13th National Conference on Mechanisms and Machines (NaCoMM07),
IISc, Bangalore, India. December 12-13, 2007

NaCoMM-2007-053

A Spatial Nine-Bar Linkage, Possible Configurations and Conditions for Paradoxical Mobility

Dominic R. Walter, Manfred L. Husty*
Institute for Basic Sciences in Engineering, Unit Geometry and CAD
University of Innsbruck, Austria

* Corresponding author (email: manfred.husty@uibk.ac.at)

Abstract

This paper is about a linkage consisting of nine bars connected by spherical joints, where the bars form a spatial hexagon with its main diagonals. Additionally the supporting lines of the diagonals have one proper point in common. This linkage is rigid in general. Wunderlich [1] stated that, given the lengths of the bars, the computation of possible configurations (up to isometries) is a problem of degree 25. By direct computation using resultants it can be shown that the number of essentially different configurations is at most 21, and that Wunderlich's result includes four solutions lying at infinity which are not interesting for practical purposes. Attempts to find a set of design parameters where all 21 solutions allow real assembling led without exception to at most 10 such solutions. Until now it is not clear if this is already the maximum. Concerning paradoxical mobility Wunderlich discussed two mechanisms, a general one and a special case of it, related to Bricard's mechanism [2]. Systematic search for all mobile linkages showed that there is another mobile one which cannot be derived from the known mechanisms by specialization.

Keywords: spatial nine-bar linkage, assembly modes, paradoxical mobility, Bricard mechanism

1 Introduction

In the following we present a complete discussion of a nine-bar linkage concerning the number of assembly modes and conditions for paradoxical mobility. The structure consists of nine bars of given lengths, which are connected by spherical joints, so that they form a spatial hexagon with the diagonals connecting opposite vertices. Additionally the supporting lines of the diagonals have to intersect in at least one proper point, as it can be seen in Fig. (1). The question, how the joints and the intersection point could be implemented for practical assembly, is not that easy and will not be answered here.

This linkage is a spatial generalization of a planar linkage consisting of a planar hexagon and its diagonals connected by revolute joints, where the diagonals need not intersect in one point. This structure had been discussed in detail by Wunderlich [3] and recently by Walter and Husty [4]. Paradoxical versions of this linkage have been found by Dixon [5] and are known as Dixon's mechanisms.

Dropping the constraint of intersecting diagonals the spatial linkage would be mobile with three degrees of freedom, because there are 12 relevant coordinates and 9 equations describing the distances. On the other hand from the condition of intersection it follows that every pair of diagonals has to be coplanar, and so there are another three equations. From this we can conclude that the linkage is rigid in general.

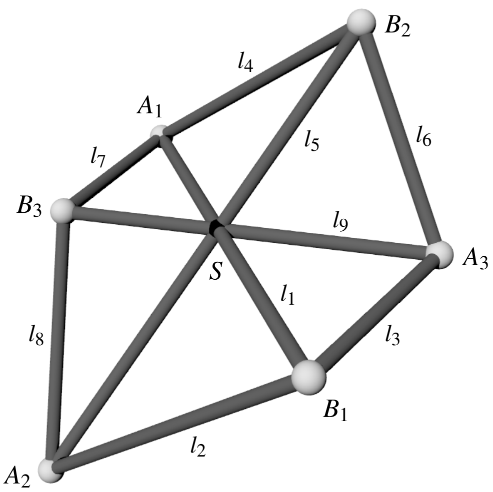

Figure 1: The spatial nine-bar linkage with given lengths l_i and intersection of diagonals at point **S**.

Using resultants it can be shown that the number of assembly modes (up to isometries) is at most 21 or 84 respectively, when also all these solutions are taken into account, which can be generated by reflections. Attempts to find a set of design parameters where all essentially different solutions allow real assembling failed. The number of real solutions never exceeded 10 and until now it is not clear if this is the maximum number.

As mentioned above the linkage is rigid generally. But on the other hand if we choose special design parameters the structure becomes mobile and we speak about "paradoxical" mobility. Wunderlich discussed two mechanisms where the second one can be seen as a special case of the first mechanism. Furthermore the second mechanism has the interesting property that all spherical joints can be replaced by revolute joints and the diagonals become redundant. The result is the well-known Bricard mechanism. Systematic search for conditions where the linkage becomes mobile showed that there exists another new nontrivial mechanism apart from these two mechanisms.

