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QUASI-ANALYTICITY CRITERIA OF SALINAS-KORENBLUM TYPE FOR GENERAL DOMAINS

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Abstract. We prove a criterion of quasi-analyticity in a boundary point of a rather general domain (not necessarily convex and simply-connected) if in a vicinity of this point the domain is close in some sense to an angle or is comparable with it.

Keywords: Carleman class, regular sequences, bilogarithmic quasi-analyticity condition.

Mathematics Subject Classification: 30D60.

1. Introduction

Let $\{M_n\}_{n=0}^{\infty}$ be a sequence of positive numbers. Some of numbers M_n can be equal to $+\infty$, but it is assumed that there exists an infinite number of finite M_n. As class $C\{M_n\}$, we call the set of all infinitely differentiable functions f defined on the segment $I = [a, b]$, $(-\infty \leqslant a < b \leqslant +\infty)$, for each of those there exists a constant K_f such that [1]

$$\sup_{a<x<b} \left|f^{(n)}(x)\right| \leqslant K_f^n M_n \qquad (n \geq 0).$$

In the general situation I can be an interval of half-interval.

In 1912 J. Hadamard posed the following question [1]: what are the numbers M_n so that for each two functions f and φ in class $C\{M_n\}$, once in some point x_0 of the interval $I = (a, b)$ for all $n \geq 0$

$$f^{(n)}(x_0) = \varphi^{(n)}(x_0),$$

it follows $f(x) \equiv \varphi(x)$ $(a < x < b)$?

It was observed that it is true if $M_n = n!$. The matter is that in this case, class $C\{n!\}$ coincides with the class of real-analytic functions on the interval (a, b) [1]. Due to the additivity of classes $C\{M_n\}$, the Hadamard problem can be reformulated as follows: what are the numbers M_n in order to class $C\{M_n\}$ to be quasi-analytic, that is, each function $f \in C\{M_n\}$ satisfying at some point $x_0 \in I$

$$f^{(n)}(x_0) = 0 \qquad (n \geq 0),$$

vanishes.

The Hadamard quasi-analyticity problem problem for the segment (interval, half-interval) I is completely solved by so-called Denjoy-Carleman theorem. One of its equivalent formulations belonging to Ostrovsky is as follows [1], [2]: *class $C\{M_n\}$ is quasi-analytic if and only if*

$$\int_1^{\infty} \frac{\ln T(r)}{r^2} dr = +\infty.$$

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Here $T(r)=\sup\limits_{n\geq 0}\frac{r^n}{M_n}$ *is the trace function for the sequence* $\{M_n\}$.

Let G be a domain in the complex plane. By $H(G,M_n)$ we denote the class of functions f analytic in the domain G and satisfying condition

$$\sup_{z\in G}\left|f^{(n)}(z)\right|\leqslant C_f M_n \qquad (n\geq 0).$$

We assume that domain G is so that all the derivatives $f^{(n)}$ $(n\geq 0)$ of a function $f\in H(G,M_n)$ can be continuously extended up to the boundary of ∂G. In this case, class $H(G,M_n)$ is called quasi-analytic at a point $z_0\in\partial G$, if $f\in H(G,M_n)$ and $f^{(n)}(z_0)=0$ $(n\geq 0)$ imply $f\equiv 0$ [3].

Let us survey briefly the results related with the quasi-analyticity problem for class $H(G,M_n)$ and let us formulate the problem we shall discuss here.

As it is known, the quasi-analyticity problem for class $H(\Delta_\gamma,M_n)$ and the angle

$$\Delta_\gamma=\{z:|argz|<\frac{\pi}{2\gamma},0<|z|<\infty\} \qquad (1<\gamma<\infty)$$

was first posed and solved by R. Salinas in 1955 [4]: *class* $H(\Delta_\gamma,M_n)$ *is quasi-analytic at the point* $z=0$ *if and only if the condition*

$$\int\limits_1^\infty\frac{\ln T(r)}{r^{1+\frac{\gamma}{1+\gamma}}}dr=+\infty$$

holds true.

It should be noticed that Ostrovsky theorem is the limiting case for R. Salinas theorem (as $\gamma\to\infty$).

The quasi-analyticity problem for class $H(K,M_n)$, where K is a circle, was solved by B.I. Korenblyum [5]. He proved the following statement: *class* $H(K,M_n)$ *is quasi-analytic at a boundary point if and only if*

$$\int\limits_1^\infty\frac{\ln T(r)}{r^{\frac{3}{2}}}=+\infty.$$

The criterion of quasi-analyticity of class $H(D,M_n)$ at a boundary point for an arbitrary convex bounded domain D was established by R.S. Yulmukhametov in [3]. Let us describe this result.

