Jordan Journal of Mathematics and Statistics (JJMS) 8(2), 2015, pp 155 - 167

EXACTNESS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS AND INTEGRATING FACTORS

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Abstract. The principle of finding an integrating factor for a none exact differential equations is extended to equations of second order. If the second order equation is not exact, under certain conditions, an integrating factor exists that transforms it to an exact one. In this paper we give explicit forms for integrating factors of the second order differential equations.

1. Introduction

The concept of exactness for a class of first order nonlinear differential equations was presented [7] with a well-defined method of solution. The notion of integrating factor were introduced to convert differential equation that is not exact into an exact one.

Second order nonlinear differential equations play an important role in Applied Mathematics, Physics, and Engineering [1, 3, 4, 6, 7, 8, 9, 11, 13, 14]. To find the general solution of a nonlinear second order differential equation is not an easy problem in the general case. In fact, a very specific class of nonlinear second order

2000 *Mathematics Subject Classification.* 34A25, 34A30.

Key words and phrases. Second order differential equation, Exact equations, nonlinear differential equations, Non-exact equations, Integrating factor.

Received: Feb. 17, 2015 Accepted: June 7, 2015 .

differential equations can be solved by using special transformations. Another approach to study the solution of nonlinear second order differential equations is the dynamical systems approach. Using this approach a qualitative solutions are given instead of the particular solution of the second order nonlinear differential equations. A class of these equations will be solved in this paper by introducing the concept of an integrating factor for second order nonlinear differential equations [2, 5].

The outline of the paper: we give mathematical formulation for the exactness of a class of second order nonlinear equations based on transforming them into a first order differential equations. Moreover, we introduce the idea of integrating factor to convert some second order nonlinear differential equations into exact differential equations. Also, we prove some related results.

2. Exactness of second order ordinary differential equations

Consider the n-th order differential equation

$$F(x, y, y', y'', ...y^{(n)}) = 0, \tag{2.1}$$

where $y^{(n)} = \frac{d^n y}{dx^n}$. The exactness of (2.1) is introduced in [10] as follows: If there exist $\Psi(x, y, y', y'', ..., y^{(n-1)})$ satisfies

$$F(x, y, y', y'', ...y^{(n)}) = \frac{d}{dx}\Psi(x, y, y', y'', ..., y^{(n-1)} = 0 \tag{2.2}$$

Then the n-th order differential equation (2.1) is reduced to (n-1)-st order differential equation

$$\Psi(x, y, y', y'', ..., y^{(n-1)}) = C. \tag{2.3}$$

Particularly, the nonlinear second order differential equation

$$a_2(x, y, y')y'' + a_1(x, y, y')y' + a_0(x, y, y') = 0, \tag{2.4}$$

is exact if a function $\Psi(x, y, y')$ exists, with the properties that

$$\frac{\partial\Psi(x,y,y')}{\partial x} = a_0(x,y,y'), \quad \frac{\partial\Psi(x,y,y')}{\partial y} = a_1(x,y,y'). \text{ and } \frac{\partial\Psi(x,y,y')}{\partial y'} = a_2(x,y,y'), \tag{2.5}$$

So, we have

$$\frac{\partial\Psi(x,y,y')}{\partial y'}y'' + \frac{\partial\Psi(x,y,y')}{\partial y}y' + \frac{\partial\Psi(x,y,y')}{\partial x} = 0.$$

Therefore, by using the chain rule, we have

$$\frac{d\Psi(x,y,y')}{dx} = 0.$$

Hence,

$$\Psi(x,y,y') = c$$

is a reduction of Equation (2.4) into a first order differential equation in the implicit form. Using the properties in Equation (2.5), the function $\Psi(x, y, y')$ is given by the following formula:

$$\Psi(x,y,y') = \int_0^x a_0(\alpha,y,y')d\alpha + \int_2^y a_1(0,\beta,y')d\beta + \int_0^{y'} a_2(0,2,\gamma)d\gamma. \tag{2.6}$$

Definition 2.1. *The nonlinear second order differential equation* (2.4) *is called exact equation if there exists a function* $\Psi(x, y, y')$ *such that* (2.5) *holds.*

It easy to show that a nonlinear second order differential equation (2.4) is exact if the conditions

$$\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}, \quad \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial y'}, \text{ and } \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y} \tag{2.7}$$

hold [12].