The paper is organized as follows. In Section 2 we deduce a system of polynomial equations which gives a complete description of the linkage. In Section 3 the system is solved and remarkably this can be done without specifying any design parameters. The result is a univariate polynomial of degree 21 with coefficients being polynomials in the design parameters. Starting with this polynomial all essentially different assembly modes can be computed. After that we explain the difference to Wunderlich's result and subsequently we look briefly at the case where the intersection $\mathbf{S}$ lies at infinity. Section 4 provides an example where two solutions lead to real coordinates and Section 5 gives necessary conditions for real solutions and a short description about the problems we had when we searched for such real solutions. In Section 6 we describe known mechanisms, the method how we sought for other mobile linkages and finally we present a new one.

2 Equations

The design of the linkage is given by the bar lengths l_i with $0 < l_i \in \mathbb{R},\ i = 1,\ldots,9$, including the information, which bars meet at which vertex. First of all the following nine equations describe the Euclidean distances between the vertices where $\mathbf{v}_{XY}$ denotes the vector pointing from vertex $\mathbf{X}$ to vertex $\mathbf{Y}$, see also Fig. (1).

$$\begin{array}{lll} \|\mathbf{v}_{B_1A_1}\|^2 = l_1^2 & \|\mathbf{v}_{B_1A_2}\|^2 = l_2^2 & \|\mathbf{v}_{B_1A_3}\|^2 = l_3^2 \\ \|\mathbf{v}_{B_2A_1}\|^2 = l_4^2 & \|\mathbf{v}_{B_2A_2}\|^2 = l_5^2 & \|\mathbf{v}_{B_2A_3}\|^2 = l_6^2 \\ \|\mathbf{v}_{B_3A_1}\|^2 = l_7^2 & \|\mathbf{v}_{B_3A_2}\|^2 = l_8^2 & \|\mathbf{v}_{B_3A_3}\|^2 = l_9^2 \end{array} \quad (1)$$

Because all vertices lie on lines through $\mathbf{S}$ we describe each vertex as the sum of $\mathbf{S}$ and a multiple of a unit vector $\mathbf{V}_i$. Additionally we can claim that $\mathbf{V}_i$ has the same direction as $\overrightarrow{\mathbf{A}_i\mathbf{B}_i}$. From this condition it follows that $\nu_i > \mu_i$.

$$\mathbf{A}_i = \mathbf{S} + \mu_i \mathbf{V}_i \qquad \mathbf{B}_i = \mathbf{S} + \nu_i \mathbf{V}_i \qquad \|\mathbf{V}_i\| = 1 \quad (2)$$

After simplification of (1), the elimination of scalar products using the substitutions

$$\mathbf{V}_i \mathbf{V}_j =: u_{ij} \quad (3)$$

we obtain the following equations:

$$(\mu_1 - \nu_1 + l_1)(\mu_1 - \nu_1 - l_1) = 0 \quad (4)$$
$$-2\nu_1 \mu_2 u_{12} + \nu_1{}^2 + \mu_2{}^2 - l_2{}^2 = 0 \quad (5)$$
$$-2\nu_1 \mu_3 u_{13} + \nu_1{}^2 + \mu_3{}^2 - l_3{}^2 = 0 \quad (6)$$
$$-2\nu_2 \mu_1 u_{12} + \nu_2{}^2 + \mu_1{}^2 - l_4{}^2 = 0 \quad (7)$$
$$(\mu_2 - \nu_2 + l_5)(\mu_2 - \nu_2 - l_5) = 0 \quad (8)$$
$$-2\nu_2 \mu_3 u_{23} + \nu_2{}^2 + \mu_3{}^2 - l_6{}^2 = 0 \quad (9)$$
$$-2\nu_3 \mu_1 u_{13} + \nu_3{}^2 + \mu_1{}^2 - l_7{}^2 = 0 \quad (10)$$
$$-2\nu_3 \mu_2 u_{23} + \nu_3{}^2 + \mu_2{}^2 - l_8{}^2 = 0 \quad (11)$$
$$(\mu_3 - \nu_3 + l_9)(\mu_3 - \nu_3 - l_9) = 0 \quad (12)$$

Due to the fact that the vectors $\mathbf{V}_i$ are oriented from $\mathbf{A}_i$ to $\mathbf{B}_i$ the second factor in equations (4), (8) and (12) cannot vanish. So we can remove these factors and solve the remaining equations for ν_1, ν_2, ν_3.

$$\nu_1 = \mu_1 + l_1 \qquad \nu_2 = \mu_2 + l_5 \qquad \nu_3 = \mu_3 + l_9 \quad (13)$$

These expressions we use to eliminate ν_1, ν_2, ν_3 in the remaining six equations. Finally we have the following system of equations in the unknowns u_{12}, u_{13}, u_{23} and μ_1, μ_2, μ_3.