Let D be a convex bounded domain in the complex plane lying in the left half-plane and $0\in\partial D$. In this case, the support function $h(\varphi)=\max\limits_{\lambda\in D}\mathrm{Re}(\lambda e^{i\varphi})$ of domain D is non-negative and vanishes on some segment $[\sigma_-,\sigma_+]$ $(-\frac{\pi}{2}<\sigma_-\leqslant 0\leqslant\sigma_+<\frac{\pi}{2})$. Let it be the maximal segment on which $h(\varphi)=0$. We let

$$\Delta_+(\varphi)=\sqrt{\varphi-\sigma_+}\left(h^{'}(\varphi)+\int\limits_0^\varphi h(\alpha)d\alpha\right), \qquad \sigma_+\leqslant\varphi\leqslant\frac{\pi}{2};$$

$$\Delta_-(\varphi)=-\sqrt{\sigma_--\varphi}\left(h^{'}(\varphi)+\int\limits_0^\varphi h(\alpha)d\alpha\right), \qquad -\frac{\pi}{2}\leqslant\varphi\leqslant\sigma_-.$$

By $v(r)$ we denote the inverse to the function

$$v_1(x)=\exp\int\limits_{x_1}^x\frac{(2\pi-\Delta_+^{-1}(y)+\Delta_-^{-1}(y))dy}{(-\pi+\Delta_+^{-1}(y)-\Delta_-^{-1}(y))y},x\to 0,x_1>0.$$

Theorem 1 ([3]). *If $h'(\sigma_\pm) = 0$, then class $H(D, M_n)$ is quasi-analytic at the point $z = 0$ if and only if*

$$\int_1^\infty \frac{\ln T(r)}{v(r)r^2} dr = +\infty.$$

The problem arises: to find quasi-analyticity criteria for general domains (not necessary bounded, convex, and simply-connected) that depend only on a given sequence $\{M_n\}$ so that for regular sequences they can be reformulated as bi-logarithmic Levinson condition. The present paper is devoted to studying this issue.

2. History of problem. Definitions and preliminaries

Let $\{M_n\}$ be a sequence of positive numbers M_n satisfying condition $M_n^{\frac{1}{n}} \to \infty$ as $n \to \infty$. We can assume that $M_0 = 1$. Sequence $\{M_n\}$ is called logarithmically convex if $M_n^2 \leqslant M_{n-1}M_{n+1}$ $(n \geq 1)$. It is well know that a logarithmically convex sequence $\{M_n\}$ is completely determined by the trace function $T(r)$ and [1], [2]

$$M_n = \sup_{r \geq 0} \frac{r^n}{T(r)} \qquad (n \geq 0).$$

Let us clarify the geometric meaning of logarithmic convexity of a sequence $\{M_n\}$. In order to do it, we find the logarithms for inequalities $M_n^2 \leqslant M_{n-1}M_{n+1}$, we obtain

$$\ln M_n \leqslant \frac{1}{2} \ln M_{n-1} + \frac{1}{2} \ln M_{n+1} \qquad (n \geq 1).$$

Hence, we see that the logarithmical convexity of sequence $\{M_n\}$ means that the point $(n, \ln M_n)$ lies not higher than the segment connecting the points $(n-1, \ln M_{n-1})$ and $(n+1, \ln M_{n+1})$ $(n \geq 1)$.

By $\{M_n^c\}$ we denote the sequence obtained from $\{M_n\}$ as a convex regularization by logarithms (see, for instance, [1], [2], [6]).

In paper [7] the quasi-analyticity criteria were given for Carleman classes $H(\Delta_\gamma, M_n)$ and the angle

$$\Delta_\gamma = \{z : |\arg z| < \frac{\pi}{2\gamma}, 0 < |z| < \infty\} \qquad (1 < \gamma < \infty)$$

explicitly in terms of a given sequence $\{M_n\}$ (or $\{M_n^c\}$). Namely, there was proven

Theorem 2 ([7]). *Class $H(\Delta_\gamma, M_n)$ is quasi-analytic at the point $z = 0$ if and only if one of following equivalent conditions*

1) $\int_1^\infty \frac{\ln T(r)}{r^{1+\frac{\gamma}{1+\gamma}}} dr = \infty$, *where* $T(r) = \sup_{n \geq 0} \frac{r^n}{M_n}$ *(R. Salinas criterion);*

2) $\sum_{n=0}^\infty \left(\frac{M_n^c}{M_{n+1}^c}\right)^{\frac{\gamma}{1+\gamma}} = \infty$;

3) $\sum_{n=0}^\infty \frac{1}{\beta_n^{\frac{\gamma}{1+\gamma}}} = \infty$, *where* $\beta_n = \inf_{k \geq n} M_k^{\frac{1}{k}}$,

holds true.

