was proved earlier by Rubinstein and Fishburn (1986) but under somewhat stronger conditions being aimed for application to purpose of preference aggregation and not to cooperative game theory.

Lemma 1. *Let* $n \geq 3$ *and let A be some fixed constant. Let* $D \subseteq \mathrm{R}^m$ *be such that* $\mathbf{0} \in D$, *for any* $x \in D$ $A\mathbf{e} - x \in D$, *and for any* $x_1, x_2 \in D$ $x_1 + x_2 \in D$, *and let* $\psi_i : D \to R^1$, $i \in \overline{1,n}$, *possess the property* $\psi_i(\mathbf{0}) = 0$. *Then*:

(1) *for the equality*

$$\sum_{i=1}^{n} \psi_i(\{x_i^j\}_{j=1}^m) = A \tag{3}$$

to hold for every set $\{x^j\}_{j=1}^m$ *of m vectors* $x^j \in R^n$, $x^j = \{x_i^{\ j}\}_{i=1}^n$, *satisfying the conditions (i) for all* $i \in \overline{1,n}$ $\{x_i^j\}_{j=1}^m \in D$ *and (ii) for all* $j \in \overline{1,m}$

$$\sum_{i=1}^{n} x_i^j = A, \tag{4}$$

the functions ψ_i, $i \in \overline{1,n}$, *have to be identical and additive, i.e. for all* $i,k \in \overline{1,n}$, $i \neq k$, $\psi_i = \psi_k = \psi$ *under some* $\psi : D \to R^1$, *and for any* $x_1, x_2 \in D$ $\psi(x_1) + \psi(x_2) = \psi(x_1 + x_2)$;

(2) *if, moreover, D forms a subspace* R^m *and the function* ψ *appears to be continuous or monotonic*, Eq. (3) *is valid for every set* $\{x^j\}_{j=1}^m$, $x^j \in R^n$, $x^j = \{x_i^{\ j}\}_{i=1}^n$, *complying with the conditions (i) and (ii), if and only if* ψ *is linear, i.e. for any* $x \in D$, $x = \{x^j\}_{j=1}^m$, $\psi(x)$ *has the form*

$$\psi(x) = \sum_{j=1}^{m} a_j x^j, \tag{5}$$

for some $\{a_j\}_{j=1}^m \in R^m$ *and besides, if* $A \neq 0$ *the coefficients* a_j, $j \in \overline{1,m}$, *are such that*

$$\sum_{j=1}^{m} a_j = 1, \tag{6}$$

and furthermore, for monotonic ψ $\{a_j\}_{j=1}^m \in R_+^m$.

Proof of Lemma 1. First we take an arbitrary $y \in D$, $y = \{y^j\}_{j=1}^{\ m}$, and fix any two distinct indices $i_1, i_2 \in \overline{1,n}$, $i_1 \neq i_2$. Consider the set of vectors $\{x^j\}_{j=1}^{\ m}$, $x^j \in R^n$, *such that for every* $j \in \overline{1,m}$

$$x_i^j = \begin{cases} y^j, & i = i_1, \\ A - y^j, & i = i_2, \\ 0, & i \neq i_1, i_2. \end{cases}$$

Evidently, the set $\{x^j\}_{j=1}^m$ *meets the conditions* (*i*) *and* (*ii*). *Hence, due to the condition* $\psi_i(\mathbf{0}) = 0$, $i \in \overline{1,n}$, *we arrive at*

$$\sum_{i=1}^{n} \psi_i(\{x_i^j\}_{j=1}^m) = \psi_{i_1}(y) + \psi_{i_2}(A\mathbf{e} - y).$$

By Eq. (3) *for all* $y \in D$ *the last is equal to A. From here since* $n \geq 3$ *and because of the arbitrariness of* $i_1, i_2 \in \overline{1,n}$, *it follows that all functions* ψ_i, $i \in \overline{1,n}$, *are identical* (*further we will denote them by* ψ), *and besides, for every* $y \in D$

$$\psi(y) = A - \psi(A\mathbf{e} - y). \tag{7}$$

Next we take two arbitrary points $y_1, y_2 \in D$, $y_i = \{y_i^j\}_{j=1}^m$, $i = 1,2$, *fix three distinct indices* $i_1, i_2, i_3 \in \overline{1,n}$ *and consider the set* $\{x^j\}_{j=1}^m$, $x^j \in R^n$, *such that for all* $j \in \overline{1,m}$

$$x_i^j = \begin{cases} y_1^j, & i = i_1, \\ y_2^j, & i = i_2, \\ A - (y_1^j + y_2^j), & i = i_3, \\ 0, & i \neq i_1, i_2, i_3. \end{cases}$$

The conditions (*i*) *and* (*ii*) *are true and therefore*

$$\sum_{i=1}^{n} \psi(\{x_i^j\}_{j=1}^m) = \psi(y_1) + \psi(y_2) + \psi(A\mathbf{e} - y_1 - y_2).$$

Whence by Eqs. (7) *and* (3) *we arrive at*

$$\psi(y_1) + \psi(y_2) = \psi(y_1 + y_2),$$

what completes the proof of the first issue.