$$E_1 := 2\,\mu_1 l_1 - 2\,(\mu_1\mu_2 + \mu_2 l_1)\,u_{12} + \mu_1^2 + \mu_2^2 + l_1^2 - l_2^2 = 0$$
$$E_2 := 2\,\mu_1 l_1 - 2\,(\mu_1\mu_3 + \mu_3 l_1)\,u_{13} + \mu_1^2 + \mu_3^2 + l_1^2 - l_3^2 = 0$$
$$E_3 := 2\,\mu_2 l_5 - 2\,(\mu_1\mu_2 + \mu_1 l_5)\,u_{12} + \mu_1^2 + \mu_2^2 + l_5^2 - l_4^2 = 0$$
$$E_4 := 2\,\mu_2 l_5 - 2\,(\mu_2\mu_3 + \mu_3 l_5)\,u_{23} + \mu_2^2 + \mu_3^2 + l_5^2 - l_6^2 = 0$$
$$E_5 := 2\,\mu_3 l_9 - 2\,(\mu_1\mu_3 + \mu_1 l_9)\,u_{13} + \mu_1^2 + \mu_3^2 + l_9^2 - l_7^2 = 0$$
$$E_6 := 2\,\mu_3 l_9 - 2\,(\mu_2\mu_3 + \mu_2 l_9)\,u_{23} + \mu_2^2 + \mu_3^2 + l_9^2 - l_8^2 = 0$$

This system is a complete description of the linkage and especially each of its solutions stands for exactly one assembly mode (up to spatial isometries).

If we have a solution of this system it is clear how coordinates for all vertices $\mathbf{A}_i$ and $\mathbf{B}_i$ can be obtained. We can set for example $\mathbf{V}_1 = (1,0,0)$ and $\mathbf{V}_2 = (V_{21}, V_{22}, 0)$ to fix an orientation of the linkage. This means that the vertices $\mathbf{A}_1, \mathbf{B}_1$ lie an a line parallel to the x-axis and $\mathbf{A}_2, \mathbf{B}_2$ in a plane parallel to the xy-plane. Then we have to solve the following system to get the remaining 5 coordinates of the vectors.

$$\begin{array}{lll} V_{21} = u_{12} & V_{21}V_{31} + V_{22}V_{32} = u_{23} & V_{21}^2 + V_{22}^2 = 1 \\ V_{31} = u_{13} & V_{31}^2 + V_{32}^2 + V_{33}^2 = 1 & \end{array}$$

In general this system has 4 solutions corresponding to reflections on the xy-plane and the xz-plane. Because these solutions can be given in closed form, we can choose one of them, for example

$$\begin{aligned} V_{21} &= u_{12} \\ V_{22} &= -\sqrt{1-u_{12}^2} \\ V_{31} &= u_{13} \\ V_{32} &= \frac{u_{12}u_{13}-u_{23}}{\sqrt{1-u_{12}^2}} \\ V_{33} &= -\sqrt{\frac{1+2u_{12}u_{13}u_{23}-u_{12}^2-u_{13}^2-u_{23}^2}{1-u_{12}^2}} \end{aligned} \tag{14}$$

It has to be mentioned that in the following we will always talk about essentially different solutions, i.e. the four possible solutions above are seen as one assembly mode.
Finally equations (13) and an arbitrarily chosen point **S** are needed to obtain the coordinates of all vertices $\mathbf{A}_i$ and $\mathbf{B}_i$. For example $\mathbf{S} = (0,0,0)$ would be a very good choice.

3 Solving the system

To solve the system of equations E_1 - E_6 resultants are used, because each unknown we want to eliminate appears only in two equations. For details about the method of elimination using resultants see [7]. All computations will be done without specifying any design parameters. First we eliminate u_{12}, u_{13}, u_{23} by computing

$$\begin{aligned} P_1(\mu_1,\mu_2) &:= \mathrm{Res}(E_1,E_3,u_{12}) \\ P_2(\mu_1,\mu_3) &:= \mathrm{Res}(E_2,E_5,u_{13}) \\ P_3(\mu_2,\mu_3) &:= \mathrm{Res}(E_4,E_6,u_{23}). \end{aligned} \tag{15}$$

where

$$\deg(P_i(\mu_j,\mu_k)) = \deg(P_i(\mu_j)) = \deg(P_i(\mu_k)) = 3.$$

It is clear that these eliminations can also be done by equating expressions. The result is the same. It can be verified easily that the triples $(0,0,0)$ and $(-l_1,-l_5,-l_9)$ are solutions of P_1, P_2 and P_3, because they cause the coefficients of the eliminated unknowns to vanish. But on the other hand these triples do not lead to valid solutions anyway. So in the end we will have to remove these solutions. Next we eliminate μ_1 by computing