We proceed to considering the question on bi-logarithmic quasi-analyticity condition for the angle. Following work [8], we introduce the adjoint sequence $\{m_n\}$, where $m_n = \frac{M_n}{n!}$. Here $\{M_n\}$ is an arbitrary sequence of numbers. Now we assume additionally that sequence $\{M_n\}$ obeys the following conditions,

a) $m_n^2 \leqslant m_{n-1}m_{n+1}$ $\quad (n \geq 1)$;

b) $\sup_n \left(\frac{m_{n+1}}{m_n}\right)^{\frac{1}{n}} < \infty$;

c) $m_n^{\frac{1}{n}} \to \infty, \quad n \to \infty.$

If conditions a)–c) hold true, sequence $\{M_n\}$ is called regular. Condition a) is the condition of logarithmic convexity for sequence $\{m_n\}$. We also note that condition b) implies that class $C\{M_n\}$ is closed w.r.t. differentiation. Condition c) yields that Carleman class $C\{M_n\}$ contains analytic function as well. For a regular sequence $\{M_n\}$ we introduce so-called associated weight [8]

$$\omega(r) = \sup_{n\geq 0} \frac{r^n}{m_n}.$$

It follows from condition a) that $M_n^2 \leqslant M_{n-1}M_{n+1}$, i.e., sequence $\{M_n\}$ is logarithmically convex (it can be checked directly). This is why in accordance with Denjoy-Carleman theorem, class $C\{M_n\}$ is quasi-analytic if and only if at least one of the following equivalent conditions [1], [2]

$$1^0. \int_1^\infty \frac{\ln T(r)}{r^2} dr = \infty; \qquad 2^0. \sum_{n=0}^\infty \frac{M_n}{M_{n+1}} = \infty$$

holds true.

For a regular sequence $\{M_n\}$, as E.M. Dyn'kin showed [8], condition 2^0 (and therefore, condition 1^0) is equivalent to bi-logarithmic Levinson condition

$$\int_0^d \ln\ln h(r) dr = +\infty,$$

where $h(r) = \omega(\frac{1}{r})$ and quantity $d > 0$ is chosen so that $h(d) \geq e$. Here

$$h(r) = \sup_{n\geq 0} \frac{1}{m_n r^n}, \qquad m_n = \frac{M_n}{n!}, \qquad r > 0.$$

It is clear that $h(r)$ is a decaying function, $\lim_{r\to 0} h(r) = \infty$. Since sequence $\{m_n\}$ is logarithmically convex, the inverse representation

$$m_n = \sup_{r>0} \frac{1}{r^n h(r)} \qquad (n \geq 0)$$

holds true.

We have

Theorem 3 ([7]). *Suppose a sequence $\{M_n\}$ $(n \geq 0)$ of positive numbers M_n is so that the changed sequence $\{M_n^*\}$, $M_n^* = M_n^{\frac{\gamma}{1+\gamma}}$ $(1 < \gamma < \infty)$ is regular. Then class $H(\Delta_\gamma, M_n)$ is quasi-analytic at the point $z = 0$ if and only if Levinson condition*

$$\int_0^d \ln\ln h(r) dr = +\infty \tag{1}$$

holds true, where

$$h_*(r) = \sup_{n\geq 0} \frac{n!}{M_n^{\frac{\gamma}{1+\gamma}} r^n}, \qquad 1 < \gamma < \infty.$$

We note that Denjoy-Carleman thereom is the limiting case of conditions 1)–3) in Theorem 2. An analogue of Theorem 3 for a segment was proven earlier by E.M. Dyn'kin under a bi-logarithmic condition which can be obtained from Levinson condition (1) if one lets formally $\gamma = \infty$.

3. Quasi-analyticity criteria

3.1. Case of convex domain. Let D be a bounded convex domain, $0 \in \partial D$, $h'(\sigma_\pm) = 0$. Then class $H(D, M_n)$ is quasi-analytic at the point $z = 0$ if and only if [3]

$$\int_1^\infty \frac{\ln T(r)}{v(r)r^2}dr = +\infty.$$

The quantities $h(\varphi)$, σ_+, σ_-, $T(r)$ were defined in Introduction. This result has another more obvious formulation. In order to provide it, we introduce certain geometric characteristics of a convex domain. As it is known, the support function

$$h(\varphi) = \max_{\lambda\in D} \mathrm{Re}(\lambda e^{i\varphi})$$

is the distance from the origin to the tangent for domain D perpendicular to the direction $\{re^{-i\varphi}, r > 0\}$. We assume that the coordinate system is chosen so that the maximal segment on which $h(\varphi) = 0$ reads as $[-\sigma, \sigma]$, where $\sigma > 0$. We note that here $\sigma < \frac{\pi}{2}$. If $\sigma = \frac{\pi}{2}$, then the domain is degenerate to a segment on the negative semi-axis.

On the boundary of domain D we choose the counterclockwise direction and introduce the arc length,

$$z = z(s),\ 0 \leqslant s < s_0,$$

where s_0 is the total length of the boundary of D. Hence, the length for the arc of the boundary from the point $z = 0$ to the point $z(s)$ (in the chosen direction) equals s.