Example 2.1. *Consider the following nonlinear second order differential equation*

$$y'' + a_1(x,y)y' + a_0(x,y) = 0, \tag{2.8}$$

where $a_1(x, y)$ *and* $a_0(x, y)$ *satisfy the condition* $\frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y}$. *Then* (2.8) *is exact.*

Example 2.2. *(The Plane Hydrodynamic Jet) Consider the second order nonlinear differential equation*

$$3\epsilon y'' + yy' = 0.$$

It is easy to that this equation is exact. By using the formula (2.6)*, we have*

$$\begin{aligned}\Psi(x, y, y') &= \int_0^y \beta d\beta + 3\epsilon \int_0^{y'} d\gamma \\ &= \frac{y^2}{2} + 3\epsilon y'.\end{aligned}$$

Hence, the equation is reduced to $\Psi(x, y, y') = c^2$*, which is equivalent to*

$$3\epsilon y' + \frac{y^2}{2} = c^2.$$

Example 2.3. *The second order nonlinear initial value problem*

$$\begin{cases} y'' + 12xy^3y' + (3y^4 - 1) = 0 \\ y(0) = 2, \quad y'(0) = 0, \end{cases} \tag{2.9}$$

is exact. Therefore, there exists a function $\Psi(x, y, y')$ *which reduces the above equation into a first order differential equation. Hence, formula* (2.6) *gives*

$$\Psi(x, y, y') = \int_{x_0}^x a_0(\alpha, y, y')d\alpha + \int_{y_0}^y a_1(x_0, \beta, y')d\beta + \int_{y_0'}^{y'} a_2(x_0, y_0, \gamma)d\gamma.$$

Since $x_0 = 0$*,* $y_0 := y(0) = 2$*, and* $y_0' := y_0'(0) = 0$*, we have*

$$\begin{aligned}\Psi(x, y, y') &= \int_0^x a_0(\alpha, y, y')d\alpha + \int_2^y a_1(0, \beta, y')d\beta + \int_0^{y'} a_2(0, 2, \gamma)d\gamma, \\ &= \int_0^x (3y^4 - 1)d\alpha + \int_0^{y'} d\gamma, \\ &= y' + (3y^4 - 1)x.\end{aligned}$$

Therefor, $\Psi(x, y, y') = c$ *reduces Equation* (2.9) *to*

$$y' + (3y^4 - 1)x = c.$$

By applying the initial data, we get $c = 0$. *Hence, Equation* (2.9) *is reduced to the following first order differential equation*

$$y' + 3xy^4 - x = 0.$$

For which an implicit solution can be obtained by separating the variables.

3. Non-exact Second Order Differential Equations and Integrating Factors

In this section, we introduce the idea of finding integrating factors for the second order differential equation (2.4) that are not exact. Also, we deduce some conditions for the existence of such integrating factor. First, we start by the following definition for the integrating factor:

Definition 3.1. *An integrating factor of Equation* (2.4) *is a non zero function* $\mu(x, y, y')$*, such that the equation*

$$\mu(x,y,y')a_2(x,y,y')y'' + \mu(x,y,y')a_1(x,y,y')y' + \mu(x,y,y')a_0(x,y,y') = 0 \tag{3.1}$$

is exact. i.e.,

$$\frac{\partial A_2}{\partial y} = \frac{\partial A_1}{\partial y'}, \quad \frac{\partial A_2}{\partial x} = \frac{\partial A_0}{\partial y'}, \quad \text{and} \quad \frac{\partial A_1}{\partial x} = \frac{\partial A_0}{\partial y}, \tag{3.2}$$

where

$$A_2(x,y,y') = \mu(x,y,y')a_2(x,y,y'),$$

$$A_1(x,y,y') = \mu(x,y,y')a_1(x,y,y'),$$

and

$$A_0(x,y,y') = \mu(x,y,y')a_0(x,y,y').$$

Example 3.1. *Consider the second order nonlinear equation*

$$xy(2x+y)y'' + (x^2+xy)y' + (3xy+y^2) = 0. \tag{3.3}$$

This equation has an integrating factor $\mu(x,y) = \frac{1}{xy(2x+y)}$, *which transforms Equation* (3.3) *into an exact differential equation. The resulting exact differential equation can be reduced into the following first order differential equation:*

$$y' + \ln\left(xy\sqrt{y+2x}\right) = c.$$

The following result gives necessary conditions for the integrating factor to be a function of x only.