Now we turn to the proof of the second one. One can easily check that for any $D \subseteq R^m$ *for every set* $\{x^j\}_{j=1}^m$, $x^j \in R^n$, $x^j = \{x_i^j\}_{i=1}^n$, satisfying the conditions (i) and (ii) the equality Eq. (3) is true for every set of identical functions $\{\psi_i\}_{i=1}^n$ of the form of Eqs. (5) and (6). Indeed,

$$\sum_{i=1}^{n} \psi(\{x_i^j\}_{j=1}^m) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_j x_i^j = \sum_{j=1}^{m} a_j \sum_{i=1}^{n} x_i^j = A.$$

Let us give evidence to the contrary. It is not difficult to verify that for any $D \subseteq R^m$ for every additive function ψ and for all $x \in D$ with rational components x^j, $j \in \overline{1, m}$, the equality Eq. (5) is valid under $a_j = \psi(e_j)$ where vectors $\{e_j\}_{j=1}^m$, $e_j \in R^m$, form a basis of R^m. The last statement can be extended easily to all vectors $x \in D$ with real components provided D is a subspace in R^m and the function ψ is either continuous or monotonic.

An equality Eq. (6) is a direct corollary of Eqs. (5), (3), (4) under condition $A \neq 0$.

The equivalence of the condition $\{a_j\}_{j=1}^m \in R_+^m$ to the monotonicity of any ψ having the form Eq. (5) is obvious. ■

Remark 1. It is worth to note that for a function ψ to possess the property $\psi(\mathbf{0}) = 0$ it suffices to be continuous and homogeneous.

Remark 2. The condition $n \geq 3$ is an essential condition in Lemma 1. Indeed, in the case $n=2$ the functions $\psi_i : R^m \to R^1$, $i=1,2$, which for every $x \in R^m$, $x=\{x^j\}_{j=1}^m$, have the form

$$\psi_1(x) = f(x)$$
$$\psi_2(x) = A - f(A\mathbf{e} - x)$$

under any $f : R^m \to R^1$ possessing the properties $f(\mathbf{0})=0$ and $f(A\mathbf{e})=A$ satisfy the conditions of Lemma 1. However, for instance for

$$f(x) = \sqrt[3]{\frac{1}{m}\sum_{j=1}^{m}(x^j)^3}, \quad f(x) = \sqrt[m]{\prod_{j=1}^{m} x^j} \quad \text{(if } m \text{ is odd), or } f(x) = \min_{j \in \overline{1,m}} x^j$$

functions ψ_1 and ψ_2 are neither identical nor additive. Note that in all three latter cases functions ψ_1 and ψ_2 are also monotonic.

Theorem 1. *Let $n \geq 3$ and a mapping $\xi : R^{2^n-1} \to R^n$ be a marginalist and efficient value possessing the null-player property which is also continuous or monotonic. Then*

(1) *for every $v \in R^{2^n-1}$ $\xi(v)$ has the form*

$$\xi(v) = \sum_{\pi \in \Pi} r_\pi x^\pi(v) \tag{8}$$

with some $\{r_\pi\}_{\pi \in \Pi} \in R^{n!}$ such that

$$\sum_{\pi \in \Pi} r_\pi = 1; \tag{9}$$

and moreover, for monotonic ξ $\{r_\pi\}_{\pi \in \Pi} \in R_+^{n!}$;

(2) *the value ξ is additive and covariant.*

Proof of Theorem 1. We use for ξ the representation in the form of Eq. (2). Fix any value v_N of the worth of grand coalition N and consider a restriction of ξ to a subclass of games $G(v_N)=\{v' \in R^{2^n-1} | v'_N = v_N\}$ with the worth of grand coalition equal to v_N. Next note that: first, the null-player property for ξ is just the condition of $\phi_i(\mathbf{0})=0$ for all $\phi_i : R^{n!} \to R^1$, $i \in N$, in Eq. (2); second, all marginal contribution vectors $x^\pi(v)$, $\pi \in \Pi$ are efficient; third, due to hypothesis of continuity or monotonicity for ξ all functions ϕ_i, $i \in N$, are either continuous or monotonic because of Remark after the Definition of Monotonicity; and fourth, for every $i \in N$ the set of all $n!$-vectors $\{x_i^\pi(v)\}_{\pi \in \Pi}$ under $v \in G(v_N)$ forms a (2^n-2)-dimensional subspace in $R^{n!}$. Whence due to Lemma 1 on $G(v_N)$ ξ has the form of Eqs. (8) and (9). Notice that $R^{2^n-1} = \cup_{v_N \in R^1} G(v_N)$. Thus, for every $v \in R^{2^n-1}$ $\xi(v)$ has the form