As in work [9], by $-\alpha_-(s)$ $(0 \leqslant s < s_0)$ we denote the slope of the tangent to the boundary of D at the point $z(s)$ w.r.t. the imaginary axis. Then function $\alpha_-(s)$ is well-defined everywhere on $[0, s_0)$ except a countable set of points s for which $z(s)$ is the angle point. We define the function $\alpha_-(s)$ by the right continuity condition. By definition, $\lim\limits_{s\to 0} \alpha_-(s) = -\sigma$. In the same way, the slope of the tangent at the point $z(s_0 - s)$ w.r.t. the direction of the imaginary axis is indicated by $\alpha_+(s)$. Then $\alpha_+(s)$ is positive, does not increase and $\lim\limits_{s\to 0} \alpha_+(s) = \sigma$. We let

$$\alpha(s) = \frac{\alpha_+(s) - \alpha_-(s)}{2},\ 0 \leqslant s < s_0.$$

Since $\lim\limits_{s\to 0} \alpha(s) = \sigma < \frac{\pi}{2}$, there exists a number $\varepsilon > 0$ such that $\alpha(s) < \frac{\pi}{2},\ 0 \leqslant s < \varepsilon$. We define

$$R(s) = \exp \int_s^\varepsilon \frac{\pi - \alpha(t)}{\frac{\pi}{2} - \alpha(t)} d\ln t, \qquad 0 \leqslant s < \varepsilon.$$

Let $\beta(s) = \pi - 2\alpha(s)$. Then function $\beta(s)$ is the angle between the tangents at the points $z(s)$ and $z(s_0 - s)$, domain D lies in this angle and function $R(s)$ casts into the form

$$R(s) = \exp \int_s^\varepsilon \frac{\pi + \beta(t)}{\beta(t)} d\ln t, \qquad 0 \leqslant s < \varepsilon.$$

We have

Theorem 4 ([9]). *Let D be a convex but not necessary bounded domain $z_0 \in \partial D$, and*

$$T(r) = \sup_{n\geq 0} \frac{r^n}{M_n}$$

is the trace function for sequence $\{M_n\}$. By $\beta(z_0, s)$ we denote the angle between the tangents to the boundary of D taken at the points separated from point z_0 by the distance s of arc of the *boundary. We let*

$$R(z_0,s)=\exp\int\limits_s^{\varepsilon}\frac{\pi+\beta(z_0,x)}{\beta(z_0,x)}d\ln x,\qquad 0\leqslant s<\varepsilon. \tag{2}$$

Then the condition

$$\int\limits_1^{\infty}\frac{\ln T(r)}{r^2R^{-1}(z_0,r)}dr=\infty \tag{3}$$

is the criterion for the quasi-analyticity of class $H(D,M_n)$ at point z_0.

In particular, by this theorem one can easily obtain aforementioned quasi-analyticity conditions for classes $H(D,M_n)$ in the case D is a circle or an angle $\pi\alpha,\ 0<\alpha\leqslant 1$.

Our aim is to show that if a convex domain D satisfies some integral condition (depending on the geometry of the domain) at a boundary point z_0, then condition (3) has a simpler formulation.

Fix a point $z_0\in\partial D$. Then the defined above angle $\beta(z_0,s)$, non-decaying, tends to $\pi\alpha$ ($0<\alpha\leqslant 1$) as parameters s tends to zero. Taking into consideration that $\beta(z_0,s)\equiv\pi\alpha$ for the angle, we extract the term $\frac{1+\alpha}{\alpha}$ from the integrand in formula (2),

$$\frac{\pi+\beta(z_0,s)}{\beta(z_0,s)}=\frac{1+\alpha}{\alpha}+\frac{\pi\alpha-\beta(z_0,s)}{\alpha\beta(z_0,s)}.$$

Then for sufficiently small s the integral $\int\limits_s^{\varepsilon}\frac{\pi\alpha-\beta(z_0,x)}{\alpha\cdot\beta(z_0,x)}\cdot\frac{dx}{x}$ differs by a small error from the quantity $\frac{1}{\pi\alpha^2}\int\limits_s^{\varepsilon}\frac{\pi\alpha-\beta(z_0,x)}{x}dx$. Hence, if the integrals $\int\limits_s^{\varepsilon}\frac{\pi\alpha-\beta(z_0,x)}{x}dx$ are uniformly bounded for all s, $0<s<\varepsilon$, then the quasi-analyticity criterion for class $H(D,M_n)$ at point $z_0\in\partial D$ becomes

$$\int\limits_1^{\infty}\frac{\ln T(r)}{r^{\frac{\alpha+2}{\alpha+1}}}dr=+\infty.$$

Indeed, it follows from the fact that in this case

$$R(s)=\exp\left[\int\limits_s^{\varepsilon}\frac{1+\alpha}{\alpha}d\ln x\right]\cdot\exp\left[\int\limits_s^{\varepsilon}\frac{\pi\alpha-\beta(z_0,x)}{\alpha\cdot\beta(z_0,x)}d\ln x\right],$$

and as $s\to 0$

$$R(s)=r\sim\left(\frac{\varepsilon}{s}\right)^{\frac{\alpha+1}{\alpha}}\cdot\exp\left(\frac{c}{\pi\alpha^2}\right),$$

where

$$c=\lim_{s\to 0}\int\limits_s^{\varepsilon}\frac{\pi\alpha-\beta(z_0,x)}{x}dx=\int\limits_0^{\varepsilon}\frac{\pi\alpha-\beta(z_0,x)}{x}dx.$$