$$T(\mu_2,\mu_3) := \mathrm{Res}(P_1,P_2,\mu_1).$$

We have

$$\deg(T(\mu_2,\mu_3)) = \deg(T(\mu_2)) = \deg(T(\mu_3)) = 9$$

and T consists of 1278 monomials. The final step is to eliminate μ_2 to obtain a univariate in μ_3. We will use a special procedure to do this, because attempts to compute the resultant with *Maple* and *Singular* failed for reasons of time. First the resultant of two polynomials of degree 3 resp. 9 is computed in general. We obtain a sum of 1222 monomials. Then each monomial is evaluated with the coefficients of $T(\mu_2)$ and $P_3(\mu_2)$ and simplified. Finally we sum up all these expressions and we obtain the univariate polynomial

$$R(\mu_3) = \mathrm{Res}(T,P_3,\mu_2).$$

R is of degree 23 and consists of 2.603.982 monomials. After removing the factors μ_3 and $\mu_3 + l_9$ which belong to the two parasitic solutions mentioned above, we get a univariate polynomial of degree 21 consisting of 2.353.262 monomials where the coefficients are polynomials in the parameters $l_1,\ldots,l_9$.

All in all we conclude that for arbitrary bar lengths the nine-bar linkage has at most 21 essentially different assembly modes. Wunderlich stated that the computation of possible configurations is a problem of degree 25 at most. To get this result he used Bézout's theorem. And indeed, the system of equations he dealt with has four solutions at infinity, which we can disregard however. So our work can be seen as a verification of Wunderlich's result.

For these computations we assumed that the intersection point **S** must be a proper point. Could it be possible that **S** lies at infinity? This would mean that all diagonals have to be parallel. Using this parallelism a relatively short system of equations can be set up, where coordinates of vertices are the unknowns. It can be shown that this case can only happen if the design parameters fulfil an equation of degree 16, which can be split into four factors of degree 4. Each vanishing factor leads to another solution. So, in the worst case one gets 17 solutions where **S** is a proper point, using equations E_1 - E_6, and another four solutions where **S** lies at infinity. This happens for example when we use the following bar lengths:

$$l_1 = \frac{5}{2},\ l_2 = \frac{1}{2}\sqrt{310},\ l_3 = \frac{1}{2}\sqrt{1542},\ l_4 = \sqrt{61},\ l_5 = 3$$

$$l_6 = \sqrt{497},\ l_7 = \sqrt{129},\ l_8 = \sqrt{257},\ l_9 = 7$$

4 An example

Now we can compute an example. When we use the following design parameters

$$l_1 = 10,\ l_2 = 5,\ l_3 = \sqrt{51},\ l_4 = \frac{4}{3}\sqrt{10},\ l_5 = \frac{7}{3}\sqrt{13}$$

$$l_6 = \frac{1}{3}\sqrt{571},\ l_7 = \frac{5}{2}\sqrt{3},\ l_8 = \frac{3}{2}\sqrt{11},\ l_9 = \frac{9}{2}\sqrt{3}$$

we get 21 different solutions. The values for μ_1,μ_2,μ_3 can be seen in Table 1, the corresponding values for u_{12}, u_{13}, u_{23} in Table 2.
We used $\mathbf{V}_1 = (1,0,0)$ and $\mathbf{V}_2 = (V_{21},V_{22},0)$ to fix an orientation of the linkage. The remaining coordinates of the vectors $\mathbf{V}_2$ and $\mathbf{V}_3$ were computed using formulas (14). For every solution the position of the intersection point **S** was

chosen such that $\mathbf{A}_1 = (0,0,0)$ and $\mathbf{B}_1 = (l_1,0,0)$. For the coordinates of the vertices $\mathbf{A}_2, \mathbf{B}_2, \mathbf{A}_3$ and $\mathbf{B}_3$ see Tables 3 and 4.

As one can see 17 solutions have real values for the six parameters and at first one might think that we get 17 real assembly modes. But to get the coordinates of the vectors $\mathbf{V}_i$ we have to use (14), expressions containing squareroots. That is the reason why only the first two solutions lead to real assembly modes, which can be seen in Fig. (2) and Fig. (3).

The conditions which have to be fulfilled to obtain solutions for real assembly follow in the next section.