Remark 3.1. *Through out this paper, we use the notation* $\partial_\eta f := \dfrac{\partial f}{\partial \eta}$.

The following Lemma gives the necessary conditions for an integrating factor in x to exist .

Lemma 3.1. *Assume that Equation* (2.4) *is not an exact equation. Then, it has an integrating factor*

$$\mu(x) = \exp\left\{\int^x \frac{\partial_y a_0 - \partial_x a_1}{a_1} dx\right\} = \exp\left\{\int^x \frac{\partial_{y'} a_0 - \partial_x a_2}{a_2} dx\right\}$$

if and only if

$$\frac{\partial_y a_0 - \partial_x a_1}{a_1} \text{ and } \frac{\partial_{y'} a_0 - \partial_x a_2}{a_2} \text{ depend only on } x,$$

$$\frac{\partial_y a_0 - \partial_x a_1}{a_1} = \frac{\partial_{y'} a_0 - \partial_x a_2}{a_2},$$

and

$$\partial_y a_2 = \partial_{y'} a_1.$$

Proof. Assume that Equation (2.4) has an integrating factor $\mu(x)$. Therefore, conditions (3.2) hold. Hence, we get the following algebraic equations:

$$\mu \frac{\partial a_2}{\partial y} = \mu \frac{\partial a_1}{\partial y'},$$

$$a_2 \mu' + \mu \frac{\partial a_2}{\partial x} = \mu \frac{\partial a_0}{\partial y'},$$

and

$$a_1 \mu' + \mu \frac{\partial a_1}{\partial x} = \mu \frac{\partial a_0}{\partial y}.$$

Using the first equation, we have a non zero integrating factor, if $\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}$. The last two equations implies that

$$\frac{\mu'}{\mu} = \frac{\frac{\partial a_0}{\partial y'} - \frac{\partial a_2}{\partial x}}{a_2} = \frac{\frac{\partial a_0}{\partial y} - \frac{\partial a_1}{\partial x}}{a_1}.$$

By integrating the above equation with respect to x, we get

$$\mu(x) = \exp\left\{ \int^x \frac{\partial_y a_0 - \partial_x a_1}{a_1} dx \right\} = \exp\left\{ \int^x \frac{\partial_{y'} a_0 - \partial_x a_2}{a_2} dx \right\}.$$

□

Similarly, we can get the following results:

Lemma 3.2. *The integrating factor of Equation* (2.4) *in terms of* y *is given by*

$$\mu(y) = \exp\left\{ \int^y \frac{\partial_{y'} a_1 - \partial_y a_2}{a_2} dy \right\} = \exp\left\{ \int^y \frac{\partial_x a_1 - \partial_y a_0}{a_0} dy \right\},$$

provided that

$$\frac{\partial_{y'} a_1 - \partial_y a_2}{a_2} \text{ and } \frac{\partial_x a_1 - \partial_y a_0}{a_0} \text{ depend only on } y,$$

$$\frac{\partial_{y'} a_1 - \partial_y a_2}{a_2} = \frac{\partial_x a_1 - \partial_y a_0}{a_0},$$

and

$$\partial_x a_2 = \partial_{y'} a_0.$$

Lemma 3.3. *The integrating factor of Equation* (2.4) *in terms of* y' *is given by*

$$\mu(y') = \exp\left\{\int^{y'} \frac{\partial_y a_2 - \partial_{y'} a_1}{a_1} dy'\right\} = \exp\left\{\int^{y'} \frac{\partial_x a_2 - \partial_{y'} a_0}{a_0} dy'\right\},$$

provided that

$$\frac{\partial_y a_2 - \partial_{y'} a_1}{a_1} \text{ and } \frac{\partial_x a_2 - \partial_{y'} a_0}{a_0} \text{ depend only on } y',$$

$$\frac{\partial_y a_2 - \partial_{y'} a_1}{a_1} = \frac{\partial_x a_2 - \partial_{y'} a_0}{a_0},$$

and

$$\partial_x a_1 = \partial_y a_0.$$

Example 3.2. *Consider the nonlinear second order differential equation*

$$e^{-x} y'' + (\cos(y) + 2e^{-x}\cos(y))y' + (\sin(y) - 2ye^{-x}\sin(x)) = 0.$$

This equation is not exact and satisfies the conditions of Lemma 3.1. So, an integrating factor for this equation, in x*, exists. The integrating factor is given by* $\mu(x) = e^x$*. Hence,*