Therefore, as $r\to\infty$,

$$R^{-1}(r)\sim\exp\left(\frac{c}{\pi\alpha^2}\cdot\frac{\alpha}{\alpha+1}\right)\varepsilon r^{-\frac{\alpha}{\alpha+1}},$$

and condition (3) casts into the form

$$\int\limits_1^{\infty}\frac{\ln T(r)}{r^2r^{-\frac{\alpha}{\alpha+1}}}dr=\int\limits_1^{\infty}\frac{\ln T(r)}{r^{\frac{\alpha+2}{\alpha+1}}}dr=+\infty.$$

Thus, for convex domains, for which quantity $\beta(z_0, s)$ obeys the restriction

$$\sup_s \int\limits_s^\varepsilon \frac{\pi\alpha - \beta(z_0, x)}{x} dx < \infty, \tag{4}$$

the quasi-analyticity criterion for class $H(D, M_n)$ at a point $z_0 \in \partial D$ coincides with Salinas quasi-analyticity criteria for the angle $\Delta_\alpha = \{z : |\arg z| < \frac{\pi\alpha}{2}\}$ $(0 < \alpha < 1)$ and Korenblyum one for half-plane Δ_1.

We have

Theorem 5. *Let D be a convex but necessary bounded domain, $z_0 \in \partial D$, and*

$$T(r) = \sup_{n \geq 0} \frac{r^n}{M_n}$$

is the trace function for sequence $\{M_n\}$. By $\beta(z_0, s)$ we denote the angle between tangents to the boundary of D taken at the points separated from point z_0 by the length s along the boundary. Suppose that at point z_0, the condition

$$\sup_s \int\limits_s^\varepsilon \frac{\pi\alpha - \beta(z_0, x)}{x} dx < \infty, \qquad \pi\alpha = \lim_{s \to 0} \beta(z_0, s) \quad (0 < \alpha \leqslant 1)$$

holds true. Then class $H(D, M_n)$ is quasi-analytic at point z_0 if and only if

$$\int\limits_1^\infty \frac{\ln T(r)}{r^{\frac{\alpha+2}{\alpha+1}}} dr = +\infty. \tag{5}$$

Remark 1. Condition (4) holds true if, for instance,

$$|\pi\alpha - \beta(z_0, s)| = O\left(s^\gamma\right), \ \gamma > 0$$

or

$$|\pi\alpha - \beta(z_0, s)| = O\left(\frac{1}{|\ln s|^\gamma}\right), \ \gamma > 1 \ \ s \to 0.$$

Remark 2. For regular sequences $\{M_n^{\frac{\gamma}{1+\gamma}}\}$, there was obtained the bi-logarithmic quasi-analyticity condition in the angle which was equivalent to condition (5) as $\alpha = \frac{1}{\gamma}$. Therefore, by Theorem 5, for convex domain with additional condition (4) at point $z_0 \in \partial D$, the bi-logarithmic quasi-analyticity condition at this point reads exactly the same as for an angle,

$$\int\limits_0^d \ln\ln h_*(r) dr = +\infty, \qquad h_*(r) = \sup_{n \geq 0} \frac{n!}{r^n \cdot M_n^{\frac{\gamma}{1+\gamma}}}, \qquad 1 < \gamma < \infty. \tag{6}$$

Theorem 5 imply several corollaries.

Corollary 1. *Let $\Delta_\alpha = \left\{z : |\pi - \arg z| < \frac{\pi\alpha}{2}\right\}$ be the angle $\pi\alpha$ $(0 < \alpha < 1)$ with vertex at the point $z = 0$. Then, obviously, $\beta(s) \equiv \pi\alpha$, and Condition (4) holds true.*

If we let $\alpha = \frac{1}{\gamma}$, then condition (5) coincides with R. Salinas quasi-analyticity criterion for an angle

$$\Delta_\gamma = \left\{z : |\arg z| < \frac{\pi}{2\gamma}, 0 < |z| < \infty\right\} \qquad (1 < \gamma < \infty).$$

Corollary 2. *Let $K = \{z : |z + R| < R\}$ be a circle. It can be checked that in this case*

$$\beta(s) = \pi - 2\frac{s}{R},$$

and $\beta(s) \uparrow \pi$ $(\alpha = 1)$ $s \to 0$. Since $\frac{\pi-\beta(x)}{x} = \frac{2}{R}$, condition (4) holds true at each point of ∂K, while relation (5) in this case (as $\alpha = 1$) becomes Korenblyum criterion.