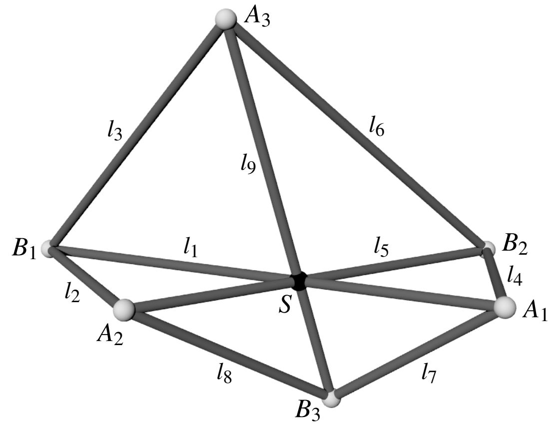

Figure 2: Real assembly of the first solution.

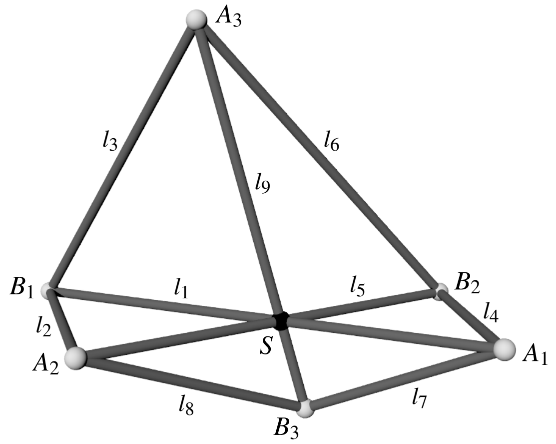

Figure 3: Real assembly of the second solution.

5 Conditions for real assembly

Here we will give a short description of the arising problems when we search for solutions which lead to real coordinates of the vertices. According to (2) and (3) the parameters u_{ij} can be written as

$$u_{ij} = \mathbf{V}_i \mathbf{V}_j = \frac{\mathbf{V}_i \mathbf{V}_j}{\|\mathbf{V}_i\| \|\mathbf{V}_j\|} = \cos\left(\measuredangle(\mathbf{V}_i, \mathbf{V}_j)\right)$$

We have the following three necessary conditions for the parameters u_{12}, u_{13}, u_{23}

$$\begin{aligned} 1 - u_{12}^2 &\geq 0 \\ 1 - u_{13}^2 &\geq 0 \\ 1 - u_{23}^2 &\geq 0 \end{aligned}$$

But there is another condition which can be taken from (14), namely

$$1 + 2u_{12}u_{13}u_{23} - u_{12}^2 - u_{13}^2 - u_{23}^2 \geq 0$$

If these four conditions are fulfilled we can be sure that we will get real coordinates using equations (14). It can easily be verified that only the first two solutions of Table 2 fulfil them all.

So, to find design parameters where all 21 solutions allow real assembly we have to make sure that the parameters μ_i are real and, moreover, that the resulting parameters u_{ij} satisfy the conditions above.

First of all a vast number of randomly chosen design parameter sets was generated and for each of them the solutions were computed. The number of solutions with real coordinates (up to isometries) varied from 0 to 10. Then it was tried to increase the number of such solutions by small perturbations, based on a set of design parameters with 10 real solutions, but this was not successful.

Another attempt to increase the actual maximum number was to apply the algorithm which Dietmaier [6] used to get 40 real postures of the Stewart-Gough platform. The main idea of this algorithm is to decrease the distance between complex solutions until they become a double solution and then to push them apart in the real domain. This is done by small changes of the design parameters where the changes are the solutions of a linear program. Dietmaier mentions that possibly the algorithm becomes stationary. This was the case which happened here. Starting with a design parameter set with 8 real assembly modes the number could only be increased up to 10. Then the algorithm became stationary. Computations using other parameters are still in progress.

Maybe 10 is the absolute maximum, but this is neither proven nor rebutted, and the search for this maximum number is subject to future work.

6 Paradoxical mobile mechanisms

Although the nine-bar linkage at hand is rigid in general nevertheless for special design parameters it becomes mobile. Such parameters can be described by a set of equations they have to fulfil.

Because of symmetry there are 12 possible permutations of the lengths (incl. identity). It follows that if we have one system of equations describing a mobile linkage, possibly we get up to 11 other systems which describe the same type of mobility. In the following we give only one version of these systems. Furthermore when we search for mobile mechanisms we will concentrate on mechanisms which allow real assembling.