$$\xi(v) = \sum_{\pi \in \Pi} r_\pi(v_N) x^\pi(v)$$

with some $\{r_\pi(v_N)\}_{\pi \in \Pi} \in R^{n!}$ such that

$$\sum_{\pi \in \Pi} r_\pi(v_N) = 1,$$

and for monotonic ξ $\{r_\pi(v_N)\}_{\pi \in \Pi} \in R_+^{n!}$.

But because of marginalism all $r_\pi(v_N)$, $\pi \in \Pi$, *must be independent of* v_N. *Therefore, for all* $v \in R^{2^n-1}$ $\xi(v)$ *has the form of Eqs.* (8) *and* (9).

The additivity of ξ *follows directly from Eq.* (8) *and the definition of vectors* $x^\pi(v)$, $\pi \in \Pi$.

Finally, the covariance of $\xi(v)$ *runs out from Eq.* (9). *Indeed, for any* $\alpha > 0$ *and* $\beta \in R^n$ for all $v \in R^{2^n-1}$ for every $i \in N$

$$\xi_i(\left\{\alpha v_S + \sum_{j \in S} \beta_j\right\}_{\substack{S \subset N \\ S \neq \emptyset}} = \sum_{\pi \in \Pi} r_\pi(\alpha x_i^\pi(v) + \beta_i) = \alpha \sum_{\pi \in \Pi} r_\pi x_i^\pi(v) + \beta_i \sum_{\pi \in \Pi} r_\pi =$$
$$\alpha \xi_i(v) + \beta_i. \qquad \blacksquare$$

A direct corollary to Theorem 1 is the following axiomatization without the linearity property for random-order values.

Theorem 2. *Let* $n \geq 3$ *then the only set of marginalist, efficient, and monotonic values possessing the null-player property is the set of random-order values.*

It turns out that the null-player property may be avoided via replacing by covariance.

Theorem 3. *Let* $n \geq 3$ *then the only set of marginalist, efficient, monotonic, and covariant values is the set of random-order values.*

The validity of Theorem 3 arises from Theorems 1 and 2 and the next lemma.

Lemma 2. *Every marginalist, monotonic, and covariant value possesses the null-player property.*

Proof of Lemma 2. As usual, for ξ we use a representation in the form of Eq. (2). Due to covariance of ξ and the definition Eq. (1) of $x^\pi(v)$, $\pi \in \Pi$, for every $i \in N$ for all $x \in R^{n!}$ for any $\alpha > 0$ and $\beta \in R^1$

$$\alpha \phi_i(x) + \beta = \phi_i(\{\alpha x_j + \beta\}_{j=1}^{n!}). \tag{10}$$

From where first, setting $\beta = 0$ it follows that every ϕ_i, $i \in N$, is the first degree homogeneous and second, it is not hard to verify the continuity of any ϕ_i. Indeed, if for any vector y by $(y)_M$ and $(y)_m$ we denote its maximum and minimum component, respectively, then because the monotonicity of ξ and since Eq. (10) for all $i \in N$ and every $x, \Delta x \in R^{n!}$

$$\phi_i(x) + (\Delta x)_m \leq \phi_i(x + \Delta x) \leq \phi_i(x) + (\Delta x)_M.$$

Whereby the continuity of every ϕ_i, $i \in N$, follows immediately. To complete the proof it remains to refer to Remark 1 of Lemma 1. $\blacksquare$

Remark 1. The marginalism is an essential condition in Lemma 2: not every monotonic and covariant value possesses the null-player property. Indeed, the value ξ *defined for all* $v\in R^{2^n-1}$ for any $i\in N$ by formula

$$\xi_i(v)=v_N-\sum_{\substack{j\in N\\ j\neq i}} v_{\{j\}}$$

is monotonic and covariant but it does not possess the null-player property and is not a marginalist one.

Remark 2. Setting $\alpha=1$ and $\beta=\bar{x}$ in Eq. (10), and noticing that $\mathrm{Sh}_i(v)=\overline{x^{\pi}}(v)$ one can easily see that every marginalist and covariant value ξ for all $v\in R^{2^n-1}$ for every $i\in N$ has the form

$$\xi_i(v)=\mathrm{Sh}_i(v)+\phi_i(\{x_i^{\pi}(v)-\mathrm{Sh}_i(v)\}_{\pi\in\Pi})$$

under some homogeneous of the first degree function $\phi_i\colon R^{n!}\to R^1$.