3.2. Special domains. Consider the domains of special form, lunes K^α. As a lune K^α, following work [10], we treat the intersection of exterior or interior for two circles of arbitrary but the same radius so that their circumferences pass the origin O and intersect by the angle $\pi\alpha$ $(0 < \alpha < 2)$. As K^1, we treat the exterior or interior for a circumference passing point O.

Let us show that for the lune K^α obtained as the intersection of two interiors for two circles, condition (4) holds true. In order to do it, we shall make use of the following

Lemma 1. *Let us draw the tangent at the point A to a circumference of an arbitrary radius R passing through point O with the center located below axis Ox. Let also β_1 $(0 < \beta_1 < \frac{\pi}{2})$ be the angle between the tangent and the negative direction of axis Ox and $\beta_1 \to \gamma$ as $A \to O$. Then*

$$\gamma - \beta_1 = \frac{\breve{AO}}{R},$$

where $\breve{AO}$ is the length of arc of the circumference between the points A and O.

Indeed, we observe that $(\pi - \gamma) + \beta_1 = \pi - \alpha$. It yields $\gamma - \beta_1 = \alpha$. Taking into consideration that $\alpha = \frac{\breve{AO}}{R}$, we obtain the desired identity $\gamma - \beta_1 = \frac{\breve{AO}}{R}$.

Let K^α be the lune formed by the intersection of the interior of two circles. Obviously, it is a convex set. We shall assume that K^α is located in the left half-plane and is symmetric w.r.t. axis Ox. Then by Lemma 1 we obtain that

$$\pi\alpha - \beta(s) = 2\frac{s}{R}.$$

Hence,

$$\int_s^\varepsilon \frac{\pi\alpha - \beta(x)}{x} dx = \frac{2}{R}(\varepsilon - s) \qquad (0 < s < \varepsilon),$$

and condition (4) holds true for K^α.

Finally, we formulate the last corollary.

Corollary 3. *For a convex lune K^α $(0 < \alpha < 2)$, condition (4) holds true everywhere. For lunes K^α $(0 < \alpha < 2)$, the quasi-analyticity criterion for at point O coincides with R. Salinas criterion for the angle*

$$\Delta_\alpha = \left\{ z : |\pi - \arg z| < \frac{\pi\alpha}{2} \right\}.$$

By Theorem 5 one can also get quasi-analyticity criteria for classes $H(G, M_n)$ for non-convex domains G satisfying certain additional restrictions.

Let G be a domain in the complex plain not containing infinity. We shall say that domain G satisfies condition (A), if its boundary C consists of a finite number of piecewise-smooth closed simple curves $c_1, c_2, \ldots, c_n$, each of which has a piecewise-continuous curvature and contains at most finite number of angle points and all interior angles (w.r.t. domain G) are not equal to 0 or 2π. We denote the interior angle between one-sided tangents to C at a point z by $\pi\alpha(z)$. Let $\alpha = \min\limits_{z \in C} \alpha(z) > 0$. Then a domain G satisfying condition A possesses the feature [10]: for each point $z \in \partial G$, there exist lunes $K_1^{\alpha(z)}$ and $K_2^{\alpha(z)}$ such that

$$K_1^{\alpha(z)} \subset G \subset K_2^{\alpha(z)}.$$

Here $K_1^{\alpha(z)}$ is a convex lune formed by the intersection of interiors, while $K_2^{\alpha(z)}$ is a lune formed by intersection of exterior for two circles of the same but sufficiently small radius such that their circumferences pass point z.

Classes $H(K_1^{\alpha(z)}, M_n)$ and $H(K_2^{\alpha(z)}, M_n)$ are quasi-analytic or not at a point $z \in C$ simultaneously [10]. Therefore, taking into account Corollary 3 and applying Theorem 5, we obtain: *all three classes $H(G, M_n)$, $H(K_1^{\alpha(z)}, M_n)$, and $H(K_2^{\alpha(z)}, M_n)$ are quasi-analytic at a point $z \in C$ if and only if*

$$\int_1^{\infty} \frac{\ln T(r)}{r^{\frac{\alpha(z)+2}{\alpha(z)+1}}} dr = +\infty. \tag{7}$$

We note that if a point $z \in C$ is a point of smoothness for the boundary of domain G (i.e., $\alpha(z) \equiv 1$), the quasi-analyticity criterion for class $H(G, M_n)$ at this point reads as follows,

$$\int_1^{\infty} \frac{\ln T(r)}{r^{\frac{3}{2}}} dr = +\infty.$$

If we take into consideration Remark 2, for regular sequences $\{M_n^{\frac{1}{\alpha+1}}\}$, condition (7) is equivalent to bi-logarithmic condition (6) as $\gamma = \frac{1}{\alpha}$.

We note that the quasi-analyticity criterion for class $H(G, M_n)$, where G is a domain satisfying condition A, was proven in a different way in work [10].