6.1 Wunderlich's mechanisms

In [1] Wunderlich discussed two mobile linkages. The first one has diagonals of equal length and additionally the lengths of opposite sides of the hexagon are equal. So the first mechanism can be described by the following five conditions:

$$l_5 = l_1,\ l_9 = l_1$$

$$l_4 = l_2,\ l_7 = l_3,\ l_8 = l_6 \tag{16}$$

This system is invariant to all possible permutations of the lengths and only four lengths can be arbitrarily chosen. If we substitute lengths fulfilling (16) into equations E_1 - E_6 it can be shown that the ideal generated by this system of polynomial equations is now of dimension 1 and the linkage is mobile with one degree of freedom.
Wunderlich's second mechanism is a special case of the first one because it can be generated by adding equations

$$l_7 = l_4,\ l_8 = l_4,\ l_1 = \sqrt{3}\, l_4$$

to (16) and we finally have the following eight equations to describe that mechanism:

$$l_1 = \sqrt{3}\, l_4,\ l_5 = \sqrt{3}\, l_4,\ l_9 = \sqrt{3}\, l_4$$

$$l_2 = l_4,\ l_3 = l_4,\ l_6 = l_4,\ l_7 = l_4,\ l_8 = l_4 \tag{17}$$

Here only one length can be arbitrarily chosen. This mechanism is also mobile with one degree of freedom and it has the interesting property that all joints can be replaced by revolute joints. Thus all diagonals become redundant and can be removed. The remaining six-bar mechanism is the Bricard mechanism which is discussed e.g. in [2]. A proof for this property can also be found in [1].

6.2 Search for new mechanisms

To find all mobile linkages two cases have to be considered. In the first case we claim that the number of solutions for μ_1, μ_2, μ_3 is finite. It follows that for mobility one of the unknowns u_{12}, u_{13}, u_{23} has to have infinitely many solutions. Regarding equations E_1 - E_6 this is only possible if the coefficients of one of these unknowns vanish. Further computations show that this case leads only to trivial mobile linkages, where vertices coincide and where the mechanism itself consists of two triangles sharing one side.

The second case is the main part of the search. Here we assume that at least one unknown μ_i has infinitely many solutions. It follows immediately from (15) that this has to hold also for the other two unknowns because all leading monomials are not mixed. All together we have to search for design parameters where all three univariate polynomials in the unknowns μ_1, μ_2, u_3 vanish.

To find out for which lengths those univariate polynomials vanish we do not use the univariate polynomials themselves because they are to large. Let us recall how the resultants and the final univariate polynomials can be computed from equations (15).

$$\begin{aligned} T_{23} &:= T(\mu_2, \mu_3) = \mathrm{Res}(P_1, P_2, \mu_1) \\ T_{13} &:= T(\mu_1, \mu_3) = \mathrm{Res}(P_1, P_3, \mu_2) \\ T_{12} &:= T(\mu_1, \mu_2) = \mathrm{Res}(P_2, P_3, \mu_3) \end{aligned} \tag{18}$$

$$\begin{aligned} R(\mu_3) &:= \mathrm{Res}(T_{23}, P_3, \mu_2) \\ R(\mu_2) &:= \mathrm{Res}(T_{12}, P_1, \mu_1) \\ R(\mu_1) &:= \mathrm{Res}(T_{13}, P_2, \mu_3) \end{aligned} \tag{19}$$

So e.g. equation $R(\mu_3)$ vanishes identically if T_{23} and P_3 have a factor in common. The degrees of these polynomials are

$$\deg(T_{23}(\mu_2, \mu_3)) = 9 \qquad \deg(P_3(\mu_2, \mu_3)) = 3$$

and we can split this task into three different cases.
Either P_3 is a factor of T_{23} or they have a factor of degree 2 resp. 1 in common. Each of these cases can be discussed now using a general ansatz and comparison of coefficients. E.g. in the last case this means that we write P_3 and T_{23} as product of a linear polynomial and a general polynomial of degree 2 resp. 8. Comparison of coefficients gives a set of equations containing design parameters and the coefficients of the two general polynomials. Then these coefficients are eliminated.

In this way we obtain three systems of equations in the design parameters. The same procedure can be done with the two pairs T_{12} and P_1 resp. T_{13} and P_2 and then all combinations can be generated to get $3 \cdot 3 \cdot 3 = 27$ systems.

Now each system can be solved. We used factorization and methods from algebraic geometry to split each system into smaller systems. All solutions containing equations like $l_k = 0$ were removed, moreover solutions containing equations like $l_i = l_j + l_k + l_m$, because such an equation means that the linkage contains a rigid quadrangle from which it follows that the whole linkage is rigid. After elimination of solutions describing special cases of others and removal of solutions which can be generated from others by permutation of the lengths, only two solutions remained. One of them is Wunderlich's first mechanism and the second one is a new one. It is clear that Wunderlich's second mechanism did not appear because it is a special case of the first one.