$$y'' + (e^x \cos(y) + 2\cos(y))y' + (e^x \sin(y) - 2y\sin(x)) = 0$$

is exact, and therefore, it reduces to the following first order differential equation

$$y' + e^x \sin(y) + 2y\cos(y) = 0.$$

Example 3.3. *Consider the nonlinear second order differential equation*

$$(1 + y^2)yy'' + g(y)y' + (1 + y^2)y = 0,$$

where $g(y)$ *is an arbitrary function in* y*. This equation is not exact. In fact, it has an integrating factor* $\mu(y) = \frac{1}{y(1+y^2)}$ *which transforms this equation into the following exact second order differential equation*

$$y'' + \frac{g(y)}{y(1 + y^2)} y' + 1 = 0.$$

Using the above technique, this equation can be reduced to the following first order differential equation:

$$y' + \int^{y} \frac{g(\xi)}{\xi(1+\xi^2)} d\xi + x = c$$

Example 3.4. *Consider the nonlinear second order differential equation*

$$(1+x)(1+y)y'' + (1+x)y' + (1+y) = 0.$$

This equation is not exact and satisfies the conditions of Lemma 3.3, so an integrating factor in y' *exists. It is easy to obtain, using the conclusion of Lemma 3.3, that* $\mu(y') = e^{y'}$ *is an integrating factor of this equation. Therefore,*

$$e^{y'}(1+x)(1+y)y'' + e^{y'}(1+x)y' + e^{y'}(1+y) = 0.$$

is exact. This fact reduces this equation to

$$e^{y'}(1+x)(1+y) = c.$$

we are looking for an integrating factor of the form $\mu(\alpha(x)\beta(y)\gamma(y'))$, where $\alpha(x), \beta(y)$ and $\gamma(y')$ are arbitrary functions in x, y, and y', respectively. For such an integrating factor to exist, we have the following theorem:

Theorem 3.1. *Assume that Equation* (2.4) *is not an exact equation. Then, an integrating factor* $\mu(\alpha(x)\beta(y)\gamma(y'))$ *of Equation* (2.4) *is given by*

$$\begin{aligned}
\mu(\xi) = \mu(\alpha(x)\beta(y)\gamma(y')) &= \exp\left\{\int^{\xi} \frac{\partial_{y'}a_1 - \partial_y a_2}{\alpha(x)\left[\beta'(y)\gamma(y')a_2 - \beta(y)\gamma'(y')a_1\right]} d\xi\right\} \\
&= \exp\left\{\int^{\xi} \frac{\partial_y a_0 - \partial_x a_1}{\gamma(y')\left[\alpha(x)\beta'(y)a_1 - \alpha'(x)\beta(y)a_0\right]} d\xi\right\} \\
&= \exp\left\{\int^{\xi} \frac{\partial_x a_2 - \partial_{y'}a_0}{\beta(y)\left[\alpha(x)\gamma'(y')a_0 - \alpha'(x)\gamma(y')a_2\right]} d\xi\right\},
\end{aligned}$$

if and only if

$$\begin{aligned}\frac{\partial_{y'}a_1-\partial_y a_2}{\alpha(x)\left[\beta'(y)\gamma(y')a_2-\beta(y)\gamma'(y')a_1\right]} &= \frac{\partial_y a_0-\partial_x a_1}{\gamma(y')\left[\alpha(x)\beta'(y)a_1-\alpha'(x)\beta(y)a_0\right]}\\ &= \frac{\partial_x a_2-\partial_{y'}a_0}{\beta(y)\left[\alpha(x)\gamma'(y')a_0-\alpha'(x)\gamma(y')a_2\right]},\end{aligned}$$

and they depend on $\xi(x,y,y') := \alpha(x)\beta(y)\gamma(y')$.

Proof. The proof is a direct consequence of conditions (3.2). □

Using the above theorem, and by either assuming $\gamma(y')=1$, $\beta(y)=1$, or $\alpha(x)=1$, we can deduce that the integrating factors are $\mu(\alpha(x)\beta(y))$, $\mu(\alpha(x)\gamma(y'))$ and $\mu(\beta(y)\gamma(y'))$, respectively. The results are listed in the following corollaries:

Corollary 3.1. *An integrating factor,* $\mu(\alpha(x)\beta(y))$, *of Equation* (2.4)*is given by*

$$\begin{aligned}\mu(\alpha(x)\beta(y)) &= \exp\left\{\int^{\xi}\frac{\partial_{y'}a_1-\partial_y a_2}{\alpha(x)\beta'(y)a_2}d\xi\right\}\\ &= \exp\left\{\int^{\xi}\frac{\partial_{y'}a_0-\partial_x a_2}{\alpha'(x)\beta(y)a_2}d\xi\right\}\\ &= \exp\left\{\int^{\xi}\frac{\partial_y a_0-\partial_x a_1}{\alpha(x)\beta'(y)a_1-\alpha'(x)\beta(y)a_0}d\xi\right\},\end{aligned}$$

if and only if

$$\frac{\partial_{y'}a_1-\partial_y a_2}{\alpha(x)\beta'(y)a_2}=\frac{\partial_{y'}a_0-\partial_x a_2}{\alpha'(x)\beta(y)a_2}=\frac{\partial_y a_0-\partial_x a_1}{\alpha(x)\beta'(y)a_1-\alpha'(x)\beta(y)a_0},$$

and they depend on $\xi(x,y) := \alpha(x)\beta(y)$.

Corollary 3.2. *An integrating factor, $\mu(\alpha(x)\gamma(y'))$, of Equation (2.4) is given by*

$$\begin{aligned}\mu(\xi) &= \mu(\alpha(x)\gamma(y'))\\ &= \exp\left\{\int^{\xi}\frac{\partial_y a_2-\partial_{y'}a_1}{\alpha(x)\gamma'(y')a_0}d\xi\right\}\\ &= \exp\left\{\int^{\xi}\frac{\partial_x a_1-\partial_y a_0}{\alpha'(x)\gamma(y')a_1}d\xi\right\}\\ &= \exp\left\{\int^{\xi}\frac{\partial_{y'}a_0-\partial_x a_2}{\alpha'(x)\gamma(y')a_2-\alpha(x)\gamma'(y')a_0}d\xi\right\},\end{aligned}$$

provided that

$$\frac{\partial_y a_2-\partial_{y'}a_1}{\alpha(x)\gamma'(y')a_0}=\frac{\partial_x a_1-\partial_y a_0}{\alpha'(x)\gamma(y')a_1}=\frac{\partial_{y'}a_0-\partial_x a_2}{\alpha'(x)\gamma(y')a_2-\alpha(x)\gamma'(y')a_0},$$

and they depend on $\xi(x,y'):=\alpha(x)\gamma(y')$.

Corollary 3.3. *An integrating factor, $\mu(\beta(y)\gamma(y'))$, of Equation (2.4) is given by*

$$\begin{aligned}\mu(\xi) &= \mu(\beta(y)\gamma(y'))\\ &= \exp\left\{\int^{\xi}\frac{\partial_y a_0-\partial_x a_1}{\beta'(y)\gamma(y')a_1}d\xi\right\}\\ &= \exp\left\{\int^{\xi}\frac{\partial_x a_2-\partial_{y'}a_0}{\beta(y)\gamma'(y')a_0}d\xi\right\}\\ &= \exp\left\{\int^{\xi}\frac{\partial_{y'}a_1-\partial_y a_2}{\beta'(y)\gamma(y')a_2-\beta(y)\gamma'(y')a_1}d\xi\right\},\end{aligned}$$

provided that

$$\frac{\partial_y a_0-\partial_x a_1}{\alpha'(y)\beta(y')a_1}=\frac{\partial_x a_2-\partial_{y'}a_0}{\alpha(y)\beta'(y')a_0}=\frac{\partial_{y'}a_1-\partial_y a_2}{\alpha'(y)\beta(y')a_2-\alpha(y)\beta'(y')a_1},$$

and they depend on $\xi(y,y'):=\beta(y)\gamma(y')$.

4. Conclusions and Remarks

In this paper, we imposed conditions on the equation

$$a_2(x,y,y')y''+a_1(x,y,y')y'+a_0(x,y,y')=0,$$

so that it is exact. In addition, we introduced an integrating factor in case where the equation is not an exact differential equation. Moreover, we presented some examples showing that this method is powerful in solving a class of second order nonlinear differential equations. For further studies, it is reasonable to improve this definition and this technique to a more complicated class of differential equations. For example, if we consider the general form of the second order nonlinear differential equation $f(x, y, y', y'') = 0$. Also, it is reasonable to improve this method to work for higher order nonlinear differential equations.

Acknowledgment

We would like to thank the editor and the referees for their valuable comments on this paper.

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