4. Existence criterion for regular minorant of non-quasi-analyticity

Let $\{M_n\}$ be a regular sequence, $\omega(r) = \max_{n \geq 0} \frac{r^n}{m_n}$ $\left(m_n = \frac{M_n}{n!}\right)$ is the associated weight [8]. Then sequence $\{M_n\}$ is completely determined by function $\omega(r)$,

$$M_n = n! \sup_{r>0} \frac{r^n}{\omega(r)}.$$

As it was said in Section 2, in this case, the condition

$$\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} < \infty \tag{8}$$

can be reformulated in terms of bi-logarithmic Levinson condition

$$\int_0^d \ln \ln H(r) dr < \infty,$$

where $H(r) = \omega(\frac{1}{r})$ and $d > 0$ is so that $H(d) > e$.

We shall sequence $\{M_n\}$ weakly regular if it obeys conditions a), b) in the definition of regular sequence $\{M_n\}$ (see Section 2). It happens that for weakly regular sequences, condition (8) has another interpretation.

Lemma 2. *Suppose the sequence $\{M_n\}$ is weakly regular. Condition (8) holds true if and only if there exists a positive continuous on $\mathbb{R}_+$ function $R = R(t)$ such that $R(t) \downarrow 0$, $tR(t) \downarrow 0$ as $t \to \infty$ and*

$$1) \frac{1}{M_n^{\frac{1}{n}}} \leqslant R(n); \qquad 2) \int_1^{\infty} R(t) dt < \infty.$$

Proof. Sufficiency is almost obvious. Indeed, since $M_n^{\frac{1}{n}} \uparrow \infty$ as $n \to \infty$ (it follows from the logarithmic convexity of sequence $\{M_n\}$ and property c)), according to Denjoy-Carleman theorem, condition can be written as [2]

$$\sum_{n=1}^{\infty} \frac{1}{M_n^{\frac{1}{n}}} < \infty. \tag{9}$$

This is why the sufficiency of lemma follows from conditions 1), 2) and properties of function $R = R(t)$.

Necessity. Letting $r(n) = M_n^{-\frac{1}{n}}$, we have

$$r(n)n = \frac{n}{M_n^{\frac{1}{n}}} = \frac{1}{m_n^{\frac{1}{n}}} \; \frac{n}{(n!)^{\frac{1}{n}}}.$$

By Stirling formula [11],

$$n! = \sqrt{2\pi n}\; n^n\; e^{-n}\; e^{\theta(n)}, \qquad |\theta(n)| \leqslant \frac{1}{12n},$$

it implies

$$r(n)n = \frac{1}{m_n^{\frac{1}{n}}} \; \frac{e^{1-\frac{\theta(n)}{n}}}{(2\pi n)^{\frac{1}{2n}}} \leqslant e^{\frac{13}{12}} \; \frac{1}{m_n^{\frac{1}{n}}}. \tag{10}$$

If we denote by $R(n)n$ the right hand side of (10), we see that $R(n)n \downarrow 0$ as $n \to \infty$. Then as $n \to \infty$,

$$R(n) = e^{\frac{13}{12}} \; \frac{1}{n} \; \left(\frac{n!}{M_n}\right)^{\frac{1}{n}} \leqslant e^{\frac{1}{6}} \; (2\pi n)^{\frac{1}{2n}} \; \frac{1}{M_n^{\frac{1}{n}}} = O\left(\frac{1}{M_n^{\frac{1}{n}}}\right).$$

Therefore, it follows from condition (9) that $\sum\limits_{n=1}^{\infty} R(n) < \infty$. Hence,

$$\frac{1}{M_n^{\frac{1}{n}}} \leqslant R(n), \qquad \sum_{n=1}^{\infty} R(n) < \infty, \qquad R(n) \downarrow 0,\; R(n)n \downarrow 0 \;\; n \to \infty.$$

The desired function is obviously $R = R(t)$ which is linear for $t \in (n, n+1)$ and it takes values $R(n)$ and $R(n+1)$ at the endpoints of the interval $(n, n+1)$. □

Lemma 2 is supplemented by

Lemma 3. *Let $\{M_n\}$ $(M_n > 0)$ be an arbitrary sequence such that there exists a continuous function $r = r(t)$ on $\mathbb{R}_+$, $r(t) \downarrow 0$, $r(t)t \downarrow 0$ as $t \to \infty$ and*

$$\frac{1}{M_n^{\frac{1}{n}}} \leqslant r(n), \qquad \int\limits_1^{\infty} r(t)dt < \infty.$$

Then there exists a weakly regular sequence $\{M_n^*\}$ such that

$$M_n^* \leqslant M_n, \qquad \sum_{n=1}^{\infty} (M_n^*)^{-\frac{1}{n}} < \infty.$$

Proof. We have

$$D_n = \left(\frac{1}{r(n)}\right)^n \leqslant M_n \qquad (n \geq 1).$$

The sequence $\left\{\frac{D_n}{n!}\right\}$ is not necessary logarithmically convex. This is why we replace it by a minorant possessing the required properties.