The new mechanism can be described by the following conditions for the design parameters:

$$l_5 = l_1$$

$$l_7 = l_8,\ l_3 = l_6,\ l_4 = l_2 \tag{20}$$

One could say that this type of mobility is the most general one, because we only need four equations to describe it. Among other things here we have conditions that adjacent sides of the hexagon have to be of the same length. Such conditions do not appear in Wunderlich's first mechanism.

7 Conclusion

A complete discussion of a nine-bar linkage was given and the maximum number of possible assembly modes could be computed. Concerning paradoxical mobility all non-trivial mobile linkages could be described and especially, a new mechanism was found.

Subject to future research is the search for the maximum number of assembly modes with real coordinates. Maybe methods from algebraic geometry could bring some additional insight.

References

[1] W. Wunderlich, "Windschiefe Gelenksechsecke mit schneidenden Diagonalen", *Rad, Jugoslav Academy of Sciences and Arts*, Vol. 382, 1978, pp. 115–127.

[2] S. Kunze, H. Stachel, "Über ein sechsgliedriges räumliches Getriebe", *Elemente der Mathematik*, Vol. 29, 1974, pp. 25–32.

[3] W. Wunderlich, "On deformable nine-bar linkages with six triple joints", Proc. K. Nederl. Akad. Wet., A 79, 1976, 257–262.

[4] D. Walter, M. Husty, "On A Nine-Bar Mechanism, Its Possible Configurations and Conditions for Flexibility", *Proceedings of IFToMM World Congress*, Besancon, 2007, 6p.

[5] A.C. Dixon, "On certain deformable frameworks", Mess. Math. 29 ,1899/1900, 1–21.

[6] P. Dietmaier, "The Stewart-Gough Platform of General Geometry can have 40 Real Postures", *Advances in Robot Kinematics: Analysis and Control, J. Lenarcic and M. L. Husty (eds.)*, 1998, pp. 7–16.

[7] D. Cox, J. Little, D. O'Shea, "Using Algebraic Geometry", Graduate Texts in Mathematics 185, Springer, 1998, pp. 71–78.

Appendix

Table 1: Solutions for μ_1, μ_2, μ_3

μ_1	μ_2	μ_3
-4.00	-3.61	-5.20
-4.38	-4.24	-5.94
101.32	-94.65	-95.53
12.37	12.82	-21.30
15.52	-19.11	-19.93
-16.80	11.63	-16.29
-1.21	-2.02	-11.13
6.44	6.80	-10.00
-4.21	-8.42	-8.02
-10.01	5.01	-7.83
4.23	-8.42	-7.69
-0.18	-12.46	-3.31
-0.73	-0.95	-2.90
-3.68	-1.72	0.77
-18.30	-17.47	0.91
-9.38	-7.10	6.51
86.02	-81.57	86.19
-12.35 - 0.99 i	-2.45 + 0.95 i	-3.96 + 0.90 i
-12.35 + 0.99 i	-2.45 - 0.95 i	-3.96 - 0.90 i
-10.01 - 0.11 i	-4.98 + 0.15 i	-8.03 - 0.43 i
-10.01 + 0.11 i	-4.98 - 0.15 i	-8.03 + 0.43 i

Table 2: Solutions for u_{12}, u_{13}, u_{23}

u_{12}	u_{13}	u_{23}
-0.55	-0.19	0.27
-0.52	-0.24	0.22
-1.01	-1.01	1.00
1.12	-0.95	-0.93
-1.02	-0.98	1.05
-0.99	1.18	-0.92
-1.59	-0.77	-0.71
1.30	-0.97	-0.88
-0.81	-0.50	12.24
-0.98	111.50	-0.85
-1.04	-0.96	-25.80
-0.93	-0.87	-1.35
-3.52	-0.81	-0.02
-0.82	-1.08	-1.75
1.20	-1.23	-1.17
-2.92	-1.02	-1.13
-1.01	1.00	-1.01
-1.15 + 0.03 i	-1.53 + 0.15 i	0.29 - 0.07 i
-1.15 - 0.03 i	-1.53 - 0.15 i	0.29 + 0.07 i
-1.37 + 0.03 i	4.16 - 7.17 i	-0.24 - 0.12 i
-1.37 - 0.03 i	4.16 + 7.17 i	-0.24 + 0.12 i