Theorem 4. *All the axioms in the hypothesis of both Theorem* 2 *and Theorem* 3 *are independent*.

The proof of Theorem 4 is provided by the next list of examples showing that if we remove anyone of axioms from the hypothesis of Theorem 2 or Theorem 3, more than the random-order values can meet the others.

(i) *Marginalism:*

(a) The center of imputation set value (CIS-value)

$$\xi_i(v)=\frac{1}{n}v_N+\frac{n-1}{n}v_i-\frac{1}{n}\sum_{\substack{j\in N\\ j\neq i}} v_{\{j\}}$$

is an efficient, monotonic, and covariant value which violates marginalism.

(b) Consider a function $r\colon R^{2^n-1}\to R^1$

$$r(v)=\frac{1}{\pi}\arctan(v_N-v_{N\setminus\{n\}}-v_{N\setminus\{n-1\}}+v_{N\setminus\{n-1,n\}})+\frac{1}{2}.$$

and two permutations: $\pi_1=1,2,\dots,n-1,n$ and $\pi_2=1,2,\dots,n,n-1$. The next non-marginalist value ξ^2 defined for all $v\in R^{2^n-1}$ by the formula

$$\xi^2(v)=r(v)x^{\pi_1}(v)+(1-r(v))x^{\pi_2}(v)$$

or which is the same

$$\xi_i^2(v)=\begin{cases} v_{\{1,\dots,i-1,i\}}-v_{\{1,\dots,i-1\}}, & i\neq n-1,n\\ r(v)(v_{N\setminus\{n\}}-v_{N\setminus\{n-1,n\}})+(1-r(v))(v_N-v_{N\setminus\{n-1\}}), & i=n-1\\ r(v)(v_N-v_{N\setminus\{n\}})+(1-r(v))(v_{N\setminus\{n-1\}}-v_{N\setminus\{n-1,n\}}), & i=n\end{cases}$$

presents an example of an efficient and monotonic value possessing the null-player property.[1] (To check monotonicity it suffices only to show that derivatives $\partial\xi^2_{n-1}/\partial v_N$, $\partial\xi^2_n/\partial v_N$, $\partial\xi^2_{n-1}/\partial v_{N\setminus\{n\}}$, $\partial\xi^2_n/\partial v_{N\setminus\{n-1\}}\geq 0$.)

(ii) *Efficiency:* The Banzhaf value gives an example of a marginalist, monotonic, and covariant value who possesses the null-player property but that is not efficient.

(iii) *Monotonicity:* Every value $\xi^3(v)$ that for each $v\in R^{2^n-1}$ $\xi_3(v)$ has the form

$$\xi^3_i(v)=\sum_{\pi\in\Pi} r_\pi x^\pi_i(v)$$

with some $\{r_\pi\}_{\pi\in\Pi}\in R^{n!}$ such that

$$\sum_{\pi\in\Pi} r_\pi = 1$$

is a marginalist, efficient, and covariant value also possessing the null-player property but if $\{r_\pi\}_{\pi\in\Pi}\not\in R^{n!}_+$ it has not be a monotonic one.

(iv) *Covariance and null-player:* Fix any $i_0\in N$ and any arbitrary permutation $\pi_0\in\Pi$ and consider the value ξ^4 defined for every $v\in R^{2^n-1}$ for all $i\in N$ as follows

$$\xi^4_i(v)=x^{\pi_0}_i(v)+\begin{cases}\dfrac{n-1}{n}, & i=i_0;\\ -\dfrac{1}{n}, & i\neq i_0.\end{cases}$$

Just defined value ξ^4 is a marginalist, efficient, and monotonic one who is neither covariant nor possesses the null-player property and violates additivity as well.

(v) $n=2$: The value

$$\xi^5_1(v)=\min_{\pi\in\Pi} x^\pi_1(v)$$
$$\xi^5_2(v)=\max_{\pi\in\Pi} x^\pi_2(v)$$

provides an example of a marginalist, efficient, monotonic and covariant value which possesses the null-player property but which is neither a random-order value nor is additive.

4. Concluding remarks

The original Weber's axiomatization for random-order values comes out of the classical one for the Shapley value via replacing symmetry by monotonicity and homogeneity which is of course true for the latter as well but is not necessary for completeness of the axiomatic system. Our axiomatization for random-order values differs from that of Young for the Shapley value in a similar way. We replace symmetry

[1] This example is due to Elena Yanovskaya.

by either monotonicity and the null-player property or monotonicity and covariance. All these properties are certainly valid for the Shapley value too, but no one of them is required for completeness of Young's system of axioms. In addition it is also worth to notice that the notions of monotonicity in these two cases are slightly distinct: Weber (1988) applied the definition of monotonicity that some other authors call positivity (in particular, Chun, 1991) which means that a value for any monotonic game is non-negative.

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