Bearing in mind Stirling formula, we have

$$\frac{D_n}{n!} = \frac{1}{n^n \Delta_n} \; \left(\frac{1}{r(n)}\right)^n,$$

where $\Delta_n = e^{-n}\; \sqrt{2\pi n}\; e^{\theta(n)}$ $\left(|\theta(n)| \leqslant \frac{1}{12n}\right)$. Since it is obvious that

$$\Delta_n \leqslant \sqrt{2\pi}\; \exp\left(\frac{1}{12n} - n + \frac{1}{2}\ln n\right) \leqslant \sqrt{2\pi} < e,$$

then

$$\frac{D_n}{n!} \geq \frac{1}{e} \left(\frac{1}{nr(n)}\right)^n > \left(\frac{1}{enr(n)}\right)^n \ (n \geq 1).$$

If we let

$$m_n^* = \frac{M_n^*}{n!} = \left(\frac{1}{enr(n)}\right)^n,$$

then $M_n^* \leqslant D_n \leqslant M_n$. Since $nr(n) \downarrow 0$ as $n \to \infty$, then $(m_n^*)^{\frac{1}{n}} \uparrow \infty$ as $n \to \infty$. We see that sequence $\{M_n^*\}$ is weakly regular.

Let us make sure that

$$\sum_{n=1}^{\infty} \frac{1}{(M_n^*)^{\frac{1}{n}}} < \infty. \qquad (11)$$

Indeed,

$$M_n^* = n! \left(\frac{1}{enr(n)}\right)^n = \sqrt{2\pi n}\, e^{-2n+\theta(n)} \left(\frac{1}{r(n)}\right)^n.$$

It yields

$$\left(\frac{1}{M_n^*}\right)^{\frac{1}{n}} \leqslant e^{\frac{25}{12}}\, r(n) \qquad (n \geq 1),$$

and condition (11) is implied by the convergence of the series $\sum\limits_{n=1}^{\infty} r(n)$. □

Remark 3. Under the hypothesis of Lemma 3, without loss of generality, one can assume that $t^2 r(t) \uparrow \infty$ as $t \to \infty$. It implied by the following statement [12].

Let $r = r(t)$ be a positive continuous on $\mathbb{R}_+$ function, $tr(t) \downarrow 0$ as $t \to \infty$, and

$$\int_1^{\infty} r(t)dt < \infty.$$

Then for each $\varepsilon > 0$ there exists a function $r_1 = r_1(t)$ satisfying conditions

1. $r(t) \leqslant r_1(t)$;
2. $tr_1(t) \downarrow 0$, $t^{1+\varepsilon} r_1(t) \uparrow$ *as* $t \to \infty$;
3. $\int\limits_1^{\infty} r_1(t)dt < \infty$.

By Remark 3, sequence $\{M_n^*\}$ constructed in Lemma 3 satisfies also regularity condition b). Thus, under the hypothesis of Lemma 3 there exists a regular sequence $\{M_n^*\}$ such that

$$M_n^* \leqslant M_n, \qquad \sum_{n=1}^{\infty} \left(\frac{1}{M_n^*}\right)^{\frac{1}{n}} < \infty.$$

Let us make sure that sequence $\{M_n^*\}$ satisfies condition b). Indeed, we have

$$a_n = \left(\frac{m_{n+1}^*}{m_n^*}\right)^{\frac{1}{n}} \leqslant \sqrt[n]{\frac{1}{(n+1)\, r(n+1)}}\ \frac{n\, r(n)}{(n+1)\, r(n+1)}.$$

But

$$\frac{n\, r(n)}{(n+1)\, r(n+1)} \leqslant \frac{(n+1)^2\, r(n+1)}{(n+1)\, r(n+1)}\ \frac{1}{n} = \frac{n+1}{n} \leqslant 2.$$

It yields

$$a_n \leqslant 2 \sqrt[n]{\frac{n+1}{(n+1)^2\, r(n+1)}} \leqslant 2 \left(\frac{n+1}{4r(2)}\right)^{\frac{1}{n}} \leqslant C < \infty,$$

and

$$\sup_n \left(\frac{m^*_{n+1}}{m^*_n}\right)^{\frac{1}{n}} < \infty.$$

Hence, we have proven

Theorem 6. *Let $M_n > 0$. There exists a regular sequence $\{M^*_n\}$ such that*

$$M^*_n \leqslant M_n, \qquad \sum_{n=1}^{\infty} \frac{M^*_n}{M^*_{n+1}} < \infty$$

if and only if there exists a positive continuous on $\mathbb{R}_+$ function $r = r(t)$, $tr(t) \downarrow 0$, $t^2 r(t) \uparrow$ as $t \to \infty$ such that

$$1) \frac{1}{M_n^{\frac{1}{n}}} \leqslant r(n) \ (n \geq 1); \qquad 2) \int_1^{\infty} r(t)dt < \infty.$$

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