Table 3: Solutions for vertices $\mathbf{A}_2$ and $\mathbf{B}_2$

$\mathbf{A}_2$	$\mathbf{B}_2$
(6.00, 3.00, 0.00)	(1.33, -4.00, 0.00)
(6.56, 3.63, 0.00)	(2.23, -3.58, 0.00)
(-5.53, 14.71 i, 0.00)	(-14.05, 13.40 i, 0.00)
(1.93, -6.34 i, 0.00)	(11.31, -10.50 i, 0.00)
(3.91, 3.48 i, 0.00)	(-4.65, 1.95 i, 0.00)
(5.30, -1.71, 0.00)	(-3.02, -2.94, 0.00)
(4.42, 2.49 i, 0.00)	(-8.94, -7.88 i, 0.00)
(2.43, -5.68 i, 0.00)	(13.39, -12.71 i, 0.00)
(11.07, 4.89, 0.00)	(4.22, 0.00, 0.00)
(5.11, -1.06, 0.00)	(-3.11, -2.85, 0.00)
(4.50, 2.29 i, 0.00)	(-4.22, 0.00, 0.00)
(11.72, 4.69, 0.00)	(3.92, 1.52, 0.00)
(4.07, 3.20 i, 0.00)	(-25.55, -25.21 i, 0.00)
(5.10, 0.97, 0.00)	(-1.84, -3.80, 0.00)
(-2.74, 11.71 i, 0.00)	(7.39, 6.07 i, 0.00)
(30.10, 19.47 i, 0.00)	(5.54, -3.60 i, 0.00)
(-3.49, 12.53 i, 0.00)	(-12.01, 11.24 i, 0.00)
(15.13 - 0.18 i, 0.70 + 1.33 i, 0.00)	(5.46 + 0.10 i, 0.15 - 3.47 i, 0.00)
(15.13 + 0.18 i, 0.70 - 1.33 i, 0.00)	(5.46 - 0.10 i, 0.15 + 3.47 i, 0.00)
(16.83 - 0.24 i, 0.35 + 4.66 i, 0.00)	(5.30, -3.22 i, 0.00)
(16.83 + 0.24 i, 0.35 - 4.66 i, 0.00)	(5.30, 3.22 i, 0.00)

Table 4: Solutions for vertices $\mathbf{A}_3$ and $\mathbf{B}_3$

$\mathbf{A}_3$	$\mathbf{B}_3$
(5.00, 1.00, 5.00)	(3.50, -0.50, -2.50)
(5.80, 0.64, 5.74)	(3.94, -0.20, -1.79)
(-4.90, 12.32 i, 4.35 i)	(-12.77, 11.32 i, 3.99 i)
(7.81, -5.49 i, 8.74)	(0.42, -3.48 i, 5.54)
(4.03, -5.98 i, 7.15)	(-3.61, -3.64 i, 4.35)
(-2.36, 26.53, 28.38 i)	(6.81, 13.83, 14.81 i)
(9.75, 17.42 i, 18.82)	(3.77, 5.22 i, 5.63)
(3.27, -4.60 i, 5.18)	(-4.30, -1.01 i, 1.14)
(8.26, 163.38, 163.23 i)	(4.33, 4.60, 4.56 i)
(-863.16, 3984.81, 4079.38 i)	(5.90, 19.00, 19.27 i)
(3.17, 758.62 i, 758.62)	(-4.33, -10.46 i, -10.40)
(3.05, -18.90, 18.83 i)	(-3.71, 25.63, -25.52 i)
(3.07, 2.45 i, 2.99)	(-3.22, -4.14 i, -5.06)
(2.85, 3.58, -3.59 i)	(-5.57, 39.90, -40.05 i)
(17.17, 0.42 i, -0.50 i)	(7.59, 4.02 i, -4.76 i)
(2.74, -9.76 i, -9.67)	(-5.22, -21.45 i, -21.25)
(0.40, 3.42 i, -5.02 i)	(8.22, 3.73 i, -5.47 i)
(18.26 - 0.98 i, 2.21 + 10.16 i, 9.19 - 1.56 i)	(6.35 + 0.20 i, 2.38 - 9.81 i, -8.69 - 2.54 i)
(18.26 + 0.98 i, 2.21 - 10.16 i, 9.19 + 1.56 i)	(6.35 - 0.20 i, 2.38 + 9.81 i, -8.69 + 2.54 i)
(-26.44 + 55.93 i, -81.54 - 53.23 i, 38.99 - 59.03 i)	(5.95 + 0.04 i, 0.11 - 5.94 i, 4.31 + 0.10 i)
(-26.44 - 55.93 i, -81.54 + 53.23 i, 38.99 + 59.03 i)	(5.95 - 0.04 i, 0.11 + 5.94 i, 4.31 - 0.10 i)